Mass from missing energy: Polynomial systems for the determination of dark matter masses at the LHC

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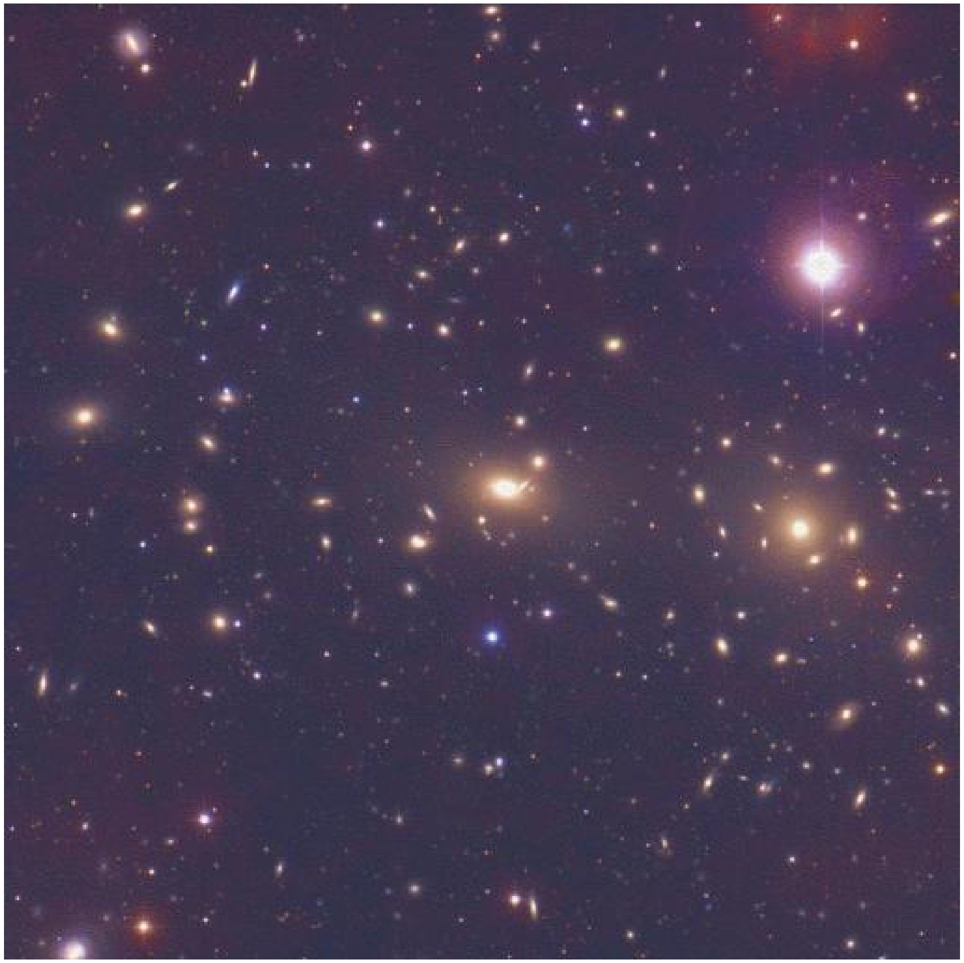
Outline

- Introduction (pretty pictures)
- Model Considerations
- Old Standby: Edges and shapes (Lester et al., "Barr" topology)
- New Development 1: Discontinuities ("Gluino Stransverse Mass" – Cho et al, Gripaios)
- New Development 2: Polynomial Systems (McElrath et al)
- New Development 3: Combining Events (Nojiri et al, McElrath et al)
- Conclusion

