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Calculational Techniques in Perturbative QCD: The Drell-Yan Process*

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Abstract

A detailed calculation of the $\mathcal{O}(\alpha_s)$ Drell-Yan reaction in the dimensional regularization scheme is presented with special emphasis on the real corrections. The concept of factorization and renormalization of parton distribution functions is discussed in connection with the $\mathcal{O}(\alpha_s)$ corrections. Some useful formulae for calculating cut diagrams and integrals in n dimensions are collected in an extensive appendix.

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1 Introduction

The answer to the question "What is a hadron?" will depend on the scale at which the hadron is observed. At the atomic scale the hadrons behave pointlike. At the strong interaction scale of the order of $\Lambda \simeq 1$ GeV the hadrons are composed from quarks and gluons (the so-called *partons* [1]), themselves elementary particles (at this scale). They obey the laws of Quantum Chromodynamics (QCD) which is a non-abelien SU(3) gauge theory [2].

Usually hadrons are probed in high energy scattering experiments. The cross sections for such experiments depend on the scattering scale Q, a typical particle mass m and the renormalization scale μ. The strong coupling constant is scale dependent. An important property of QCD is its asymptotic freedom [3], which states that the coupling between quarks and gluons vanishes for asymptotic small distances. For $\mu \to \infty$ the coupling behaves as

$$g(\mu) \sim \frac{1}{\ln\left(\frac{\mu}{\Lambda}\right)} \quad . \tag{1}$$

This allows a perturbative calculation of the scattering cross section in the limit of large scattering scales Q if we set $\mu = Q$. However, it turns out that in an n-loop calculation the coupling will appear in the combination $g^{2n}(Q)\ln^{kn}(Q/m)$ for $k = 1$ or 2 [4] which is no longer small for large Q, since $m \ll Q$ eventually. The presence of the large logarithm $\ln(Q/m)$ spoils the convergence of the perturbative expansion and indicates that contributions from long distances are important in the cross section. Asymptotic freedom is a property of the coupling at short distances only. Still, it is possible to apply perturbative QCD to the calculation of high energy scattering cross sections through the application of an appropriate *factorization theorem*, which separates short and long distance behaviours.

For our purposes we can regard factorization as an established theorem of QCD, although there are some unresolved questions, see [4, 5] for a review. A neat intuitive approach to factorization for the case of deeply inelastic lepton-hadron scattering (DIS), for which a schematic model is shown in Fig. 1, is given in [6]. In the center of mass (c.m.) system both, the electron and the hadron, move with very high energies in opposite directions, shown in Fig. 1a. The hadron of momentum p is Lorentz contracted and consists of a number of partons in a virtual state with momentum fractions $\xi_i p$ with $\xi_i \in [0, 1]$. The lifetime of the virtual state in the rest frame of the hadron will be time dilated in the c.m. system, therefore the time it takes the electron to cross the hadron is very small (and vanishes in the limit of very high energies). During the collision the electron thus sees a frozen distribution of partons in the hadron, see Fig. 1b. The partons have no time to interact with each other and thus can be described by a probability distribution of finding a parton of momentum fraction ξ_i in the hadron. The electron exchanges a large transverse four-momentum $q := k - k'$ if it comes as close as $\mathcal{O}(1/Q)$ ($Q^2 \equiv -q^2$) to one of the partons; this follows from the uncertainty principle. The interaction between the electron and the parton is called the *hard scattering* and is a perturbatively calculable short distance cross section, if Q is sufficiently large. The formation of the hadron takes

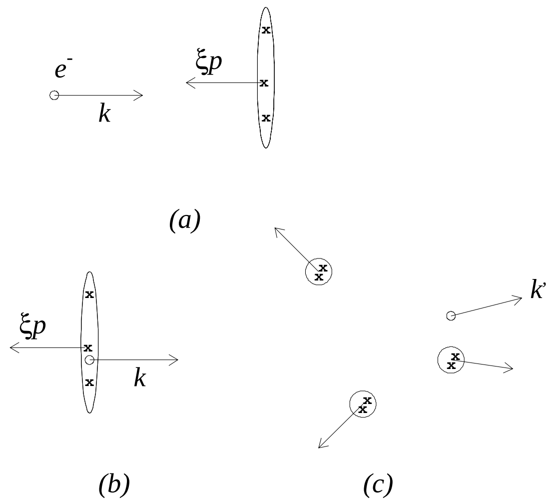

Fig. 1 Schematic picture for deep-inelastic scattering based on the parton model.

place a long time before the collision and the hadronization of the fragments, shown in Fig. 1c, takes place a long time after the collision. These long distance behaviours of the cross section therefore decouple from the short distance behaviour. The probability for the electron to find an additional parton within the hadron is suppressed by a geometrical factor $\mathcal{O}(1/Q^2)$.

Mathematically, the factorization of hard and soft processes in the hadronic cross section $d\sigma^H(S)$, where S is the c.m. energy, is written as a convolution of the hard scattering cross section $d\sigma(\xi_i S)$ with the parton density function (PDF) $f_{iA}(\xi_i)$ that describes the probability of finding a parton i with momentum fraction ξ_i inside the hadron A. The total cross section is given by summing over all partons i:

$$d\sigma^H = \sum_i \int_0^1 d\xi_i d\sigma(\xi_i S) f_{iA}(\xi_i) \quad . \tag{2}$$

To lowest order (LO) the hard scattering cross section is of order $\mathcal{O}(\alpha)$, where α is the electromagnetic coupling constant. The LO cross section is calculated by means of Quantum Electrodynamics (QED) [7]. The parton and the electron exchange a virtual photon of mass $-Q^2$. The calculation is straightforward. Since the partons obey the rules of QCD we can calculate corrections of order $\mathcal{O}(\alpha\alpha_s)$ to the LO process, where α_s is the strong coupling constant. The calculation of these next-to-leading order corrections (NLO) is not as straightforward any more, since various divergences occur.

A detailed calculation of a NLO cross section can be done relatively easy for the Drell-Yan reaction. In this reaction two hadrons collide and produce a pair of leptons. It was first calculated by S. Drell and T. Yan [8] in 1970 (see also [9, 10]). J. Smith has given a lecture on this subject at the CTEQ-DESY summer school 1995, that serves as a basis for the sections 2, 4 and 5 [14]. Some literature already exists on the subject of calculating cross sections in QCD, see e.g. [11, 12] and references therein. For an introduction to the basics of perturbative QCD see [13].

The paper is organized as follows. In section 2 we start by defining the kinematic variables of the Drell-Yan process and calculate the Born cross section. In section 3 we explain the renormalization of parton densities, starting from a discussion of the origins of the soft singularities in the hard cross section. We proceed by calculating the NLO real corrections to the Born process and explain how to extract the singular terms that appear in these corrections in section 4. We will not be concerned with the calculation of the virtual corrections since it is our main purpose in this paper to discuss the renormalization of PDF's. The final result is presented in section 5. The appendix summarizes a number of useful formulae and techniques, used throughout this paper.

2 The Born Cross Section

The Drell-Yan process for the case of muon production is

$$A(P_1) + B(P_2) \to \mu^+(k_1) + \mu^-(k_2) + X \quad , \tag{3}$$

where the incoming hadrons A and B have momenta P_1 and P_2, respectively, the outgoing muons have momenta k_1 and k_2, and X denotes any additional final-state particles. The four-momentum of the virtual photon that couples to the lepton pair is $q := k_1 + k_2$, with mass $Q^2 := q^2$. The factorization of long and short distance behaviours holds for hadron-hadron collisions in the same way as explained in section 1 for DIS. The partons are assumed to be spread randomly inside the hadrons, each parton carrying a momentum fraction $x_1, x_2 \in [0,1]$ so that their momenta are $p_1 = x_1 P_1$ and $p_2 = x_2 P_2$, respectively. The masses of the partons and the hadrons are neglected, since they are much smaller than Q^2. The hadronic cross section is written as a convolution of the hard (*partonic*) scattering cross section with the parton densities of the hadrons A and B

$$d\sigma^H(S) = \sum_{i,j} \int dx_1 dx_2 f_{iA}(x_1) d\sigma_{ij}(s) f_{jB}(x_2) \quad , \tag{4}$$

where S is the hadronic and $s = x_1 x_2 S$ the partonic c.m. energy respectively. The parton densities $f_{iA}(x)$ and $f_{iB}(x)$ are not calculable within perturbative QCD and must be determined by experiment. They are universal in the sense that $f_{iA}(x)$ is fixed by the parton i and the hadron A, regardless of the process under consideration. The factorization may be pictured as in Fig. 2. The small blobs represent the PDF's of the initial hadrons and the large blob in the center stands for the hard scattering.

