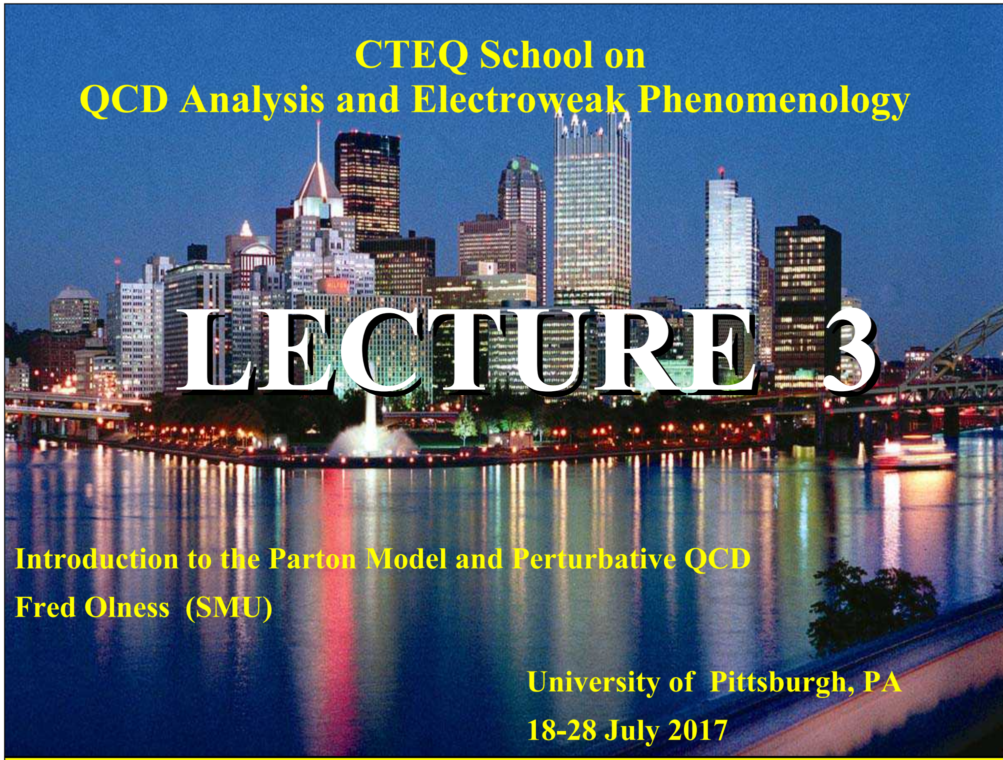

CTEQ School on
QCD Analysis and Electroweak Phenomenology

LECTURE 3

Introduction to the Parton Model and Perturbative QCD

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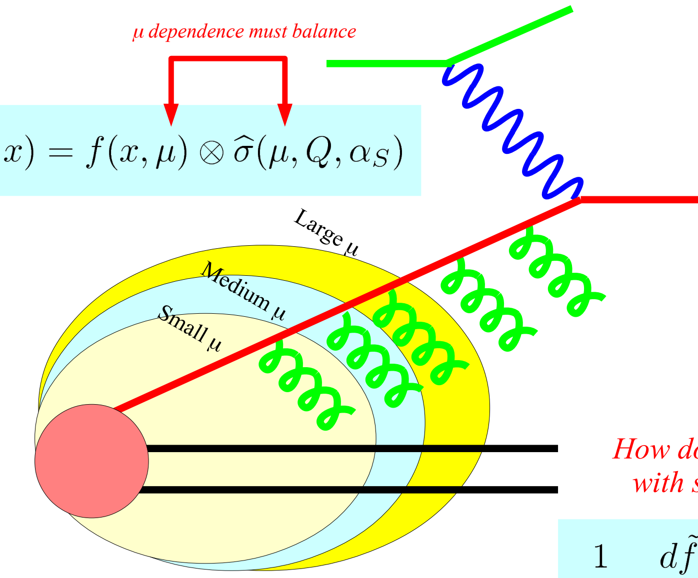
18-28 July 2017

μ dependence must balance

$$\sigma(Q,x) = f(x,\mu) \otimes \widehat{\sigma}(\mu,Q,\alpha_S)$$

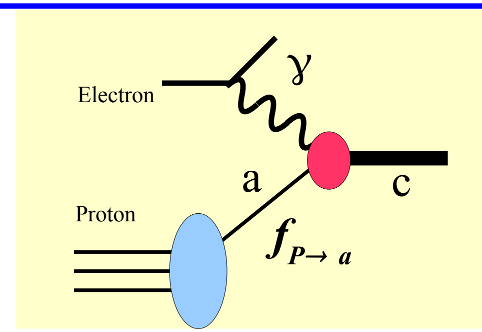


How does f change with scale μ???