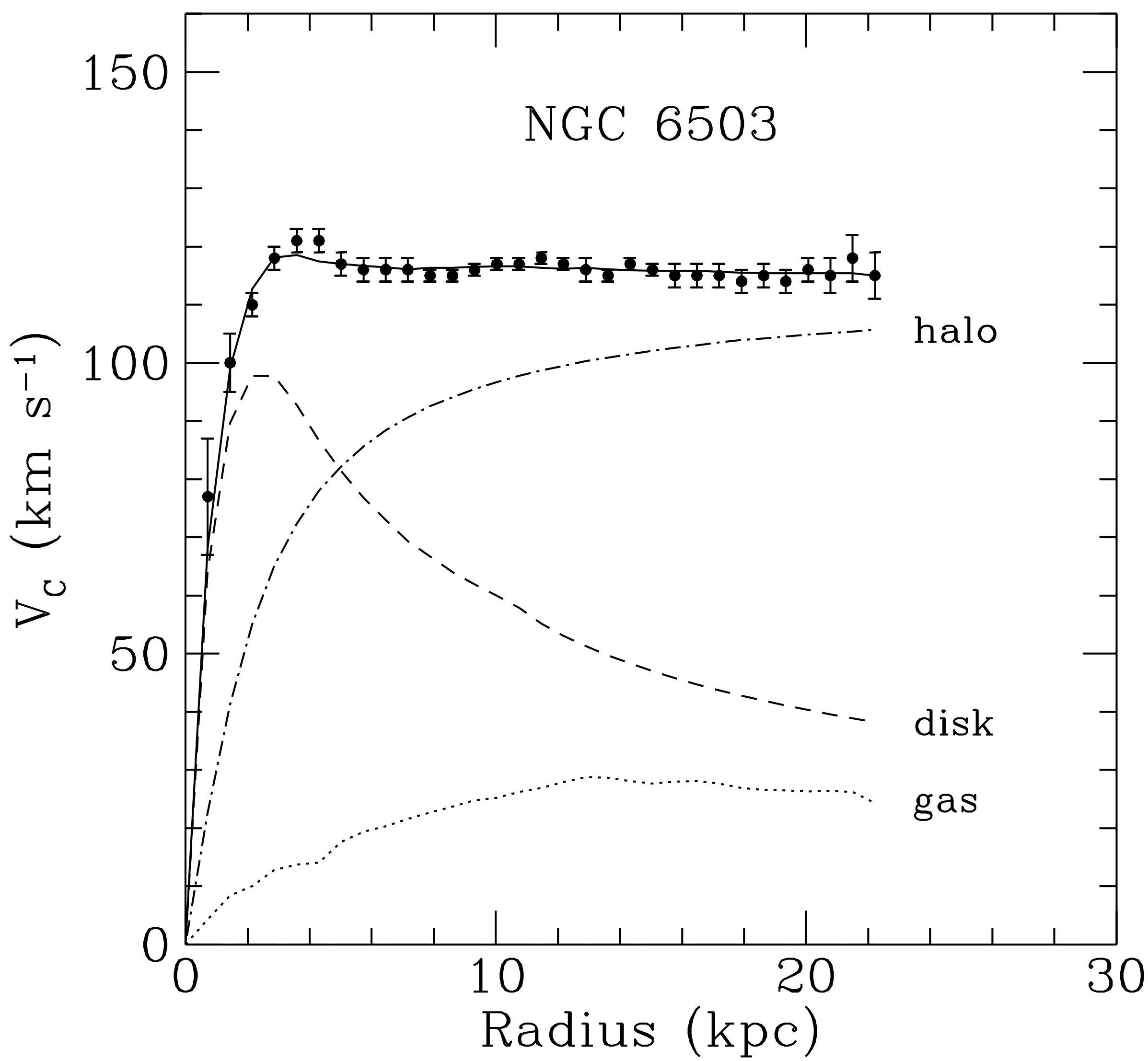
Experimental Hints for New Particles: Dark Matter

1933 Fritz Zwicky calculates the mass of the Coma cluster using the Virial Theorem using galaxies on the outer edge, and comes up with a number 400 times larger than the expected mass.

