Noncommuting Gauge Fields as a Lagrange Fluid

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Noncommuting Abelian Gauge Fields as a Lagrange Fluid

Noncommuting coordinates

$$[x^i, x^j] = i\theta^{ij} \qquad (*)$$

(Heisenberg $\xrightarrow{1930}$ Peierls $\longrightarrow$ Pauli $\longrightarrow$ Oppenheimer $\xrightarrow{1947}$ Snyder)

↖ Suppress short-distance singularities ↖ Lowest Landau Level First paper ↑

Which coordinate transformations preserve (*)?

$$\delta \boldsymbol{x} = -\boldsymbol{f}(\boldsymbol{x}) \quad \Rightarrow \quad f^i(x) = \theta^{ij}\partial_j f(\boldsymbol{x}) \quad \Rightarrow \quad \boldsymbol{\nabla}\cdot\boldsymbol{f} = 0$$

(Subgroup of) volume-preserving diffeomorphisms $\equiv$ (volume)$'$ preserving

NB: In 2-d, (volume)$'$ preserving $\equiv$ area preserving

Fluid Mechanics (Lagrange formulation, Euler formulation)

Lagrange formulation invariant against volume-preserving diffeomorphisms

Suggestion Coincidence of symmetries

leads to other similarities providing useful information

$\Rightarrow$ noncommuting U(1) gauge theory

$\sim$ fluid mechanics in Lagrange formulation

Peierls Substitution

Consider a charged (e), massive (m) particle moving on the (x, y) plane in a constant magnetic field along the z axis

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \quad \begin{cases} A_x = 0 \\ A_y = Bx \end{cases}$$

$$L = \underbrace{\tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{e}{c}\dot{x}A_x + \frac{e}{c}\dot{y}A_y}_{\text{Landau levels}} - \underbrace{V(x, y)}_{\text{impurity}}$$

separation between Landau levels $O(B/m)$

strong field ($B \to \infty$) only lowest Landau level survives

strong field $(B \to \infty) \Longleftrightarrow (m \to 0)$

$$L_0 = \frac{e}{c}Bx\dot{y} - V(x, y) \sim p\dot{q} - H(p, q)$$

$$i[x, y] = \frac{\hbar c}{eB}$$

[G. Dunne, C. Trugenberger, & R.J., PRD **42**, 661 (90); NPB **33**(C) (93)]

Alternate viewpoint

$$\langle LLL \big| [x, y] \big| L'L'L' \rangle = \langle LLL \big| xy \big| L'L'L' \rangle - \langle LLL \big| yx \big| L'L'L' \rangle$$

$$= \sum_n \Big(\langle LLL \big| x \big| n \rangle \langle n \big| y \big| L'L'L' \rangle - (\text{transpose})^* \Big)$$

$\nwarrow$ all states $\Rightarrow [x, y] = 0$

$\uparrow$ LLL states $\Rightarrow [x, y] = -i\dfrac{\hbar c}{eB}$

$\uparrow$ first N LL states $\Rightarrow [x, y] = -i\dfrac{\hbar c}{eB} \begin{bmatrix} O & & \\ & \ddots & \\ & & N+1 \end{bmatrix}$

[G. Magro, quant-ph/0302001]

Realizing non-Commutativity

A.) Define Hilbert space on which x^i act

(like p & q in quantum mechanics)

Define inner product of states.

Define trace

or

B.) Weyl-Moyal method (equivalent)

Ignore non commutativity but replace ordinary product (multiplication) of functions of $\mathbf{x}$ by "star product" $\star$

$$(f \star g)(\mathbf{x}) \equiv exp \frac{i}{2}\left(\theta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right) f(\mathbf{x})\; g(\mathbf{y})\Big|_{\mathbf{x}=\mathbf{y}}$$