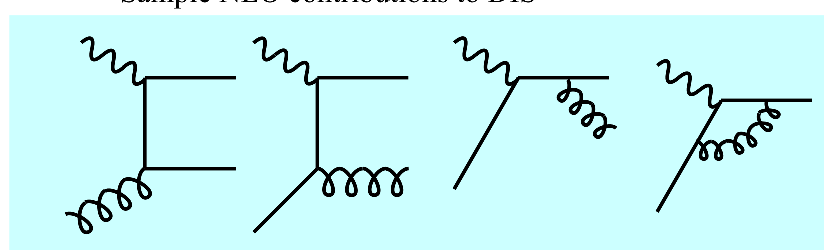
$$\frac{1}{\tilde{f}}\,\frac{d\tilde{f}}{d\ln[\mu]} = -\gamma$$

DGLAP Evolution Equation

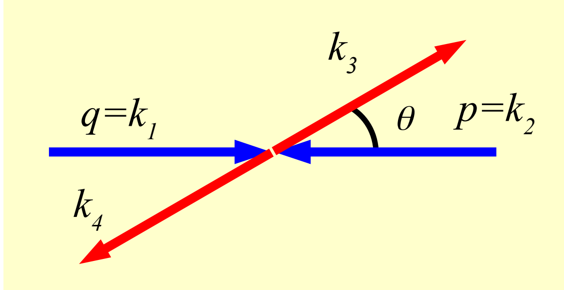
DIS
AT
NLO



Sample NLO contributions to DIS



DIS NLO Kinematics



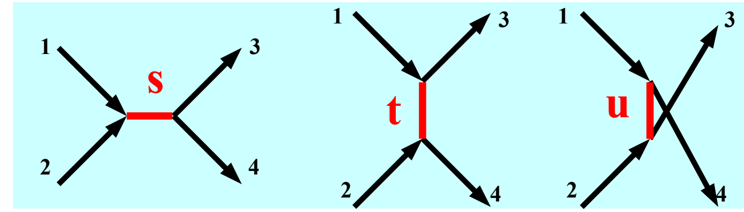
$$k_1 \equiv q^\mu = \left(\frac{s-Q^2}{2\sqrt{s}}, 0, 0, \frac{(s+Q^2)}{2\sqrt{s}}\right) \qquad -q^2 = Q^2 > 0$$

$$k_2 \equiv p^\mu = \left(\frac{s+Q^2}{2\sqrt{s}}, 0, 0, \frac{-(s+Q^2)}{2\sqrt{s}}\right) \qquad p^2 = 0$$

$$k_3^\mu = \tfrac{\sqrt{s}}{2}\,(1, +\sin\theta, 0, +\cos\theta) \qquad k_3^2 = 0$$

$$k_4^\mu = \tfrac{\sqrt{s}}{2}\,(1, -\sin\theta, 0, -\cos\theta) \qquad k_4^2 = 0$$

Mandelstam Variables {s,t,u}



$$\begin{aligned} s &= (k_1+k_2)^2 \equiv (k_3+k_4)^2 \\ t &= (k_1-k_3)^2 \equiv (k_2-k_4)^2 \\ u &= (k_1-k_4)^2 \equiv (k_2-k_3)^2 \end{aligned}$$

$$s+t+u = m_1^2+m_2^2+m_3^2+m_4^2$$

Exercise

{s,t,u} are partonic

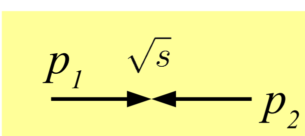
$$s = +Q^2\frac{(1-x)}{x} \qquad t = -Q^2\frac{(1-z)}{2x} \qquad u = -Q^2\frac{(1+z)}{2x}$$

$$x = \frac{Q^2}{2p\cdot q} \qquad x \subset [0,1]$$

$$z \equiv \cos\theta \qquad z \subset [-1,1]$$

Homework

1) Let's work out the general 2→2 kinematics for general masses.