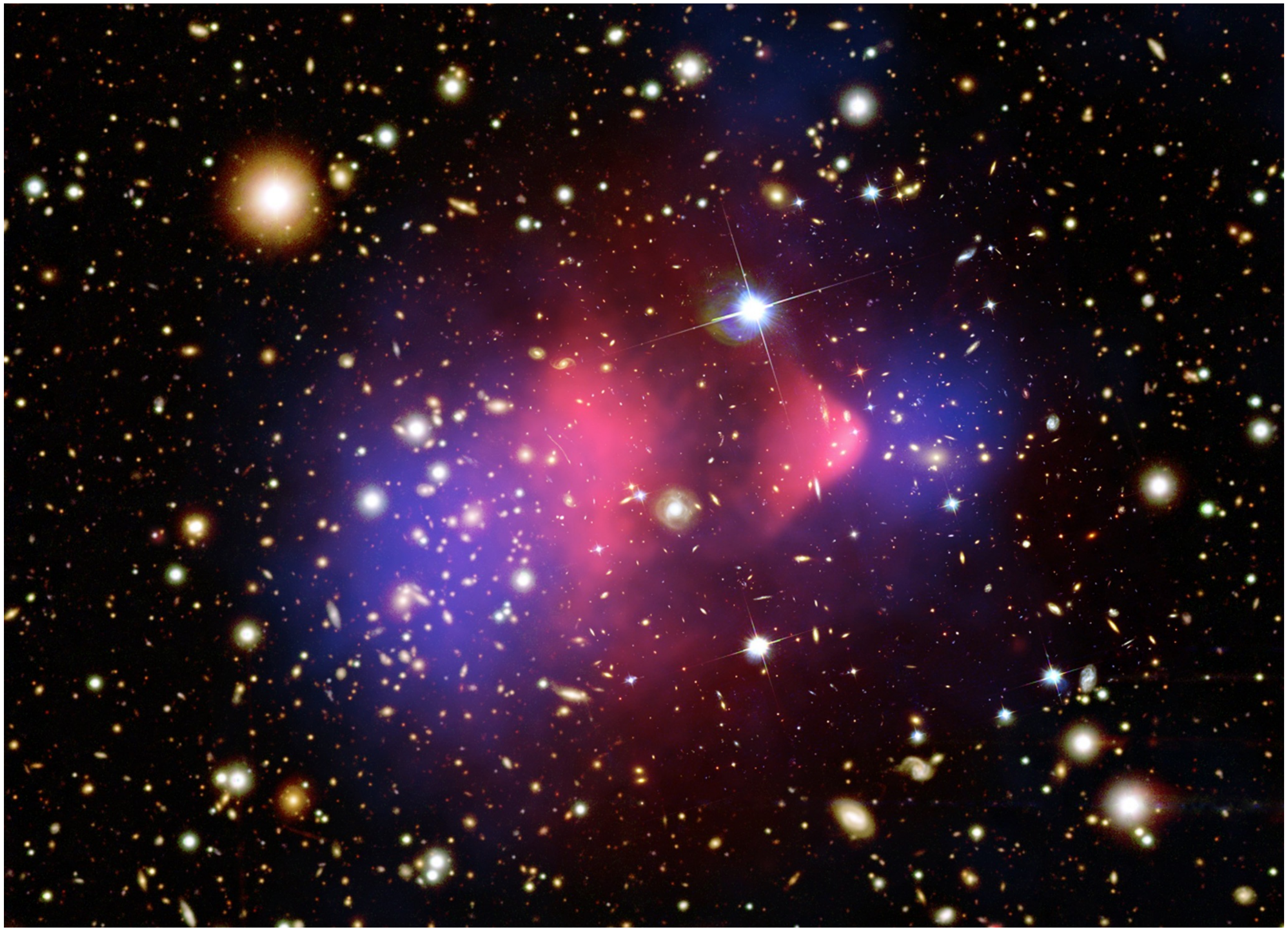


Experimental Hints for New Particles: Rotation Curves

1975 Vera Rubin notices the rotation curves of galaxies are flat at large radii. (Jungman, Kamionkowski, Griest)

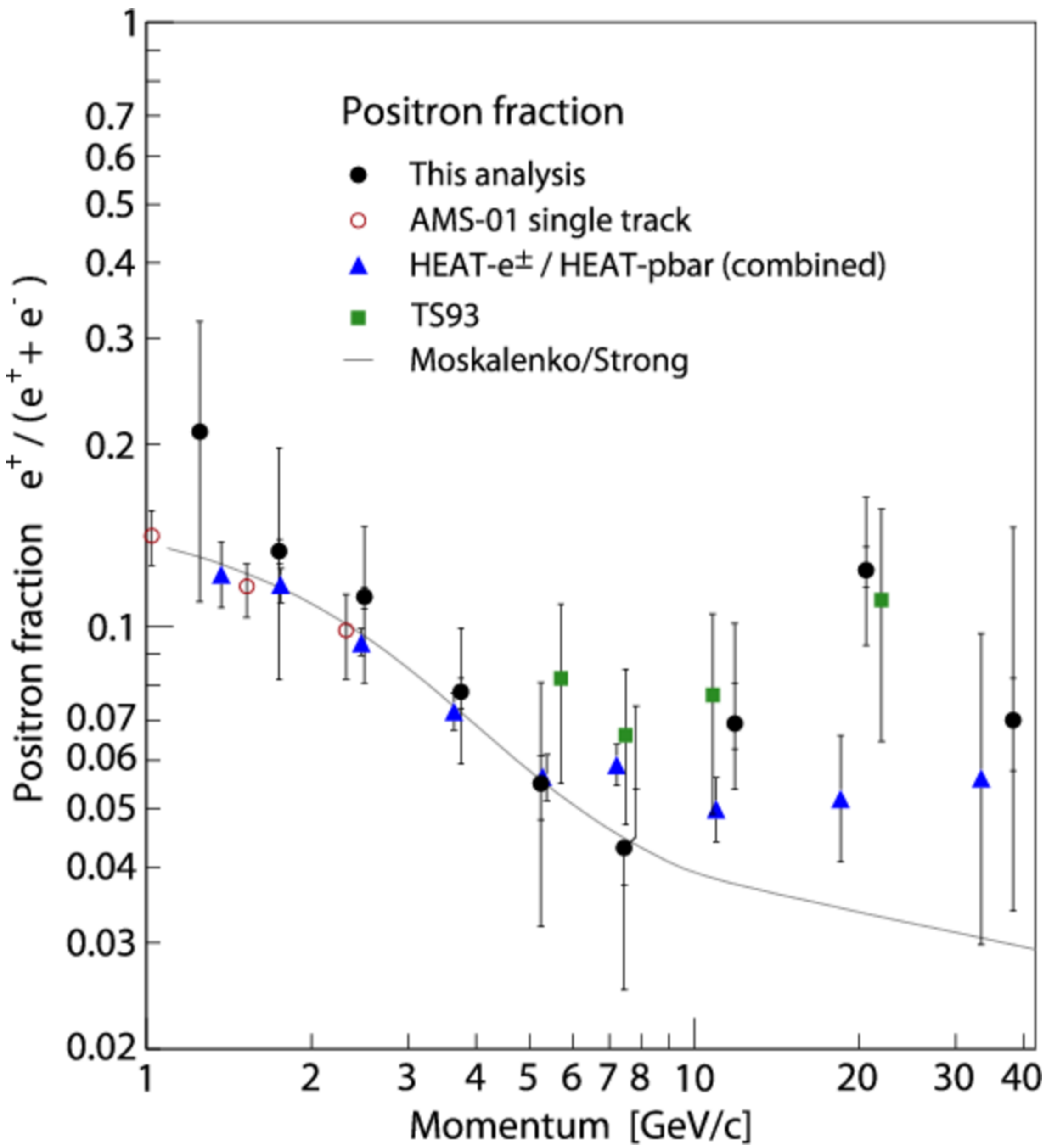


Experimental Hints for New Particles: Bullet Cluster



X-ray gas, gravitational potential [Clowe et al. astro-ph/0608407]; Note the claim of disproving MOND is disputed: [Angus, Famaey, Zhao astro-ph/0606216] requires addition of Hot Dark Matter: 2 eV neutrinos.