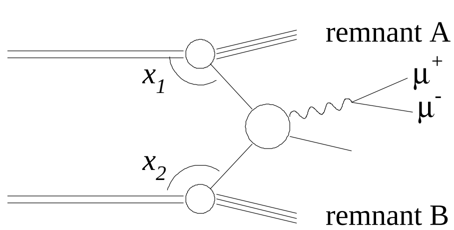

Fig. 2 Factorization of hard and soft processes in the Drell-Yan reaction.

In this section we calculate the cross section on the Born level of order $\mathcal{O}(\alpha)$. The partonic cross section for the Born process $d\sigma$ is given by the squared matrix element, summed over all incoming and outgoing spins, polarizations and colours, multiplied by the phase space of the final-state particles (which are the two outgoing muons) and divided by the flux factor [11, 12]:

$$d\sigma^b = \frac{1}{36}\frac{1}{\cdot 2s}|\overline{\mathcal{M}}_b|^2 d\mathrm{PS}^{(2)} \quad . \tag{5}$$

Here $1/(2s)$ is the flux factor and the factor $1/36$ comes from averaging over the unobserved degrees of freedom of the initial state.

In section 4 the method of *dimensional regularization* [15, 16] is used, which regularizes the divergencies in $n = 4 - 2\epsilon$ dimensions. To obtain consistent results, all calculations have to be performed in n dimensions. This also applies to the calculation of the finite Born cross section. The leptonic phase space in n dimensions is given by (see e.g. equation (85) in the appendix)

$$d\mathrm{PS}^{(2)} = \frac{1}{(2\pi)^{n-2}}\frac{d^{n-1}k_1}{2\omega_1}\frac{d^{n-1}k_2}{2\omega_2}\delta^n(p_1+p_2-k_1-k_2) \quad , \tag{6}$$

where ω_1 and ω_2 are the energies of the 4-momenta k_1 and k_2, respectively. It is useful to express the momenta k_1 and k_2 in terms of the variables $y = (1+\cos\theta)/2$, where θ is the angle between the outgoing leptons in the c.m. frame, and the c.m. energy given by $s = (p_1+p_2)^2 = (k_1+k_2)^2 = Q^2$. Integrating the phase space in $n = 4-2\epsilon$ dimensions yields the result

$$\int d\mathrm{PS}^{(2)} = \frac{1}{8\pi}\left(\frac{4\pi}{Q^2}\right)^{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \quad . \tag{7}$$

The integration over the c.m. angular variable y has already been carried out, as the Born matrix element does not depend on the angle. The details of the integration can be found in the appendix B, especially equations (94) and (95). To be able to write the cross section differential in Q^2 the term

$$1 = \int dQ^2\delta(s-Q^2) = \frac{1}{s}\int dQ^2\delta(1-Q^2/s) \tag{8}$$

is inserted into the phase space, so that

$$\frac{d\mathrm{PS}^{(2)}}{dQ^2} = \frac{1}{8\pi s}\left(\frac{4\pi}{Q^2}\right)^{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\delta(1-Q^2/s) \quad . \tag{9}$$

The matrix element $|\overline{\mathcal{M}}_b|^2$ is now calculated using the techniques explained in the appendix A. The calculation of the cut diagram drawn in Fig. 3 gives a product of two traces and two propagators:

$$|\overline{\mathcal{M}}_b|^2 = Q_f^2 e^4 \mathrm{Tr}[\not{k}_1\gamma_\mu\not{k}_2\gamma_\nu]\frac{1}{(q^2)^2}\mathrm{Tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] \quad . \tag{10}$$

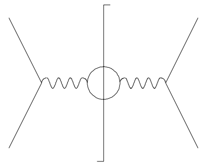

Fig. 3 The cut diagram representing the Born matrix element.

The quark charge is eQ_f, where e is the electric charge. The matrix element separates into a lepton trace and a quark trace, which we denote by $L_{\mu\nu}$ and $H^{\mu\nu}$, respectively. The phase space integral of the outgoing particles and the photon propagators are included into the lepton tensor, since these can always be separated from the hadronic part:

$$L_{\mu\nu} := \frac{1}{q^4}\mathrm{Tr}(\not{k}_1\gamma_\mu\not{k}_2\gamma_\nu)d\mathrm{PS}^{(2)} \quad . \tag{11}$$

From current conservation $q^\mu L_{\mu\nu} = 0$ can be deduced so that the lepton tensor has the form $L_{\mu\nu} = (q^2 g_{\mu\nu} - q_\mu q_\nu)L(q^2)$. The trace of $L_{\mu\nu}$ gives $L_\mu^\mu = 3q^2 L(q^2)$. Combining equations (7) and (11) the trace of the Lepton tensor is

$$L_\mu^\mu(q) = \frac{e^2}{(q^2)^2}\mathrm{Tr}[\not{k}_1\gamma_\mu\not{k}_2\gamma^\mu]d\mathrm{PS}^{(2)} = -\frac{2\alpha}{q^2}\left(\frac{4\pi}{Q^2}\right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \quad , \tag{12}$$

so that for $\epsilon \to 0$ the result $L(q^2) = -(2/3)(\alpha/q^4)$ is obtained. Current conservation also holds for the hadron tensor $H^{\mu\nu}$ so that $q_\mu H^{\mu\nu} = 0$ and $L_{\mu\nu}H^{\mu\nu} = q^2 g_{\mu\nu}H^{\mu\nu}L(q^2)$. Performing the quark trace for the hadron tensor gives

$$H_\mu^\mu = g_{\mu\nu}H^{\mu\nu} = -\frac{48\pi\alpha}{3q^2}Q_f^2 s \quad . \tag{13}$$

The Born cross section in terms of the leptonic and the hadronic tensors is given by $d\sigma^b = (1/72s)L_{\mu\nu}H^{\mu\nu}$, so that the final answer for the Born cross section in $n = 4$ space time dimensions is given by

$$\frac{d\sigma^b}{dQ^2} = \frac{4\pi\alpha^2}{9Q^2 s}\, Q_f^2\, \delta(1 - Q/s) \quad , \tag{14}$$

where the delta function comes from the phase space. The integration over Q^2 gives the total Born cross section $\sigma_B(s) := 4\pi\alpha^2/(9Q^2 s)Q_f^2$, so we can rewrite (14) simply as $d\sigma^b/dQ^2 = \sigma_B\delta(1 - Q^2/s)$.

3 Renormalization of PDF's

The Born cross section is finite. However, in the calculation of the NLO matrix elements singularities will occur in certain regions of the phase space. In Fig. 4 the Feynman diagrams for the amplitudes of the Born subprocess and the $O(\alpha_s)$ correction containing a real gluon emission are drawn. We consider only the divergence occuring for a parton with momentum p_1 emitted from hadron A. The same argument holds for the parton

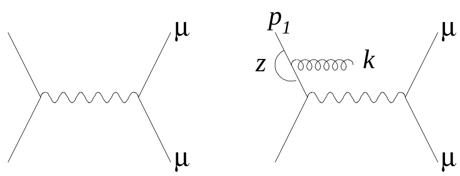

Fig. 4 Hard cross section: Born graph and real gluon emission.