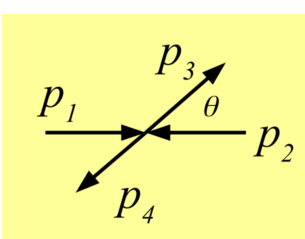
a) Start with the incoming particles.
Show that these can be written in the general form:

$$\begin{aligned} p_1 &= (E_1, 0, 0, +p) \qquad p_1^2 = m_1^2 \\ p_2 &= (E_2, 0, 0, -p) \qquad p_2^2 = m_2^2 \end{aligned}$$

... with the following definitions:

$$E_{1,2} = \frac{\hat{s} \pm m_1^2 \mp m_2^2}{2\sqrt{\hat{s}}} \qquad p = \frac{\Delta(\hat{s}, m_1^2, m_2^2)}{2\sqrt{\hat{s}}}$$

$$\Delta(a,b,c) = \sqrt{a^2+b^2+c^2-2(ab+bc+ca)}$$



Note that $\Delta(a,b,c)$ is symmetric with respect to its arguments, and involves the only invariants of the initial state: s, m_1^2, m_2^2.

b) Next, compute the general form for the final state particles, p_3 and p_4. Do this by first aligning p_3 and p_4 along the z-axis (as p_1 and p_2 are), and then rotate about the y-axis by angle θ.

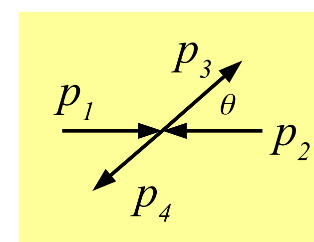
Homework Part 2

PROBLEM #2: Consider the reaction: $pp \to pp$ $(12 \to 34)$ with **CMS** scattering angle θ. The CMS energy is $\sqrt{s} = 2\,TeV$.

a) Compute the boost from the CMS frame to the rest frame of #2 (lab frame)

b) Compute the energy of #1 in the lab frame.

c) Compute the scattering angle θ_{lab} as a function of the CMS θ and invariants.



Hint: by using invariants you can keep it simple.
I.e., don't do it the way Goldstein does.

The power of invariants

Matrix element: NLO DIS

$$|\mathcal{M}|^2 = \frac{s}{-t} + \frac{-t}{s} + \frac{2uQ^2}{st}$$

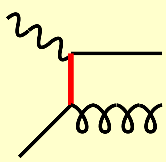
$$\simeq \frac{2(1-x)}{(1-z)} + \frac{2(1-z)}{(1-x)} + \frac{2x(1+z)}{(1-x)(1-z)}$$

For the real $2\to 2$ graphs

Singular at z=1

$z \to 1, \quad \cos\theta \to 1$

$\theta \to 0, \quad t \to 0$

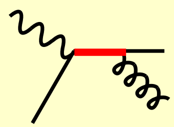


Collinear Singularity

Separate infinity, and subtract

Singular at x=1

$x \to 1, \quad s \to 0$



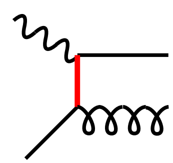
Soft Singularity

Separate infinity, cancel with virtual graphs

The Plan

Collinear Divergences

$$|\mathcal{M}|^2 \underset{z\to 1}{\longrightarrow} \frac{2}{(1-z)} \frac{(1+x^2)}{(1-x)}$$



Looks like a PDF splitting function

Plan

1) Separate ∞ at z=1

2) Subtract … *(should be part of PDF)*

Method

Need to regulate ∞

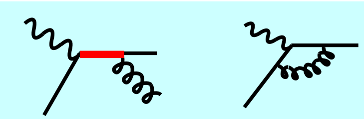
Choices

1) Dimensional Regularization

2) Quark Mass

3) θ Cut

Soft Singularities



Plan

1) Separate ∞ at x=1

2) Cancel between Real and Virtual graphs

Method

Need to regulate ∞

Choices

1) Dimensional Regularization

2) Gluon Mass

3) ...

Dimensional Regularization

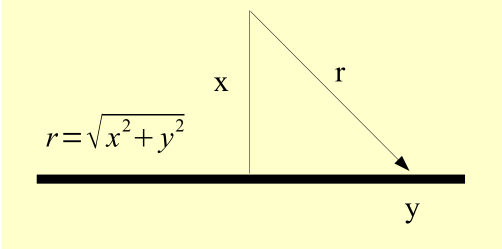
meets

Freshman E&M

M. Hans, Am.J.Phys. 51 (8) August (1983). p.694
C. Kaufman, Am.J.Phys. 37 (5), May (1969) p.560
B. Delamotte, Am.J.Phys. 72 (2) February (2004) p.170