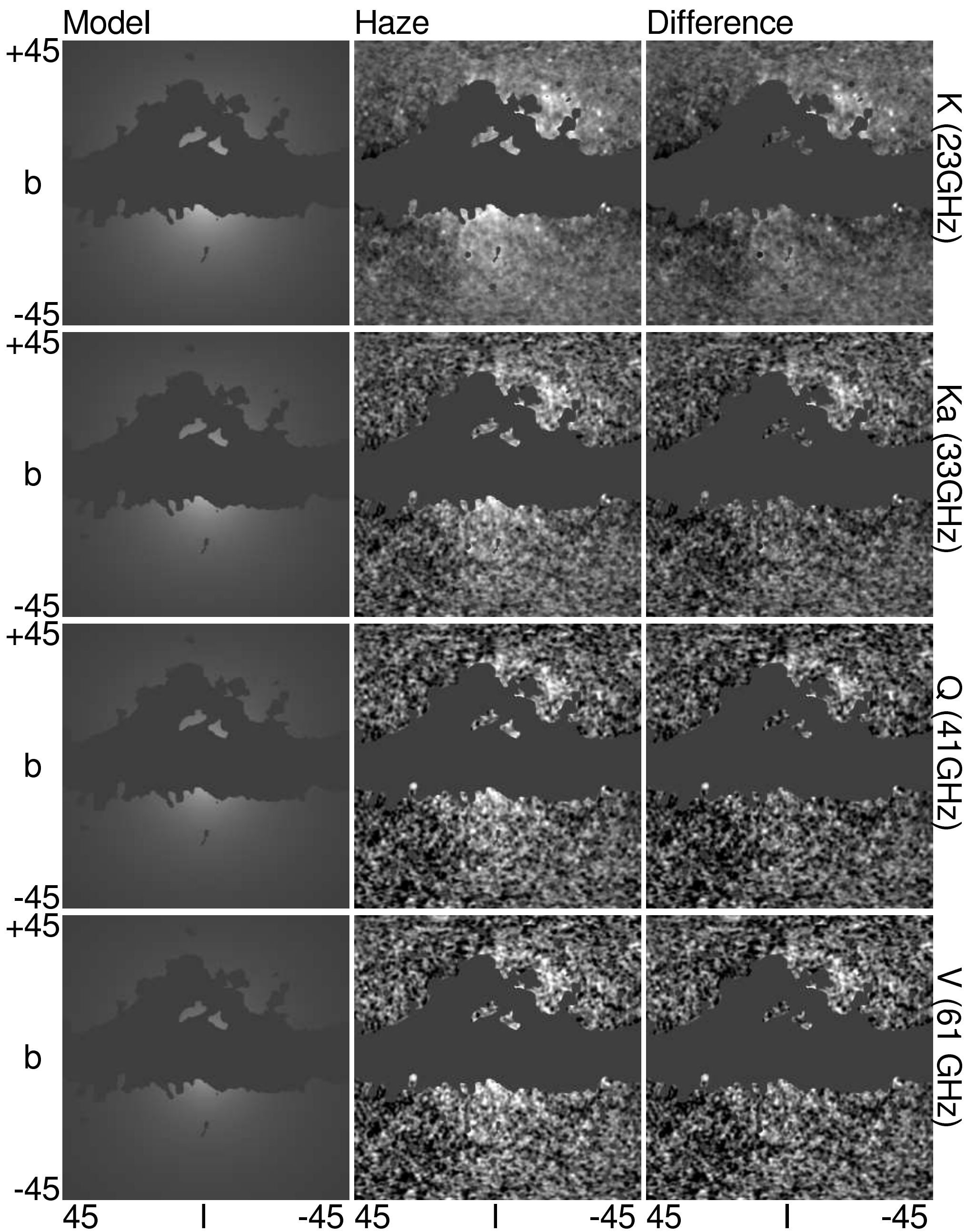
Experimental Hints for New Particles: HEAT Excess



Implies $M_\chi \gtrsim 1$ TeV

[Aguilar et al
astro-ph/0703154]