emitted from hadron B. Taking k to be the momentum of the outgoing gluon the Feynman diagram contains a propagator of the form

$$G \sim \frac{1}{(p_1-k)^2} \quad . \tag{15}$$

The propagator diverges if the particle momentum is on-mass-shell, since the limit of massless quarks is considered. The denominator

$$(p_1-k)^2 = -2p_1k = -2|p_1||k|(1-\cos\theta) \quad , \tag{16}$$

where θ is the angle between the gluon and the parton, vanishes if $\cos\theta = 1$ (which is called a *collinear* divergence) and if $|k| = 0$ (called a *soft* divergence, since the energy of the gluon vanishes). However, it can be proven that in the hard cross section the short distance (finite) parts and the long distance (singular) parts factorize in a similar way as in the hadronic cross section [4, 5, 6]. We define the *bare* partonic cross section $d\sigma$, which is calculable in perturbative QCD, a *renormalized* (finite) partonic cross section $d\bar{\sigma}$ and *transition functions* Γ_{ij} so that

$$d\sigma_{ij}(s) = \int dz_1 dz_2 \Gamma_{ik}(z_1) d\bar{\sigma}_{kl}(z_1 z_2 s) \Gamma_{jl}(z_2) \quad . \tag{17}$$

The variables $z_1, z_2 \in [0,1]$ give the momentum fraction of p_1, p_2 in the quark propagator after a gluon is emitted as can be seen in Fig. 4. The singular terms are absorbed into the transition functions in such a way, that the renormalized cross section remains finite. This factorization is pictured in Fig. 5. The residues of the transition functions Γ may

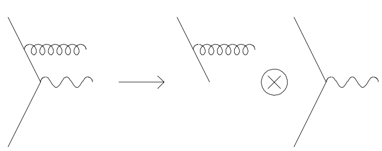

Fig. 5 The factorization theorem for the singular part of the partonic cross section.

be interpreted as the probability densities of finding a quark inside a quark. For the calculation of the hadronic cross section, *renormalized*, i.e. finite, PDF's $\bar{f}$ are needed, which we define as

$$\bar{f}(\eta) := \int_0^1 \int_0^1 dx dz f(x) \Gamma(z) \delta(\eta - xz) = \int_\eta^1 \frac{dz}{z} f\left(\frac{\eta}{z}\right) \Gamma(z) = f(\eta) \otimes \Gamma(\eta) \quad , \tag{18}$$

where the symbol $\otimes$ is introduced to write the convolution in a compact form. Now the infrared safe hadronic cross section, containing purely finite terms, reads

$$d\sigma^H_{AB}(s) = \int d\eta_1 d\eta_2 \bar{f}_{kA}(\eta_1) d\bar{\sigma}_{kl}(\eta_1\eta_2 s)\bar{f}_{lB}(\eta_2) \quad , \tag{19}$$

where the variables $\eta_1, \eta_2 \in [0,1]$ are defined as $\eta_1 = x_1 z_1$ and $\eta_2 = x_2 z_2$. The connection between the renormalized and unrenormalized hadronic cross sections can easily be seen by inserting the definition of the renormalized PDF's (18) into (19) and performing the integrations over η_1 and η_2 using the delta function in the definition (18):

$$d\sigma^H_{AB}(s) = \int dx_1 dx_2 dz_1 dz_2 f_{iA}(x_1)\Gamma_{ik}(z_1) d\bar{\sigma}_{kl}(z_1 z_2 s)\Gamma_{jl}(z_2) f_{jB}(x_2) \quad . \tag{20}$$

Making use of equation (17), equation (4) from section 2 is obtained,

$$d\sigma^H_{AB}(s) = \sum_{i,j} \int dx_1 dx_2 f_{iA}(x_1) d\sigma_{ij}(s) f_{jB}(x_2) \quad , \tag{21}$$

which now contains only unrenormalized quantities. In the case of the Born cross section, this is not dangerous, because the hard cross section does not contain singularities. But in NLO it will be necessary to make use of the renormalization prescription defined here.

How is the renormalized partonic cross section extracted from the unrenormalized, calculable one? The quantities $d\bar{\sigma}, d\sigma$ and Γ are assumed to have perturbative expansions in α_s:

$$\begin{aligned} d\bar{\sigma}(s) &= \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n d\bar{\sigma}^{(n)}(s) & (22)\\ d\sigma(s) &= \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n d\sigma^{(n)}(s) & (23)\\ \Gamma_{ik}(z) &= \delta_{ik}\delta(1-z) + \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n \Gamma^{(n)}_{ik}(z) & (24)\end{aligned}$$

The first term in the expansion of the transition function is a delta function, because no gluon is radiated from the quark in LO, so the probability of finding a quark inside a quark must be one. Inserting the expansions (22)–(24) into (17) gives up to $\mathcal{O}(\alpha_s)$

$$\begin{aligned} d\sigma^{(0)}_{ij}(s) + \frac{\alpha_s}{2\pi} d\sigma^{(1)}_{ij}(s) &= d\bar{\sigma}^{(0)}_{ij}(s) + \frac{\alpha_s}{2\pi}\Big[d\bar{\sigma}^{(1)}_{ij}(s) + \int dz_1\, \Gamma^{(1)}_{ik}(z_1) d\bar{\sigma}^{(0)}_{kj}(z_1 s) \\ &+ \int dz_2\, d\bar{\sigma}^{(0)}_{ik}(z_2 s)\Gamma^{(1)}_{kj}(z_2)\Big] \quad . & (25)\end{aligned}$$

Comparing the left hand and the right hand side in LO gives $d\bar{\sigma}^{(0)} = d\sigma^{(0)}$, i.e. the renormalized and unrenormalized Born cross section are the same. The NLO correction can be obtained by comparing the left hand and right hand side to order α_s and rearranging the terms:

$$d\bar{\sigma}^{(1)}_{ij} = d\sigma^{(1)}_{ij} - \int dz_1 \Gamma^{(1)}_{ik} d\sigma^{(0)}_{kj} - \int dz_2 \Gamma^{(1)}_{ik} d\sigma^{(0)}_{kj} \quad . \tag{26}$$

Thus the prescription for subtracting the singular parts from the bare cross section is simple: the singularity from $d\sigma^{(1)}$ is removed by the convolution of the finite Born cross section with the $\mathcal{O}(\alpha_s)$ transition functions (which of course are singular).

What follows in the next section is an explicit calculation of the bare cross section in $\mathcal{O}(\alpha_s)$. To handle the singularities a regularization procedure is needed. A number of different methods are in use. As already mentioned in section 2, we choose the dimensional regularization method. The aim of this regularization is to isolate the singularities in pole terms like $1/\epsilon$ or $1/\epsilon^2$. After the poles have been removed, the limit $\epsilon \to 0$ is taken and the four dimensional result is obtained.

4 The $\mathcal{O}(\alpha_s)$ Corrections

The $\mathcal{O}(\alpha_s)$ corrections to the Born process include real and virtual corrections. The real corrections arise from the additional radiation of a gluon from one of the partons in the initial state (the subproces $q\bar{q} \to g\mu\bar{\mu}$) or the interaction of a gluon from one of the hadrons with a quark of the other hadron (the subprocess $qg \to q\mu\bar{\mu}$), whereas the virtual corrections contain self-energy diagrams and vertex corrections. At the end of this section the structure of the result for the virtual corrections is briefly discussed.

We now calculate the cross section for the emission of a real gluon from one of the incoming quarks $q\bar{q} \to g\mu\bar{\mu}$. The same notation as in section 2 is chosen, in addition the momentum of the outgoing gluon is denoted by k_3. The partonic cross section is similar to equation (5), only that one additional outgoing parton has to be included into the phase space, so that it contains three final-state particles:

$$d\sigma^r = \frac{1}{72s}|\overline{\mathcal{M}}_r|^2 d\mathrm{PS}^{(3)} \quad . \tag{27}$$

To apply the splitting of the matrix element into a leptonic and a hadronic part, as in section 2 the three particle phase space has to be split into the two particle phase space of the muons and a two particle phase space including the intermediate photon and the outgoing gluon. The decomposition of the phase space is illustrated in Fig. 6. Mathematically speaking, the splitting is achieved by inserting [15]

$$1 = \int \frac{dQ^2}{2\pi} \int \frac{d^{n-1}q}{(2\pi)^{n-1}2E_q} \delta^{(n)}(q-k_1-k_2)(2\pi)^n \tag{28}$$

for the intermediate photon $q = k_1 + k_2$ into the three particle phase space

$$d\mathrm{PS}^{(3)} = \frac{d^{n-1}k_1}{2E_1(2\pi)^{n-1}} \frac{d^{n-1}k_2}{2E_2(2\pi)^{n-1}} (2\pi)^n \frac{d^{n-1}k_3}{2E_3(2\pi)^{n-1}} \delta^n(p_1+p_2-k_1-k_2-k_3) \quad , \tag{29}$$