Regularization, Renormalization, and Dimensional Analysis:
Dimensional Regularization meets Freshman E&M.
Olness & Scalise, arXiv:0812.3578 [hep-ph]

Infinite Line of Charge

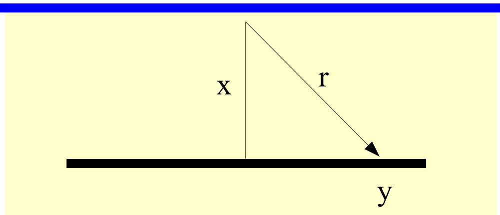


$$dV=\frac{1}{4\pi\epsilon_0}\frac{dQ}{r} \qquad \lambda=Q/y$$

$$V=\frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}dy\,\frac{1}{\sqrt{x^2+y^2}}=\infty$$

Note: ∞ can be very useful

Scale Invariance



$$\begin{aligned}V(kx)&=\\ &=\frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}dy\,\frac{1}{\sqrt{(kx)^2+y^2}}\\ &=\frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}d\left(\frac{y}{k}\right)\frac{1}{\sqrt{x^2+(y/k)^2}}\\ &=\frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}dz\,\frac{1}{\sqrt{x^2+z^2}}\\ &=V(x)\end{aligned}$$

$$V(kx)=V(x)$$

Naively Implies:
V(kx) – V(x) = 0

Note: $\infty + c = \infty$
$\therefore\quad \infty - \infty = c$

How do we distinguish this from

$\infty - \infty = c+17$

Cutoff Method

$$V=\frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{+L}dy\,\frac{1}{\sqrt{x^2+y^2}}$$

$$V=\frac{\lambda}{4\pi\epsilon_0}\log\left[\frac{+L+\sqrt{L^2+x^2}}{-L+\sqrt{L^2+x^2}}\right]$$

V(x) depends on artificial regulator L

We cannot remove the regulator L

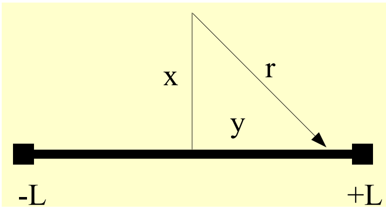
All physical quantities are independent of the regulator:

Electric Field $\quad E(x)=\frac{-dV}{dx}=\frac{\lambda}{2\pi\epsilon_0\,x}\frac{L}{\sqrt{L^2+x^2}}\rightarrow\frac{\lambda}{2\pi\epsilon_0\,x}$

Energy $\quad \delta V=V(x_1)-V(x_2)\underset{L\to\infty}{\rightarrow}\frac{\lambda}{4\pi\epsilon_0}\log\left[\frac{x_2^2}{x_1^2}\right]$

Problem solved at the expense of an extra scale L
AND we have a broken symmetry: translation invariance

Broken Translational Symmetry



Shift: y → y' = y – c

y=[+L+c, -L+c]

$$V=\frac{\lambda}{4\pi\epsilon_0}\int_{-L+c}^{+L+c}dy\;\frac{1}{\sqrt{x^2+y^2}}$$

$$V=\frac{\lambda}{4\pi\epsilon_0}\log\left[\frac{+(L+c)+\sqrt{(L+c)^2+x^2}}{-(L-c)+\sqrt{(L-c)^2+x^2}}\right]$$

V(r) depends on “y” coordinate!!!

In QFT, gauge symmetries are important. E.g., Ward identies

Dimensional Regularization

Compute in n-dimensions

$$dy\rightarrow d^n y=\frac{d\Omega_n}{2}\;y^{n-1}dy$$

$$\Omega_n=\int d\Omega_n=\frac{2\pi^{n/2}}{\Gamma(n/2)}\qquad \Omega_{1,2,3,4}=\{2,2\pi,4\pi,2\pi^2\}$$

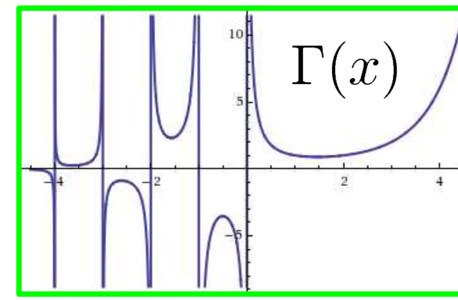
$$V=\frac{\lambda}{4\pi\epsilon_0}\int_0^{+\infty}d\Omega_n\;\frac{y^{n-1}}{\mu^{n-1}}\;\frac{dy}{\sqrt{x^2+y^2}}$$