$\star$ product associative, but not commutative

$$[x^i, x^j]_\star = x^i \star x^j - x^j \star x^i = i\theta^{ij}$$

$$trace \Leftrightarrow \int dx$$

Field Theory Defined on non-Commutative Space

e.g "φ^4"

action:

$$I = \int dx \left[\tfrac{1}{2}\partial_\mu\varphi \star \partial^\mu\varphi - \tfrac{m^2}{2}\,\varphi \star \varphi - \tfrac{\lambda}{4}\varphi \star \varphi \star \varphi \star \varphi\right]$$

equation of motion:

$$\Box\varphi + m^2\varphi + \lambda\varphi \star \varphi \star \varphi = 0$$

e.g. $U(1)$ gauge theory, A_μ

gauge transformation:

$$A_\mu \to A_\mu + \partial_\mu\theta - i[A_\mu, \theta]_\star \equiv A_\mu + D^\star_\mu\theta$$

field strength:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star$$

gauge variant:

$$F_{\mu\nu} \to F_{\mu\nu} - i[F_{\mu\nu}, \theta]_\star$$

action:

$$I = -\frac{1}{4}\int dx F^{\mu\nu} \star F_{\mu\nu}$$

equation of motion:

$$\partial_\mu F^{\mu\nu} - i[A_\mu, F^{\mu\nu}]_* \equiv D^\star_\mu F^{\mu\nu} = \text{sources}$$

like non-Abelian, commuting gauge theory

NO LOCAL GAUGE-INVARIANT QUANTITIES

Fluid mechanics in Lagrange formulation

$$
\begin{array}{l}
\text{Fluid} \\
\downarrow \text{ coordinates} \\
\boldsymbol{X}(t, \boldsymbol{x}) \\
\qquad \uparrow \text{ labels for fluid coordinates,} \\
\qquad \boldsymbol{X}(0, \boldsymbol{x}) = \boldsymbol{x} \text{ (comoving coordinates)}
\end{array}
$$

Possibility of relabeling $\boldsymbol{x}$ leads to invariance against volume-preserving transformations on $\boldsymbol{x}$: $\delta \boldsymbol{x} = -\boldsymbol{f}(\boldsymbol{x})$, $\quad \boldsymbol{\nabla} \cdot \boldsymbol{f} = 0$, provided $\boldsymbol{X}$ transforms as a scalar $\delta_f \boldsymbol{X} = f^i \partial_i \boldsymbol{X} = \theta^{ij} \partial_j f \partial_i \boldsymbol{X}$

Introduce Poisson bracket with help of θ^{ij}

(assumed to be nonsingular: $\theta^{ij} \omega_{jk} = \delta^i_k$)

$$\{\mathcal{O}_1, \mathcal{O}_2\} = \theta^{ij} \frac{\partial}{\partial x^i} \mathcal{O}_1 \frac{\partial}{\partial x^j} \mathcal{O}_2$$

$$\{x^i, x^j\} = \theta^{ij}$$

$$\delta_f \boldsymbol{X} = \{\boldsymbol{X}, f\}$$

Introduce vector potential $\widehat{\boldsymbol{A}}$ for evolution of $\boldsymbol{X}$

$$X^i(t, \boldsymbol{x}) = x^i + \theta^{ij} \widehat{A}_j(t, \boldsymbol{x})$$

$$\delta_f \boldsymbol{X} \Rightarrow \delta_f \widehat{\boldsymbol{A}} = \boldsymbol{\nabla} f + \{\widehat{\boldsymbol{A}}, f\}$$

(like a gauge transformation in noncommuting gauge theory)

$$\{X^i(t, \boldsymbol{x}), X^j(t, \boldsymbol{x})\} = \theta^{ij} + \theta^{ik} \theta^{j\ell} \widehat{F}_{k\ell}$$

$$\widehat{F}_{k\ell}(\boldsymbol{x}) = \partial_k \widehat{A}_\ell - \partial_\ell \widehat{A}_k + \{\widehat{A}_k, \widehat{A}_\ell\}$$

(like a field strength in noncommuting gauge theory)

(b) Seiberg-Witten map

Can replace noncommuting gauge fields by (nonlocal) function of commuting gauge fields

$\widehat{\boldsymbol{A}} = \widehat{\boldsymbol{A}}(\boldsymbol{A})$

$\widehat{\boldsymbol{A}}(\boldsymbol{A}^g) = \widehat{\boldsymbol{A}}^{G(\boldsymbol{A},g)}(\boldsymbol{A}) \Rightarrow$ determines functional relationship between $\widehat{\boldsymbol{A}}$ and $\boldsymbol{A}$ *