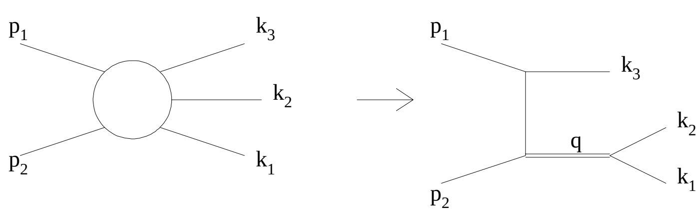

Fig. 6 Splitting of the three particle phase space.

which yields

$$
\begin{aligned}
d\mathrm{PS}^{(3)} &= \frac{dQ^2}{2\pi}\; d\mathrm{PS}_H^{(2)}\; d\mathrm{PS}_L^{(2)} \\
&= \frac{dQ^2}{2\pi}\left\{\frac{d^{n-1}k_3}{2E_3(2\pi)^{n-1}}\frac{d^{n-1}q}{2E_q(2\pi)^{n-1}}\delta^n(p_1+p_2-k_3-q)(2\pi)^n\right\} \\
&\times \left\{\frac{d^{n-1}k_1}{2E_1(2\pi)^{n-1}}\frac{d^{n-1}k_2}{2E_2(2\pi)^{n-1}}(2\pi)^n\delta^n(q-k_1-k_2)\right\} \quad .
\end{aligned} \tag{30}
$$

The leptonic phase space $d\mathrm{PS}_L^{(2)}$ is the same as in the previous section, namely equation (7). The details of calculating the hadronic phase space are given in appendix B and from equation (94) it follows that

$$
d\mathrm{PS}_H^{(2)} = \frac{1}{8\pi}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\frac{(s-Q^2)^{1-2\epsilon}}{s^{1-\epsilon}}\int_0^1 dy[y(1-y)]^{-\epsilon} \quad . \tag{31}
$$

The integration over y, which parametrizes the angle between the gluon and the photon, has to be done later, since the matrix elements will depend on y. With the achieved separation of the phase spaces, the partonic cross section (27) can be rewritten as

$$
d\sigma^r = \frac{1}{72s}L_{\mu\nu}H^{\mu\nu}d\mathrm{PS}_H^{(2)} \quad , \tag{32}
$$

where the lepton tensor is the same as for the Born case, including the photon propagators and the leptonic phase space.

We proceed by calculating the trace of the hadron tensor. The cut diagrams necessary for calculating the matrix elements are given in Fig. 7. Using the techniques explained in appendix A we write down the traces which have to be calculated. Perfoming all summations in diagram (c) we are left with

$$
|\overline{\mathcal{M}}_r^{(c)}|^2 = -e^2 3C_F g^2\mu^{2\epsilon}J_c Q_f^2 \quad , \tag{33}
$$

where

$$
J_c = \frac{1}{tu}\mathrm{Tr}\,[\not{p}_2\gamma^\alpha(\not{p}_2-\not{k})\gamma_\mu\not{p}_1\gamma_\alpha(\not{p}_1-\not{k})\gamma^\mu] \quad . \tag{34}
$$

Calculating the trace with the help of an algebraic computer program [21, 22] the result

$$
J_c = -8\frac{sQ^2}{tu}(1-\epsilon) + 8\epsilon(1-\epsilon) \tag{35}
$$

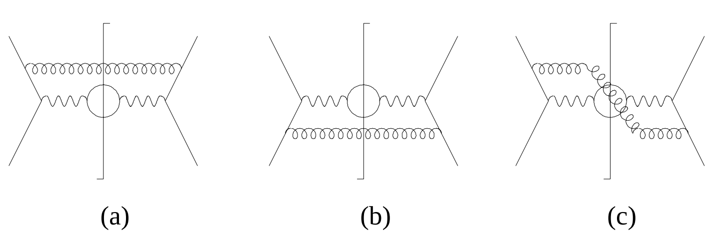

Fig. 7 The cut diagrams for real gluon emision.

is found. The diagram (c) occurs twice in the cross section. In the same manner the diagram (a) is calculated, which gives

$$|\overline{\mathcal{M}}_r^{(a)}|^2 = -e^2 3C_F g^2 \mu^{2\epsilon} J_a Q_f^2 \qquad \text{with} \qquad J_a = 8\frac{u}{t}(1-\epsilon)^2 \quad . \tag{36}$$

The diagram (b) is obtained by crossing $t \leftrightarrow u$ from J_a. Collecting the results, taking diagram (c) twice,

$$\begin{aligned} -\frac{1}{3}H_\mu^\mu &= \frac{1}{3}\Big(2\,|\overline{\mathcal{M}}_r^{(c)}|^2 + |\overline{\mathcal{M}}_r^{(a)}|^2 + |\overline{\mathcal{M}}_r^{(a)}(t\leftrightarrow u)|^2\Big) \\ &= -e^2 C_F g^2 \mu^{2\epsilon}(1-\epsilon)Q_f^2\left[\frac{16sQ^2}{tu} + 8(1-\epsilon)\left(\frac{t}{u}+\frac{u}{t}\right) - 16\epsilon\right] \end{aligned} \tag{37}$$

is found. It is useful to express the Mandelstam variables in terms of Q^2, $y = (1+\cos\theta)/2$ and $z := Q^2/s$ by using the relations

$$s = (p_1+p_2)^2 = \frac{Q^2}{z} \quad , \tag{38}$$

$$\begin{aligned} t &= (p_1-k_3)^2 = -2p_1k_3 = -2\frac{\sqrt{s}}{2}\,|k_3|(1-\cos\theta) \\ &= -\frac{Q^2}{z}(1-z)(1-y) \quad , \end{aligned} \tag{39}$$

$$u = (p_2-k_3)^2 = -\frac{Q^2}{z}(1-z)y \quad . \tag{40}$$

Note that indeed $s+t+u = Q^2$. Now the partonic cross section for real gluon emission is found by multiplying the flux factor, the phase space integral and the matrix elements and using the definition of σ_B:

$$\begin{aligned} \frac{\alpha_s}{\pi}\frac{d\sigma^r}{dQ^2} &= 4\sigma_B\alpha_s C_F\mu^{2\epsilon}\int d\mathrm{PS}_H^{(2)}\left[\frac{2sQ^2}{tu} + (1-\epsilon)\left(\frac{t}{u}+\frac{u}{t}\right) - 2\epsilon\right] \\ &= \sigma_B\frac{\alpha_s}{2\pi}C_F\left(\frac{4\pi\mu^2}{Q^2}\right)^\epsilon \frac{z^\epsilon(1-z)^{1-2\epsilon}}{\Gamma(1-\epsilon)}\int_0^1 dy[y(1-y)]^{-\epsilon}\times \\ &\times\ (1-z)\left[\frac{2z}{(1-z)^2y(1-y)} + (1-\epsilon)\left(\frac{1-y}{y}+\frac{y}{1-y}\right) - 2\epsilon\right] \quad . \end{aligned} \tag{41}$$

In this equation the possible divergencies are explicit. For $\epsilon \to 0$, the terms in the big bracket are divergent for $y \to 0$ and $y \to 1$, so that the integral over the phase space variable y would be infinite. However, the singular terms from the integrals can be extracted if ϵ is kept finite. The y-integrals can be evaluated using the Beta function introduced in the appendix A. Using the properties of the Gamma function [20] (not to be confused with the transition functions) the following integrals are found:

$$\int_0^1 dy\frac{[y(1-y)]^{-\epsilon}}{y(1-y)} = \frac{\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)} = -\frac{2}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \quad , \tag{42}$$

$$\int_0^1 dy[y(1-y)]^{-\epsilon} = \frac{1}{1-2\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \quad , \tag{43}$$

$$\int_0^1 dy[y(1-y)]^{-\epsilon}\frac{y}{(1-y)} = \int_0^1 dy[y(1-y)]^{-\epsilon}\frac{(1-y)}{y} = -\frac{(1-\epsilon)}{\epsilon(1-2\epsilon)}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \; . \tag{44}$$