Each term is individually dimensionaless

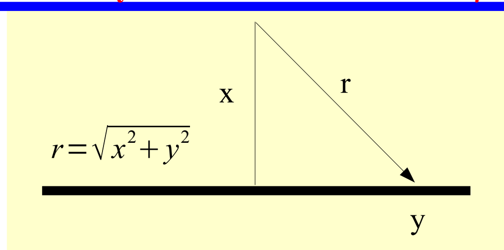
New scale μ

$n=1-2\epsilon$

$$V=\frac{\lambda}{4\pi\epsilon_0}\left(\frac{\mu^{2\epsilon}}{x^{2\epsilon}}\frac{\Gamma[\epsilon]}{\pi^{\epsilon}}\right)$$



Why do we need an extra scale μ ???



$$dV=\frac{1}{4\pi\epsilon_0}\;\frac{dQ}{r}\qquad \lambda=Q/y$$

$$V=\frac{\lambda}{4\pi\epsilon_0}\;f(x)$$

Dimensional Regularization

All physical quantities are independent of the regulators:

Electric Field
$$E(x)=\frac{-dV}{dx}=\frac{\lambda}{4\pi\epsilon_0}\left[\frac{2\epsilon\,\mu^{2\epsilon}\,\Gamma[\epsilon]}{\pi^{\epsilon}\,x^{1+2\epsilon}}\right]\underset{\epsilon\to 0}{\rightarrow}\frac{\lambda}{2\pi\epsilon_0}\;\frac{1}{x}$$

Energy
$$\delta V=V(x_1)-V(x_2)\underset{\epsilon\to 0}{\rightarrow}\frac{\lambda}{4\pi\epsilon_0}\;\log\left[\frac{x_2^2}{x_1^2}\right]$$

Problem solved at the expense of an extra scale μ **AND** regulator ε

Translation invariance is preserved!!!

Dimensional Regularization respects symmetries

Renormalization

$$V \to \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon} + \ln\left[\frac{e^{-\gamma_E}}{\pi}\right] + \ln\left[\frac{\mu^2}{x^2}\right]\right]$$ Original

$$V \to \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon} + \ln\left[\frac{e^{-\gamma_E}}{\pi}\right] + \ln\left[\frac{\mu^2}{x^2}\right]\right]$$ MS

$$V \to \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon} + \ln\left[\frac{e^{-\gamma_E}}{\pi}\right] + \ln\left[\frac{\mu^2}{x^2}\right]\right]$$ MS-Bar

Physical quantities are independent of renormalization scheme!

$$V_{\overline{MS}}(x_1) - V_{\overline{MS}}(x_2) = \delta V = V_{MS}(x_1) - V_{MS}(x_2)$$

But only if performed consistently:

$$V_{\overline{MS}}(x_1) - V_{MS}(x_2) \neq \delta V \neq V_{MS}(x_1) - V_{\overline{MS}}(x_2)$$

Connection to QFT

This was the potential from our "Toy" calculation:

$$V \to \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon} + \ln\left[\frac{e^{-\gamma_E}}{1\ \pi}\right] + \ln\left[\frac{\mu^2}{x^2}\right]\right]$$

This is a partial result from
a real NLO Drell-Yan Calculation:
Cf., B. Potter

$$\frac{D(\epsilon)}{\epsilon} = \left(\frac{4\pi\mu^2}{Q^2}\right)\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \to \left[\frac{1}{\epsilon} + \ln\left[\frac{e^{-\gamma_E}}{4\pi}\right] + \ln\left[\frac{\mu^2}{Q^2}\right]\right]$$

Recap

Regulator provides unique definition of V, f, ω

- Cutoff regulator L:
  - simple, but does NOT respect symmetries
- Dimensional regulator ε:
  - respects symmetries: translation, Lorentz, Gauge invariance
  - introduces new scale μ

All physical quantities (E, dV, σ) are independent of the regulator

- AND the new scale μ
- Renormalization group equation: dσ/dμ=0

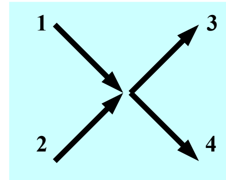
We can define renormalized quantities (V,f,ω)

- Renormalized (V,f,ω) are scheme dependent and arbitrary
- Physical quantities (E,dV, σ) are unique and scheme independent
  - if we apply the scheme consistently