[JHEP 9009, 032 (99)]

N.B. $G(\boldsymbol{A}, g)$ is a noncommuting 1-cocycle

[S.-Y. Pi & R.J., PLB **534**, 181 (02)]

Interesting and useful for extracting physical, gauge-invariant information

e.g., plane wave solutions to noncommuting Maxwell theory

[Z. Guralnik, S.-Y. Pi, A. Polychronakos, & R.J., PLB **515**, 450 (01)]

$$* \Rightarrow \quad \widehat{A}_\mu = A_\mu - \tfrac{1}{2}\theta^{\alpha\beta} A_\alpha\big(\partial_\beta A_\mu + F_{\beta\mu}\big) + O(\theta^2)$$

$$\widehat{\mathcal{L}}_{\mathrm{EM}} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{8}\theta^{\alpha\beta}F_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\theta^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}F^{\mu\nu} + O(\theta^2)$$

equations of motion support plane waves with Lorentz-noninvariant dispersion law

CS modification of Electromagnetism

CS: 3-d (Euclidean) $\Rightarrow$ embed in 4-d space-time physics
(violates Lorentz boosts, CTP, ...)
[Carroll, Field, RJ, PRD **41**, 123 (90)]
Abelian gauge theory CS

$$CS(A) = \tfrac{1}{4}\varepsilon^{ijk} F_{ij} A_k = \tfrac{1}{2}\mathbf{A}\cdot\mathbf{B}$$

$$\begin{aligned}
I &= \int d^4x \left(-\tfrac{1}{4}\, F^{\mu\nu} F_{\mu\nu} + \tfrac{\mu}{2}\, \mathbf{A}\cdot\mathbf{B}\right) \\
&= \int d^4x \left(-\tfrac{1}{4}\, F^{\mu\nu} F_{\mu\nu} + \tfrac{1}{2}\, v_\mu \underbrace{{}^*F^{\mu\nu} A_\nu}\right) \qquad v_\mu = (\mu, \mathbf{0}) \\
& CS \text{ current} : \partial_\mu({}^*F^{\mu\nu} A_\nu) = \tfrac{1}{2}\, {}^*F^{\mu\nu} F_{\mu\nu} \\
&= \int d^4x (-\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} - \tfrac{1}{4}\theta\, {}^*F^{\mu\nu} F_{\mu\nu}) \qquad v_\mu = \partial_\mu \theta \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \theta = \mu t
\end{aligned}$$

Only Ampère's law is modified

$$-\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = \mathbf{J} + \mu \mathbf{B}$$

$\Rightarrow$ gauge invariant, 2 polarizations, each travels with velocity
$\neq c$ (Lorentz boost invariance lost)
$\neq$ each other (parity lost)

$\Rightarrow$ Faraday rotation, light from distant galaxies shows no such effect in Nature.

Exploiting the fluid: noncommuting gauge theory connection

(replace bracket with $-i \times$ commutator)

(a) Coordinate transformations

Noncovariance problem with coordinate transformations

(in commuting gauge theory)

$$\delta_f x^\mu = -f^\mu(\boldsymbol{x}), \delta_f A_\mu = L_f A_\mu \equiv f^\alpha \partial_\alpha A_\mu + \partial_\mu f^\alpha A_\alpha$$

$$\delta_f F_{\mu\nu} = L_f F_{\mu\nu} \equiv f^\alpha \partial_\alpha F_{\mu\nu} + \partial_\mu f^\alpha F_{\alpha\nu} + \partial_\nu f^\alpha F_{\mu\alpha}$$

Not covariant!

Cure [R.J., PRL **41**, 1635 (78)]

$$\begin{aligned} L_f A_\mu &= f^\alpha\big(\partial_\alpha A_\mu - \partial_\mu A_\alpha - i[A_\alpha, A_\mu]\big) \\ &\qquad + f^\alpha \partial_\mu A_\alpha + i f^\alpha [A_\alpha, A_\mu] + \partial_\mu f^\alpha A_\alpha \\ &= f^\alpha F_{\alpha\mu} + \underbrace{D_\mu(f^\alpha A_\alpha)}_{\uparrow \text{drop gauge transformation}} \\ \delta'_f A_\mu &= f^\alpha F_{\alpha\mu} \\ \delta'_f F_{\mu\nu} &= f^\alpha D_\alpha F_{\mu\nu} + \partial_\mu f^\alpha F_{\alpha\nu} + \partial_\nu f^\alpha F_{\mu\alpha} \quad \text{covariant!} \end{aligned}$$