Inserting these integrals into (41) and the simple expression

$$\frac{\alpha_s}{\pi}\frac{d\sigma^r}{dQ^2} = -\sigma_B\frac{\alpha_s}{\pi}C_F D(\epsilon)\ \frac{2z^\epsilon}{\epsilon}\ \left[2(1-z)^{-1-2\epsilon}z + (1-z)^{1-2\epsilon}\right] \tag{45}$$

is obtained, where the definition

$$D(\epsilon) := \left(\frac{4\pi\mu^2}{Q^2}\right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \tag{46}$$

has been used. Expanding this function to order ϵ, using $a^\epsilon = e^{\epsilon\ln a} \simeq 1 + \epsilon\ln a$, yields

$$D(\epsilon) \simeq 1 + \epsilon(\ln 4\pi - \gamma_E + \ln\frac{\mu^2}{Q^2}) \tag{47}$$

with $D(\epsilon \to 0) = 1$. Now the poles can be extracted from equation (45). Only the first term in the brackets is singular for $z \to 1$ in the case $\epsilon \to 0$. To get a finite answer, use of the plus distribution function as defined in the appendix C is made. First,

$$\int_0^1 \frac{dz}{(1-z)^{1+2\epsilon}} = \frac{\Gamma(1)\Gamma(-2\epsilon)}{\Gamma(1-2\epsilon)} = -\frac{1}{2\epsilon} \quad . \tag{48}$$

The convolution of $(1-z)^{-1-2\epsilon}$ with a test function $f(z)$ gives

$$\int_0^1 \frac{f(z)\ dz}{(1-z)^{1+2\epsilon}} = \int_0^1 dz\frac{f(z)-f(1)+f(1)}{(1-z)^{1+2\epsilon}} = \int_0^1 dz\frac{f(z)-f(1)}{(1-z)^{1+2\epsilon}} - \frac{1}{2\epsilon}f(1) \quad , \tag{49}$$

where use of the integral (48) has been made. Expanding $(1-z)^{-1-2\epsilon}$ to order ϵ gives

$$(1-z)^{-1-2\epsilon} = \frac{1}{1-z}e^{-2\epsilon\ln(1-z)} = \frac{1}{1-z}(1-2\epsilon\ln(1-z)+\ldots) \quad , \tag{50}$$

which, inserted into (49), yields

$$\begin{aligned}\int_0^1 \frac{f(z)\ dz}{(1-z)^{1+2\epsilon}} &= \int_0^1 dz\frac{f(z)-f(1)}{(1-z)} - 2\epsilon\int_0^1 dz\left[f(z)-f(1)\right]\frac{\ln(1-z)}{(1-z)} \\ &- \frac{1}{2\epsilon}\int_0^1 dz f(z)\delta(1-z) \quad .\end{aligned} \tag{51}$$

The first and the second integral on the right hand side of this equation can be written by using the definition of the plus distribution function. Abstracting from the integrals, the equation reads

$$\frac{1}{(1-z)^{1+2\epsilon}} = -\frac{1}{2\epsilon}\delta(1-z) + \left(\frac{1}{1-z}\right)_+ - 2\epsilon\left(\frac{\ln(1-z)}{1-z}\right)_+ + O(\epsilon^2) \quad . \tag{52}$$

The left hand side of (52) convoluted with a test function gives the same result as the convolution of the right hand side with the same test function, as can be seen from (51). We are now ready to write down the final regulated result for the partonic real gluon emission cross section. Going back to (45) and expanding the second term inside the brackets as well as z^ϵ to order ϵ and using (52), we find

$$\begin{aligned}\frac{\alpha_s}{\pi}\frac{d\sigma^r}{dQ^2} &= \sigma_B\frac{\alpha_s}{\pi}C_F D(\epsilon)\left[\frac{2}{\epsilon^2}\delta(1-z)-\frac{2}{\epsilon}\frac{(1+z)^2}{(1-z)_+}\right.\\ &+ \left. 4(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - 2\left(\frac{1+z^2}{1-z}\right)\ln z\right] \quad . \end{aligned} \tag{53}$$

The last term does not give a singularity for $z\to 1$ since $\lim_{z\to 1}\ln(z)/(1-z) = const.$

The question that arises, is how to get rid of the pole terms $1/\epsilon$ and $1/\epsilon^2$. The $O(\alpha_s)$ corrections to the Born cross section include not only the real corrections but also the virtual corrections that are given by

$$d\sigma^v = \frac{1}{72s}|\overline{\mathcal{M}}_v|^2 d\mathrm{PS}^{(2)} \quad . \tag{54}$$

The matrix elements for the virtual corrections are obtained by multiplying the Feynman diagrams for the Born amplitude with the graphs containing vertex and quark self energy corrections, see Fig. 8. Since these corrections contain internal lines the phase space of the final state contains only two final-state particles and is the same as in the Born case. The

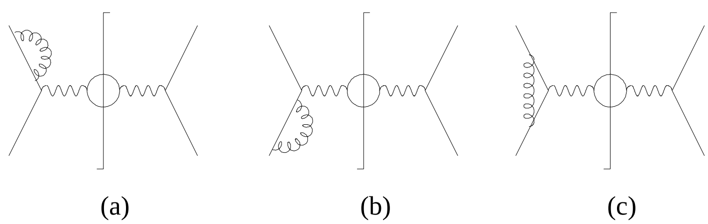

Fig. 8 Cut diagrams for the virtual corrections.

momenta in the loops in the virtual corrections are not fixed and have to be integrated out. These loop integrals are divergent for large loop momenta and give rise to ultra-violet (UV) divergencies. They can be regulated in $1/\epsilon$ poles and removed by adding a counter term to the QCD Lagrangian, where the singularities are absorbed by a redefinition of the quark charge, quark field and gluon field. There are also infrared (IR) divergencies from the loop integrals. If all contributions of the virtual corrections are added together there are some cancellations among the IR contributions, and the result is [9]

$$\frac{\alpha_s}{\pi}\frac{d\sigma^v}{dQ^2} = \sigma_B\frac{\alpha_s}{\pi}C_F D(\epsilon)\delta(1-z)\left[-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}+\frac{2\pi^2}{3}+O(\epsilon)\right] \quad . \tag{55}$$

Adding the real and the virtual corrections $d\sigma^r$ and $d\sigma^v$ cancels the $1/\epsilon^2$ pole. The virtual corrections are multiplied by a delta function which reflects the fact that the final state contains only two particles. This means that poles from the virtual corrections can only cancel contributions from the real corrections that are multiplied by a similar delta function. For the remaining poles the renormalization prescription of the PDF's discussed in section 3 has to be applied.

5 The Finite Result

Adding the real and virtual corrections $d\sigma^r$ and $d\sigma^v$ from equations (53) and (55) the unrenormalized, i.e. bare partonic cross section to order $\mathcal{O}(\alpha_s)$ is found to be

$$\frac{d\sigma^{(1)}}{dQ^2} = \frac{d\sigma^r}{dQ^2} + \frac{d\sigma^v}{dQ^2} = \sigma_B D(\epsilon)\left(-\frac{2}{\epsilon}P_{qq}(z) + R(z)\right) \quad , \tag{56}$$

where we have defined the functions

$$P_{qq}(z) = C_F\left[\frac{(1+z)^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right] \tag{57}$$

and

$$R(z) = C_F\left[\delta(1-z)\left(\frac{2\pi^2}{3} - 8\right) + 4(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - 2\left(\frac{1+z^2}{1-z}\right)\ln z\right] \quad . \tag{58}$$

The function $P_{qq}(z)$ is one of the so-called *Altarelli-Parisi splitting functions* [17]. It describes the probability of finding a quark inside a quark if a gluon is radiated. Equation (26) from section 3 describes how to find the renormalized cross section from the bare cross section:

$$\frac{d\bar{\sigma}^{(1)}(z)}{dQ^2} = \frac{d\sigma^{(1)}(z)}{dQ^2} - \int dz_1 \Gamma^{(1)}(z_1)\frac{d\sigma^{(0)}(z_1)}{dQ^2} - \int dz_2 \Gamma^{(1)}(z_2)\frac{d\sigma^{(0)}(z_2)}{dQ^2} \tag{59}$$