Apply Dimensional Regularization to QFT

D-Dimensional Phase Space

$$d\sigma = \frac{1}{2s}\,|\mathcal{M}|^2\,d\Gamma$$



$$d\Gamma_i = \frac{d^D k_i}{(2\pi)^D}\,(2\pi)\,\delta(k_i^2)$$ 1-particle

$$d\Gamma = d\Gamma_3\,d\Gamma_4\,(2\pi)^D\,\delta^D(k_1+k_2-k_3-k_4)$$ Final state

$$d\Gamma = \frac{1}{16\pi}\left(\frac{s}{16\pi}\right)^{-\epsilon}\frac{(1-z^2)^{-\epsilon}}{\Gamma[1-\epsilon]}\,dz$$ Final state

$$g \to g\,\mu^\epsilon$$ Enter, μ scale

All the pieces

$$d\Gamma = \frac{1}{16\pi}\left(\frac{16\pi\mu^2}{Q^2}\right)^{+\epsilon}\frac{1}{\Gamma[1-\epsilon]}\,\frac{x^\epsilon}{(1-x)^\epsilon}\,(1-z^2)^{-\epsilon}\,dz$$

Homework: Part 1

#1) Show:

$$\frac{d^3p}{(2\pi)^3\,2E} = \frac{d^4p}{(2\pi)^4}\,(2\pi)\,\delta^+(p^2-m^2)$$

This relation is often useful as the RHS is manifestly Lorentz invariant

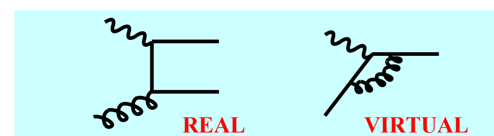
#2) Show that the 2-body phase space can be expressed as:

$$d\Gamma = \frac{d^3p_3}{(2\pi)^3\,2E_3}\,\frac{d^3p_4}{(2\pi)^3\,2E_4}\,(2\pi)^4\,\delta(p_1+p_2-p_3-p_4) = \frac{d\cos(\theta)}{16\pi}$$

Note, we are working with massless partons, and θ is in the partonic CMS frame

Soft Singularities

Soft Singularities



$$\frac{x^\epsilon}{(1-x)^\epsilon}\,\frac{1}{(1-x)} = \frac{1}{(1-x)_+} - \frac{1}{\epsilon}\,\delta(1-x)$$

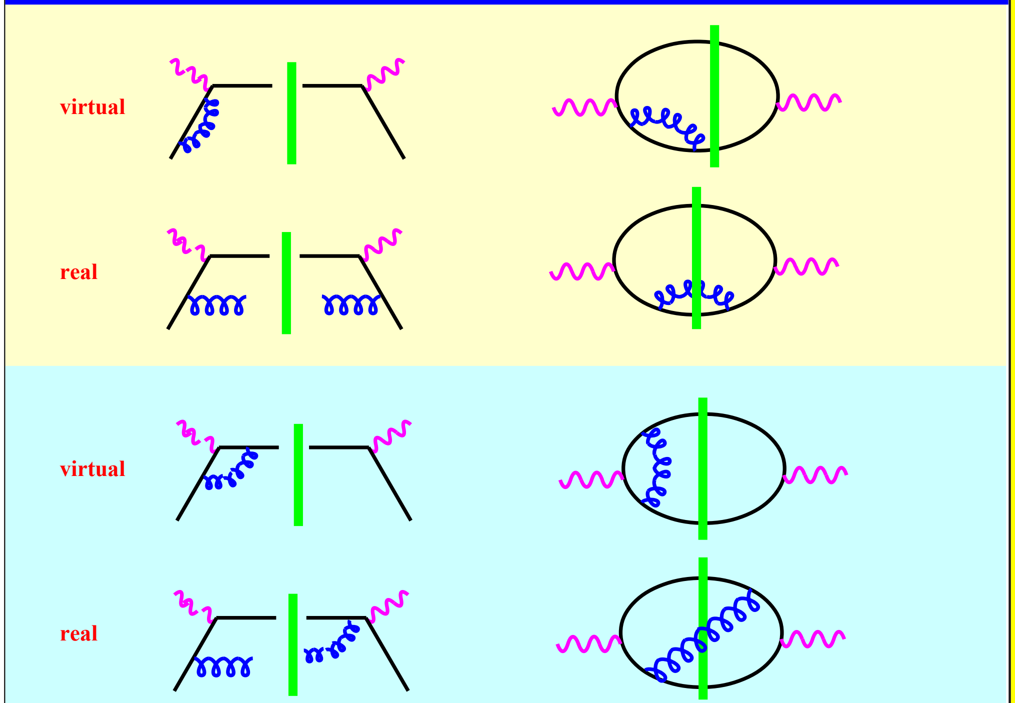
- $\frac{x^\epsilon}{(1-x)^\epsilon}$: From phase space
- $\frac{1}{(1-x)}$: Soft Singularity
- $\frac{1}{(1-x)_+}$: Finite remainder
- $\frac{1}{\epsilon}\,\delta(1-x)$: To be canceled by virtual diagram