(in noncommuting gauge theory noncovariance more severe)

using fluids as guide, find

$\delta'_f \widehat{A}_\mu = \frac{1}{2}\{f^\alpha(X), \widehat{F}_{\alpha\mu}\}_+$ plus reordering terms

Note: f^α is restricted to be either linear or volume preserving

f^α enters anticommutatator evaluated at $X^i = x^i + \theta^{ij}\widehat{A}_j$

[S.-Y. Pi and R.J., PRL **88**, 111603 (02);

S.-Y. Pi, A.P. Polychronakos, and R.J., Ann. Phys. **301**, 157 (02)]

Seiberg-Witten map from fluid mechanics

[S.-Y. Pi, A. Polychronakos, & R.J., Ann. Phys. **301**, 174 (02)]

Fluid mechanics in Euler formulation

Euler variables:

density $\rho(t, \boldsymbol{r})$ and current $\boldsymbol{j}(t, \boldsymbol{r})$ or velocity $\boldsymbol{v}(t, \boldsymbol{r})$ [$\boldsymbol{j} = \rho \boldsymbol{v}$]

[R.J., *Lectures on Fluid Dynamics* (Springer, 2002)]

Relation between Euler variables and Lagrange variables

$$\rho(t, \boldsymbol{r}) = \int \mathrm{d}x\, \delta(\boldsymbol{X}(t, \boldsymbol{x}) - \boldsymbol{r})$$

$$\boldsymbol{j}(t, \boldsymbol{r}) = \int \mathrm{d}x\, \dot{\boldsymbol{X}}(t, \boldsymbol{x}) \delta(\boldsymbol{X}(t, \boldsymbol{x}) - \boldsymbol{r})$$

(make $\boldsymbol{X}$ an independent variable, called $\boldsymbol{r}$;

$$\frac{1}{\rho} = \det \frac{\partial X^i}{\partial x^j}\bigg|_{\substack{\boldsymbol{x} \text{ and } \boldsymbol{X} \equiv \boldsymbol{r} \\ \text{exchanged}}} \Bigg)$$

$$\partial_t \rho + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

Continuity equation follows from above definitions

Derivation of Seiberg-Witten map in (2+1)-d

$$j^\mu = \int \mathrm{d}^2x \begin{pmatrix} 1 \\ \dot{\boldsymbol{X}} \end{pmatrix} \delta(\boldsymbol{X} - \boldsymbol{r})$$

$$\partial_\mu j^\mu = 0$$

$\varepsilon_{\mu\nu\alpha} j^\alpha$ is 2-form that satisfies Bianchi identity

$$\propto \partial_\mu a_\nu - \partial_\nu a_\mu + \text{constant}$$

$$\varepsilon_{ij} \rho \propto \partial_i a_j - \partial_j a_i + \text{constant}$$

Passage to noncommuting gauge theory:

X^i is operator $= x^i + \theta^{ij} \widehat{A}_j = x^i + \theta \varepsilon^{ij} \widehat{A}_j$

$\int \mathrm{d}^2 x$ is $\theta \times$ trace

Need ordering – prescribe Weyl ordering by Fourier transform

$$\int \mathrm{d}^2 r\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} (\partial_i a_j - \partial_j a_i) = -\varepsilon^{ij} \left[\int \mathrm{d}x\, e^{i\boldsymbol{k}\cdot\boldsymbol{X}} - (2\pi)^2 \delta(\boldsymbol{k}) \right]$$

$$= -\varepsilon^{ij} \left[\int \mathrm{d}x\, e^{i[k_i x^i + \theta k_i \varepsilon^{ij} \widehat{A}_j]} - (2\pi)^2 \delta(\boldsymbol{k}) \right]$$

$$\equiv \text{(inverse) Seiberg-Witten map}$$

can be extended to higher dimensions;
reproduces [Y. Okawa and H. Ooguri, PRD **64**, 046009 (2001)]