$(z = Q^2/s)$. The Born cross section is

$$\frac{d\sigma^{(0)}(\hat{s})}{dQ^2} = \sigma_B(\hat{s})\delta(1 - Q^2/\hat{s}) \quad , \tag{60}$$

where now $\hat{s} = z_1 s$ or $\hat{s} = z_2 s$. The integrations are performed using the delta functions in (60), so that

$$\frac{d\bar{\sigma}^{(1)}(z)}{dQ^2} = \frac{d\sigma^{(1)}(z)}{dQ^2} - 2\,\Gamma^{(1)}(z)\sigma_B(s) \quad . \tag{61}$$

The transition function is defined as

$$\Gamma^{(1)}(z) := -\frac{1}{\epsilon}\,D(\epsilon)P_{qq}(z) \quad , \tag{62}$$

in order to cancel the $1/\epsilon$ pole term multiplying P_{qq} in (56). With this definition, the final finite partonic cross section for the subprocess $q\bar{q} \to g\mu\bar{\mu}$ from (61) is

$$\frac{d\bar{\sigma}^{(1)}}{dQ^2} = \sigma_B R(z) \tag{63}$$

(where, as there are no singular terms left, the limit $\epsilon \to 0$ with $D(0) = 1$ is taken). The final result for the total Drell-Yan reaction also contains a contribution from the subprocess $qg \to q\mu\bar{\mu}$ that will not be considered here. The calculation is similar to that of section 4 and the result can be found in [9, 10].

The definition (62) is not unique. Since Γ is not directly observable any kind of renormalization scheme can be adopted, which means that finite parts from $R(z)$ can be moved into $\Gamma(z)$. The convention used here is called the *Minimal Subtraction* ($\overline{\mathrm{MS}}$) renormalization scheme [18]. The renormalized PDF's are convoluted with the transition functions, so they also depend on the renormalization scheme. If one uses the PDF's in another calculation, these have to be done with the same scheme to yield consistent results. In addition, the final result (63) will depend on the choice of the renormalization constant μ. The constant μ should be large, so that the perturbative expansion makes sense, but is otherwise arbitrary. In a calculation to all orders this dependence cancels, so one expects the NLO result to be less dependent on μ than the LO result. Normally $\mu^2 \sim Q^2$ is used, since Q is the large scale in the process. One actually finds a reduced dependence on the renormalization scale in NLO. Finally, to make any theoretical prediction, actually two experimental results are needed. One experiment (e.g. DIS) can be used to fix the PDF's. These are then included in the calculation of the cross section of another process, e.g. the Drell-Yan reaction.

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A Calculating Matrix Elements

In this section the matrix elements for real gluon emission are calculated by using the technique of cut diagrams. In Fig. 9 the Feynman diagrams are shown that are necessary for the calculation. In the main text we use the notation shown below as the last of the equivalent symbols, where the incoming particles are not connected. All cut lines are

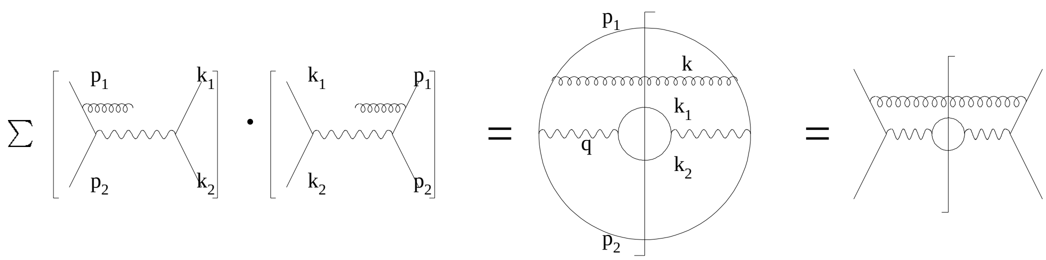


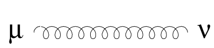
Fig. 9 The Feynman diagrams of the squared matrixelement for real gluon emission and their representation by cut diagrams.

external particles (i.e. ingoing or outgoing) that are on mass shell. Lines that are not cut are virtual particles which are represented by propagators. The Feynman rules used in the calultion of the above cut diagram are collected in Fig. 10 [12]. In $n = 4 - 2\epsilon$ space time dimensions the coupling constant has dimensions $[g] = \epsilon$. Normally g is taken to be dimensionless. To obtain this, the coupling constant g is replaced by $g \to g\mu^\epsilon$ with the renormalization scale μ with dimension $[\mu] = 1$. For $\epsilon \to 0$ the usual dimensionless result is restored.

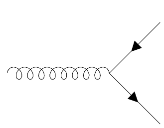
Each closed loop corresponds to a Dirac trace over the loop-momenta. In addition the gluon or photon vertices $(-ie\gamma^\mu)$ of that loop have to be written down. The outer loop over the quark lines in Fig. 9 gives the contribution

$$
\begin{aligned}
&\mathrm{Tr}\quad \left(\not{p}_2(-ie\gamma^\mu)\frac{-i(\not{p}_1 - \not{k})}{(p_1-k)^2}(-ig_s\gamma^\nu T^a)\not{p}_1(+ig_s\gamma^\rho T^a)\frac{+i(\not{p}_1 - \not{k})}{(p_1-k)^2}(+ie\gamma^\sigma)\right) \\
&= \quad \frac{e^2 g_s^2}{(p_1-k)^4}\mathrm{Tr}\left(\not{p}_2\gamma^\mu(\not{p}_1 - \not{k})\gamma^\nu\not{p}_1\gamma^\rho(\not{p}_1 - \not{k})\gamma^\sigma\right)\mathrm{tr}(T^a T^a) \quad .
\end{aligned}
$$

Quarkpropagator: $\frac{i(\not{p}+m)}{p^2-m^2}$



Gluonpropagator: $\frac{i}{k^2}(-g_{\mu\nu})$



Quark–Gluon vertex: $-ig\mu^\epsilon(T_a\gamma_\mu)$

Fig. 10 The QCD Feynman rules for calculating matrix elements in the Feynman gauge.

Note that everything on the right hand side of the cut has to be taken complex conjugated (vertices have opposite sign etc.). Since the gluon line is connecting the two quark–gluon vertices, the same Gell-Mann matrices T^a are taken. This additional trace over the Gell-Mann matrices is distinct from the Dirac trace and given by $\mathrm{tr}(T_a T_a) = 1/2$. For the fermion loop with momenta k_1 and k_2 the same procedure is followed and the trace $e^2 \mathrm{Tr}(\not{k}_1 \gamma_\mu \not{k}_2 \gamma_\sigma)$ is found. The two photon propagators with momenta $q = k_1 + k_2$ give

$$\frac{-i}{(k_1+k_2)^2} \frac{i}{(k_1+k_2)^2} = \frac{1}{q^4} \quad , \tag{64}$$

and the cut (external) gluonline gives $\epsilon_\nu(k)\epsilon^*_\rho(k)$. For summing over outgoing gluons the rule $\sum \epsilon_\nu(k)\epsilon^*_\rho(k) = g_{\nu\rho}$ is used. Collecting the results, the squared matrixelement can be written down:

$$|\mathcal{M}|^2 = \frac{1}{2} \frac{e^4 g_s^2}{(p_1-k)^4} \frac{1}{(k_1+k_2)^4} \mathrm{Tr}\,(\not{p}_2 \gamma^\mu (\not{p}_1 - \not{k}) \gamma^\nu \not{p}_1 \gamma_\nu (\not{p}_1 - \not{k}) \gamma^\sigma)\, \mathrm{Tr}\,(\not{k}_1 \gamma_\mu \not{k}_2 \gamma_\sigma) \quad . \tag{65}$$

The square of the matrix element separates into two tensors, one for the trace over the quarks and the other for the trace over the leptons, namely

$$H^{\mu\rho} = \mathrm{Tr}\,(\not{p}_2 \gamma^\mu (\not{p}_1 - \not{k}) \gamma^\nu \not{p}_1 \gamma_\nu (\not{p}_1 - \not{k}) \gamma^\sigma) \quad , \tag{66}$$

and

$$L_{\mu\rho} = \mathrm{Tr}\,(\not{k}_1 \gamma_\mu \not{k}_2 \gamma_\sigma) \quad . \tag{67}$$