This only makes sense under the integral

$$\frac{f(x)}{(1-x)_+} = \frac{f(x)-f(1)}{(1-x)}$$

$$\int_0^1 dx\,f(x)\,\frac{x^\epsilon}{(1-x)^{1+\epsilon}} = \int_0^1 dx\,\frac{f(x)-f(1)}{(1-x)} - \frac{1}{\epsilon}\int_0^1 dx\,\delta(1-x)\,f(x)$$

KLN (Kinoshita, Lee, Nauenberg) Theorem



Collinear Singularities

Collinear Singularity

$$\int_{-1}^{1} dz\,(1-z^2)^{-\epsilon}\,|\mathcal{M}|^2 \simeq -\underbrace{\frac{1}{\epsilon}\,\frac{(1+x^2)}{(1-x)}}_{\text{This looks like part of the PDF}} + \underbrace{\frac{1-4x+4(1+x^2)\,\ln 2}{2(1-x)}}_{\text{This is finite for z=[-1,1]}}$$

This looks like part of the PDF

This is finite for z=[-1,1]

... looks like a splitting kernel

Key Points

1) Subtract
2) This is defined by the scheme
3) Need to match schemes of ω and PDF
   ... MS, MS-Bar, DIS, ...
4) Note we have regulator ε and extra scale μ

How do we know

what goes in ω and PDFs ???

Compute NLO Subtractions

for a <u>partonic</u> target

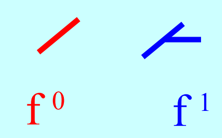
Application of Factorization Formula at Leading Order (LO)

Basic Factorization Formula $\sigma = f \otimes \omega + \mathcal{O}(\Lambda^2/Q^2)$

Higher Twist

At Zeroth Order:

$$\sigma^0 = f^0 \otimes \omega^0 + O(\Lambda^2/Q^2)$$



Use: $f^0 = \delta$ for a parton target.

Therefore: $$\sigma^0 = f^0 \otimes \omega^0 = \delta \otimes \omega^0 = \omega^0$$

$$\sigma^0 = \omega^0$$

Warning: *This trivial result leads to many misconceptions at higher orders*

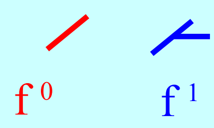
Application of Factorization Formula at NLO

Basic Factorization Formula $\sigma = f \otimes \omega + \mathcal{O}(\Lambda^2/Q^2)$

At First Order:

$$\sigma^1 = f^1 \otimes \omega^0 + f^0 \otimes \omega^1$$

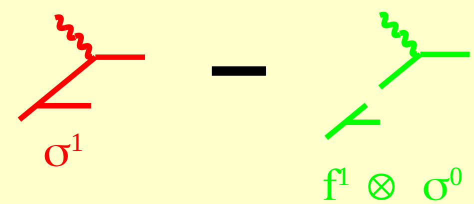
$$\sigma^1 = f^1 \otimes \sigma^0 + \omega^1$$



We used: $f^0 = \delta$ for a parton target.

Therefore: $$\omega^1 = \sigma^1 - f^1 \otimes \sigma^0$$

$\omega^1 =$



$$f^1 \sim \frac{\alpha_s}{2\pi} P^{(1)}$$

$P^{(1)}$ defined by scheme choice

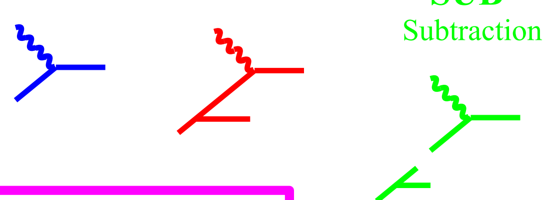
Application of Factorization Formula at NLO

Combined Result:

Complete NLO Term: ω^1

$$\omega^0 + \omega^1 = \sigma^0 + \sigma^1 - f^1 \otimes \sigma^0$$

TOT, LO, NLO, SUB Subtraction



TOT = LO + NLO - SUB

HOMEWORK PROBLEM: NNLO WILSON COEFFICIENTS

Use the Basic Factorization Formula $\sigma = f \otimes \omega \otimes d + \mathcal{O}(\Lambda^2/Q^2)$