Thus the squared matrix element can be written as a multiplication of the two tensors:

$$|\mathcal{M}|^2 = \frac{1}{2} \frac{e^4 g_s^2}{(p_1-k)^4} \frac{1}{(k_1+k_2)^4} H^{\mu\rho} L_{\mu\rho} \quad . \tag{68}$$

From current conservation $q^\mu L_{\mu\nu} = 0$ is deduced so that the lepton tensor has the form $L_{\mu\nu} = (q^2 g_{\mu\nu} - q_\mu q_\nu) L(q^2)$. Performing the lepton trace $L(q^2) = -(2/3)(\alpha/q^4)$ is found. Current conservation also holds for the hadron tensor $H^{\mu\nu}$ so that $q_\mu H^{\mu\nu} = 0$ and $L_{\mu\nu} H^{\mu\nu} = q^2 g_{\mu\nu} H^{\mu\nu} L(q^2)$. Therefore the trace of the hadron tensor has to be calculated. To calculate the traces appearing in (68) the following Dirac algebra is used, where the indizes run from 0 to $n = 4 - 2\epsilon$ [12]:

$$\begin{aligned}
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu &= 2 g^{\mu\nu} \\
g^{\mu\nu} g_{\mu\nu} &= n \\
\gamma_\mu \not{a} \gamma^\mu &= \gamma_\mu a_\nu \gamma^\nu \gamma^\mu \\
&= \gamma_\mu a_\nu [-\gamma^\mu \gamma^\nu + 2 g^{\mu\nu}] = \not{a}(2-n) = -2(1-\epsilon)\not{a} \\
\gamma_\mu \not{a} \not{b} \gamma^\mu &= 4ab - 2\epsilon \not{a} \not{b} \\
\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu &= -2 \not{c} \not{b} \not{a} + 2\epsilon \not{a} \not{b} \not{c} \\
\mathrm{Tr}[\not{a} \not{b}] &= 4(ab) \\
\mathrm{Tr}[\not{a} \not{b} \not{c} \not{d}] &= 4[(ab)(cd) - (ac)(bd) + (ad)(bc)]
\end{aligned}$$

It is tedious (and even impracticable for higher orders), to do the traces by hand. Fortunately the trace formulas are implemented in computer programs, see e.g. [21, 22]. Using the definition of the Mandelstam variables

$$
\begin{aligned}
s &= (p_1+p_2)^2 = 2p_1p_2 \quad , \\
t &= (p_1-k)^2 = -2p_1k \quad , \\
u &= (p_2-k)^2 = -2p_2k \quad ,
\end{aligned}
$$

finally $H^\mu_\mu = 8(1-\epsilon)^2 u/t$ is found.

B Phase Space Integration

Gamma Function

The Gamma function is defined as [20]

$$
\Gamma(z) = \int\limits_0^1 dt e^{-t} t^{z-1}, \qquad (\mathrm{Re} z > 0) \quad . \tag{69}
$$

The function is shown in Fig. 11. It has poles at negative integers. Some basic properties of Γ are

$$
\begin{aligned}
\Gamma(n+1) &= n! \qquad (n = \text{integer}) \quad , & (70)\\
\Gamma(z+1) &= z\Gamma(z) \qquad (\mathrm{Re}(z) > 0) \quad , & (71)\\
\Gamma(z)\Gamma\left(\frac{1}{2}\right) &= 2^{z-1}\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right) \quad . & (72)
\end{aligned}
$$

Equation (72) is called the *doubling formula.* Some values of the Γ–function and its derivatives are

$$
\begin{aligned}
\Gamma(1) &= \Gamma(2) = 1, \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad , & (73)\\
\Gamma'(1) &= -\gamma_E, \qquad \Gamma''(1) = \gamma_E^2 + \frac{\pi^2}{6} \quad , & (74)
\end{aligned}
$$

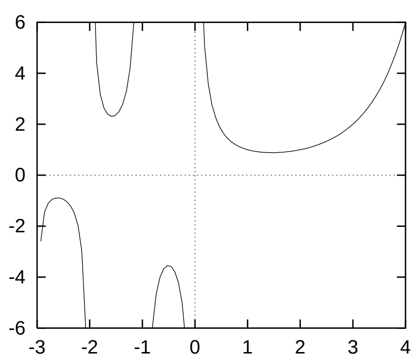


Fig. 11 Graph of the Gamma function.

where γ_E is the Euler-Mascheroni constant. The Taylor expansion of Γ gives, using (74),

$$\Gamma(1+\epsilon) = 1 - \gamma_E\epsilon + \frac{\epsilon^2}{2}(\gamma_E^2 + \frac{\pi^2}{6}) + \ldots \quad , \tag{75}$$

or, from (71),

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \frac{\epsilon}{2}(\gamma_E^2 + \frac{\pi^2}{6}) + \ldots \quad . \tag{76}$$

In n-dimensional integrals the *Beta function*

$$B(\alpha,\beta) := \int_0^1 dy y^{\alpha-1}(1-y)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{77}$$

frequently occurs. Substituting $y = \sin^2\theta$ the useful formula

$$\int_0^{\pi/2} d\theta(\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1} = \frac{1}{2}B(\alpha,\beta) = \frac{1}{2}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{78}$$

is obtained.

Polar Coordinates in n Dimensions

We use the notation $l^2 = l_0^2 - |l|^2 = m_l^2$, with m_l being the mass of a fourvector l_μ, $|l|$ being the length of the three-vector and l_0 being the energy. To begin with we will consider polar coordinates in n (n integer) dimensions, namely $|l|, \phi, \theta_1, \ldots, \theta_{n-2}$. We define

$$d\Omega_{n-1} := d\phi d\theta_1 \sin\theta_1 \ldots d\theta_{n-2}\sin^{n-2}\theta_{n-2} \quad . \tag{79}$$

The n-dimensional phase space in polar coordinates is then given by

$$d^n l = d|l| \; |l|^{n-1} d\Omega_{n-1} \quad . \tag{80}$$

To perform the integration over $d^n l$ equation (78) is used with $\beta = \frac{1}{2}$:

$$\int_0^{\pi} d\theta \sin^n\theta = \sqrt{\pi}\; \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})} \quad . \tag{81}$$

Using this result the integral over the angles

$$\int d\Omega_{n-1} = 2\pi(\sqrt{\pi})^{n-2}\frac{\Gamma(1)}{\Gamma(\frac{3}{2})}\frac{\Gamma(\frac{3}{2})}{\Gamma(2)}\cdot\ldots\cdot\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} = 2\frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} \tag{82}$$

is obtained. The intermediate Gamma functions cancel each other, except for the first numerator and the last denominator. From this line and the defintion of $d\Omega_{n-1}$ equation (79)

$$\int d\Omega_{n-1} = \int d\Omega_{n-2}d\theta_{n-2}\sin^{n-2}\theta_{n-2} = \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}\int d\theta_{n-2}\sin^{n-2}\theta_{n-2} \tag{83}$$

is found. Finally, the integration over the n-dimensional phase space gives

$$\int d^n l = \int dl \; |l|^{n-1} d\Omega_{n-1} = 2\pi \int dl |l|^{n-1} \frac{\pi^{n/2}}{\Gamma(n/2)} \quad . \tag{84}$$

The above formulae hold for any complex n, in particular for $n = 4 - 2\epsilon$.