At Second Order *(NNLO):*

Therefore: $$\sigma^2 = f^2 \otimes \omega^0 \otimes d^0 + \ldots$$
$$+ f^1 \otimes \omega^1 \otimes d^0 + \ldots$$

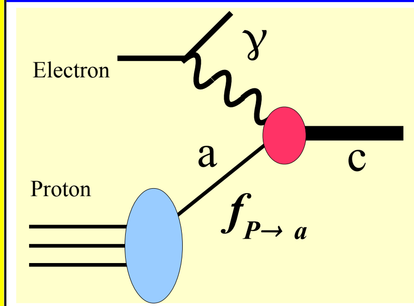
$$\omega^2 = ???$$

Include Fragmentation Functions *d*

Compute ω^2 at second order.
Make a diagrammatic representation of each term.

Do we get different answers with different schemes???

Pictorial Demonstration of Scheme Consistency

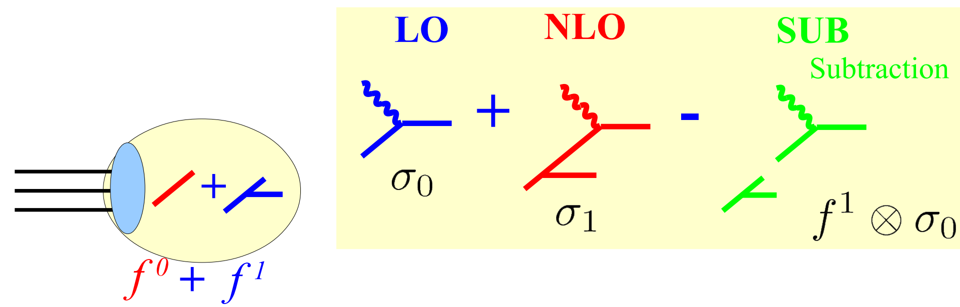


Parton Model

$$\sigma(Q^2) = f(\mu, \alpha_s) \otimes \widehat{\omega}(Q^2, \mu^2, \alpha_s)$$

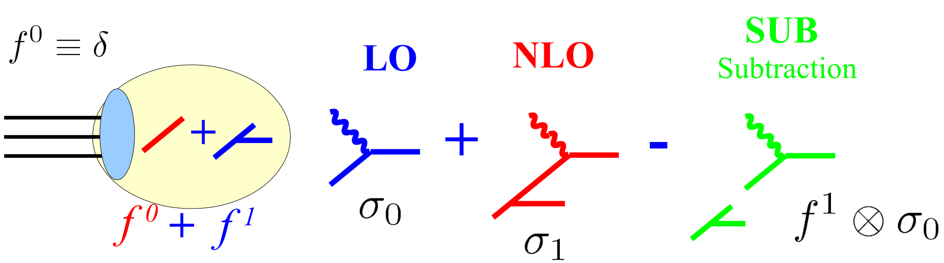
Evolution Equation

$$\frac{df}{\ln[\mu^2]} = P \otimes f$$



Pictorial Demonstration of Scheme Consistency

$f^0 \equiv \delta$



Complete NLO Term

$$[\delta + f^1] \otimes [\sigma_0 + \sigma_1 - f^1 \otimes \sigma_0]$$

$$f^1 \sim \frac{\alpha_s}{2\pi} P^{(1)}$$

$P^{(1)}$ defined by scheme choice

$$\sigma_0 + \sigma_1 + f^1 \otimes \sigma_0 - f^1 \otimes \sigma_0 + \mathcal{O}(\alpha_s^2)$$

From NLO Subtraction

From PDF Evolution

Contains BOTH collinear and non-collinear region

QCD is Bullet-proof

Do we get different answers with different schemes???

NO !!!

End of lecture 3: Recap

- NLO Theoretical Calculations:
  - Essential for accurate comparison with experiments
- We encounter singularities:
  - Soft singularities: cancel between real and virtual diagrams
  - Collinear singularities: "absorb" into PDF
- Regularization and Renormalization:
  - Regularize & Renormalize intermediate quantities
  - Physical results independent of regulators (e.g., L, or μ and ε)
  - Renormalization introduces scheme dependence (MS-bar, DIS)
- Factorization works:
  - Hard cross section $\widehat{\sigma}$ or ω is not the same as σ
  - Scheme dependence cancels out (if performed consistently)

END OF LECTURE 3