Two Particle Phase Space

We calculate a two particle phase space for the case that one outgoing particle is massive. Consider two particles with momenta p_1 and p_2 and masses $p_1^2 = 0$ and $p_2^2 = m^2$, from which $|p_1| = E_1$ and $|p_2|^2 = E_2^2 - m^2$ follows. The two particle phase space is given by [12]

$$\int d\mathrm{PS}^{(2)} = \int \frac{d^{n-1}p_1}{2E_1(2\pi)^{n-1}} \frac{d^{n-1}p_2}{2E_2(2\pi)^{n-1}} (2\pi)^n \delta^n(P - p_2 - p_1) \quad . \tag{85}$$

The delta function guarantees the energy-momentum conservation. The phase space is best calculated in the cm frame of the particles p_1 and p_2 with cm energy $\sqrt{s}$:

$$P_\mu = p_{1_\mu} + p_{2_\mu} = (\sqrt{s}, 0, 0, 0) \quad . \tag{86}$$

To integrate over the delta function the identity

$$\begin{aligned} \int dp_2 \delta(p_2^2 - m^2) &= \int dp_2 \delta(E_2^2 - (|p_2|^2 + m^2)) \\ &= \int dp_2 \frac{1}{2\sqrt{|p_2|^2 + m^2}} \left[\delta \left(E_2 + \sqrt{|p_2|^2 + m^2} \right) \right. \\ &+ \left. \delta \left(E_2 - \sqrt{|p_2|^2 + m^2} \right) \right] = \frac{1}{2E_2} \end{aligned} \tag{87}$$

is used. Inserting this into (85), using $p_2^2 = (P - p_1)^2 = s - 2E_1\sqrt{s}$

$$\begin{aligned} \int d\mathrm{PS}^{(2)} &= \int \frac{d^{n-1}p_1}{2E_1(2\pi)^{n-1}} \frac{d^n p_2}{(2\pi)^{n-1}} (2\pi)^n \delta((P - p_1)^2 - m^2) \delta^n(P - p_2 - p_1) \\ &= \frac{1}{(2\pi)^{n-2}} \int \frac{d^{n-1}p_1}{2E_1} \delta(s - 2E_1\sqrt{s} - m^2) \end{aligned} \tag{88}$$

is obtained. Using polarcoordinates in $n - 1$ dimensions

$$d^{n-1}p_1 = |p_1|^{n-2} d|p_1| d\Omega_{n-2} = E_1^{n-2} dE_1 d\Omega_{n-2} \quad , \tag{89}$$

the formula

$$\int d\mathrm{PS}^{(2)} = \frac{1}{(2\pi)^{n-2}} \int dE_1 \frac{E_1^{n-3}}{2} \delta(s - 2E_1\sqrt{s} - m^2) \int d\Omega_{n-2} \tag{90}$$

is calculated. Using formula (83)

$$\int d\mathrm{PS}^{(2)} = \frac{1}{(2\pi)^{n-2}} \frac{2\pi^{(n-2)/2}}{\Gamma(\frac{n-2}{2})} \int dE_1 \frac{E_1^{n-3}}{2} \delta(s - 2\sqrt{s}E_1 - m^2) \int_0^\pi d\theta \sin^{n-3}\theta \tag{91}$$

is obtained. With the use of

$$\delta(s - 2\sqrt{s}E_1 - m^2) = \frac{1}{2\sqrt{s}}\, \delta\left(E_1 - \frac{s-m^2}{2\sqrt{s}}\right) \quad , \tag{92}$$

the integration over the delta function can be performed, yielding

$$\int d\mathrm{PS}^{(2)} = \frac{\pi^{(n-2)/2}}{(2\pi)^{n-2}\Gamma(\frac{n-2}{2})} \frac{1}{2\sqrt{s}} \left(\frac{s-m^2}{2\sqrt{s}}\right)^{n-3} \int_0^\pi d\theta \sin^{n-3}\theta \quad . \tag{93}$$

Substituting $y = (1+\cos\theta)/2$

$$\int d\mathrm{PS}^{(2)} = \frac{1}{8\pi} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \frac{(s-m^2)^{1-2\epsilon}}{s^{1-\epsilon}} \int_0^1 dy [y(1-y)]^{-\epsilon} \tag{94}$$

is found. If the matrix elements are independant of y, the integral over y is performed with the help of (77):

$$\int_0^1 dy[y(1-y)]^{-\epsilon} = \frac{\Gamma^2(1-\epsilon)}{(1-2\epsilon)\Gamma(1-2\epsilon)} \quad . \tag{95}$$

C The Plus Function

Consider a function $F(x)$ that is singular at $x = 1$. The integral of $F(x)$ over the interval $[0,1]$ will diverge. To overcome this problem, the *plus function* is defined as

$$F_+(x) := \lim_{\beta \to 0} \left(F(x)\theta(1-\beta-x) - \delta(1-\beta-x) \int_0^{1-\beta} dy F(y) \right) \quad . \tag{96}$$

It is clear from the defintion, that $F_+(x) = F(x)$ for $x < 1-\beta$. Strictly speaking, the plus function is a distribution. It is well behaved only when it is convoluted with a smooth test function $G(x)$:

$$\int_0^1 dx F_+(x) G(x) = \int_0^1 dx F(x)[G(x) - G(1)] \quad . \tag{97}$$

The property

$$\int_0^1 dx F(x)_+ = 0 \tag{98}$$

of $F_+(x)$ can be easily obtained from (97). If the lower integration boundary is not equal to zero, the convolution with a test function yields

$$\int_a^1 dx F_+(x) G(x) = \int_a^1 dx F(x)[G(x) - G(1)] + G(1) \int_0^a dx F(x) \quad . \tag{99}$$

References

[1] J.D. Bjorken and E.A. Paschos, Phys. Rev. **185** (1969) 1975; R.P. Feynman, Phys. Rev. Lett. **23** (1969) 1415; *Photon-Hadron Interactions* (Benjamin, New York, 1972).

[2] H. Fritzsch, M. Gell-Mann, Proc. XVI Int. Conf. on High Energy Physics, eds. J.D. Jackson et al. (Chicago 1972).

[3] D.J. Gross, F. Wilczek, Phys. Rev. Lett. **30** (1973) 1343;
H.D. Politzer, Phys. Rev. Lett. **30** (1973) 1346.

[4] J.C. Collins, D.E. Soper and G. Sterman in A.H. Mueller, ed., *Perturbative Quantum Chromodynamics* (World Scientific, 1989).

[5] J.C. Collins and D.E. Soper, Ann. Rev. Nucl. Part. Sci. 37 (1987) 383.

[6] G. Sterman et. al., *Handbook of Perturbative QCD*, Rev. Mod. Phys., **67**, No.1 (1995) 168.

[7] J.D. Bjorken and S.D. Drell, *Relativistic Quantum Mechanics* (Mc-Graw-Hill, New York, 1964).

[8] S. Drell, T. Yan, Phys. Rev. Lett. **25** (1970) 316.

[9] G. Altarelli, R.K. Ellis, G. Martinelli, Nucl. Phys. **B143** (1978) 521 (E: **B146** (1978) 544); Nucl. Phys. **B157** (1979) 461.

[10] B. Humpert, W.L. van Neerven, Nucl. Phys. B**184** (1981) 225.

[11] T. Muta, *Foundations of Quantum Chromodynamics* (World Scientific, 1987).

[12] R. D. Field, *Applications of Perturbative QCD* (Addison Wesley, 1989).

[13] D.E. Soper, *Basics of Perturbative QCD*, talk given at the SLAC Summer Institute (August 1996), preprint, hep-ph/9702203.

[14] J. Smith, *The $O(\alpha_s)$ Drell–Yan Reaction*, Lecture notes from the CTEQ–DESY Summer School on QCD–Analysis and Phenomenology (Bad Lauterberg 1995).

[15] G. 't Hooft and M. Veltmann, Nucl. Phys. **B44** (1972) 189;
C.G. Bollini and J.J Giambiagi, Nuovo Cim. **12B** (1972) 20.

[16] W.J. Marciano, Phys. Rev. **D12** (1975) 3861.

[17] G. Altarelli and G. Parisi, Nucl. Phys. **B126** (1977) 298.

[18] W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, Phys. Rev. **D18** (1978) 3998.

[19] E. Byckling, K. Kajantie, *Particle Kinematics* (Jon Wiley, 1972).

[20] M. Abramovitz, I.A. Stegun, *Handbook of Mathematical Functions* (Verlag Harri Deutsch, 1984).

[21] E. Yehudai, preprint Fermilab–Pub–92/22–T (1992).

[22] A.C. Hearn, REDUCE User's Manual, Version 3.4, RAND Publication CP78 (1991).