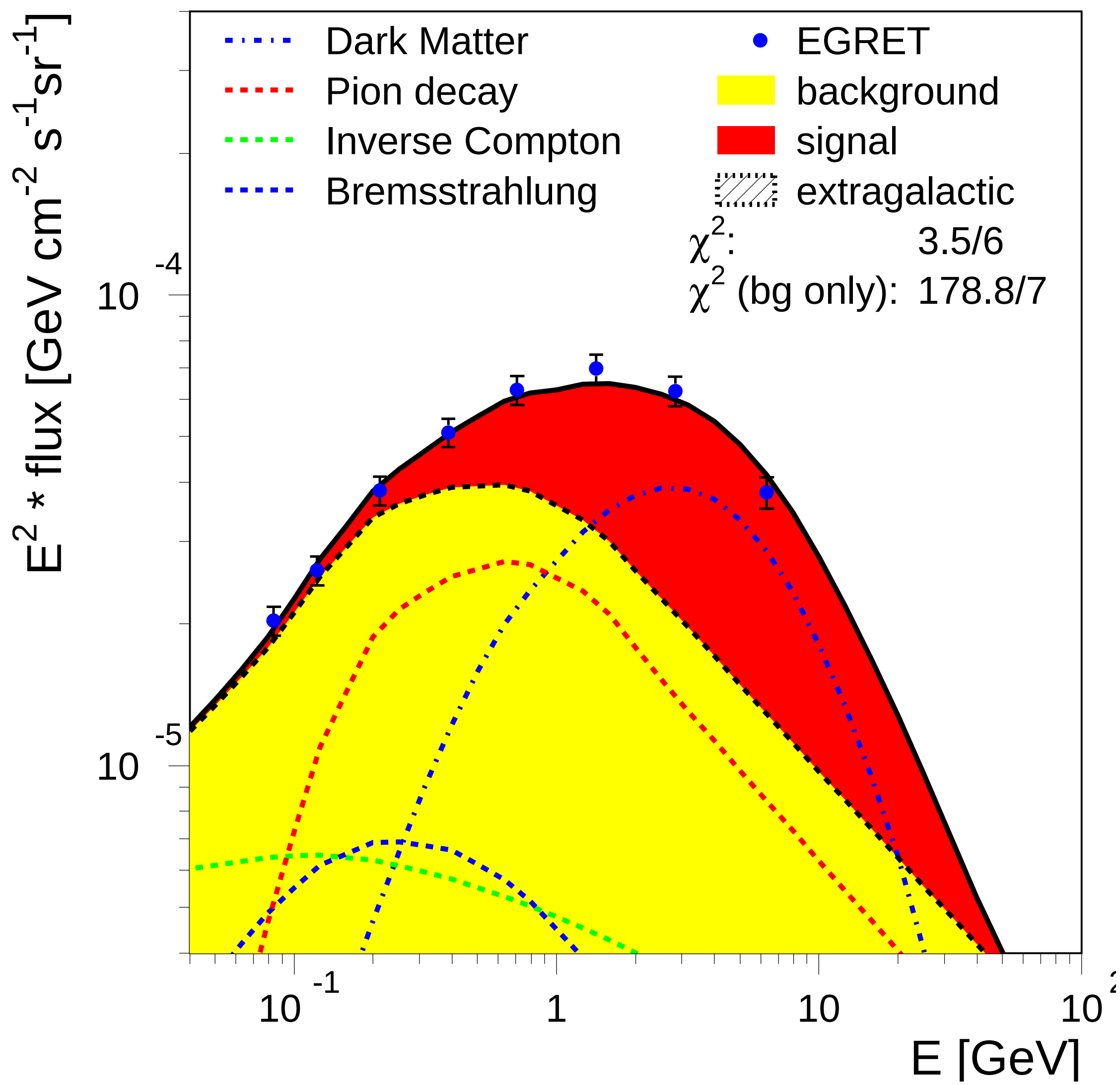
Experimental Hints for New Particles: WMAP Haze



[Finkbeiner, astro-ph/0409027]

Implies $M_\chi \gtrsim 100$ GeV

Experimental Hints for New Particles: EGRET Excess



[de Boer et al, `astro-ph/0408272`]

Implies $M_\chi \simeq 50 - 100$ GeV.

Inconsistent with galactic positron measurements; requires unusual dark matter distribution.

Motivation

If our current particle picture of Dark Matter is correct, the LHC is likely to be a Dark Matter factory. Realistic models containing a Dark Matter particle tend to be very similar.

- A symmetry is added to keep Dark Matter stable $\rightarrow$ Dark Matter is produced in pairs.

- Symmetries which keep Dark Matter stable are often taken from other sources (because we prefer as simple a model as possible), such as:

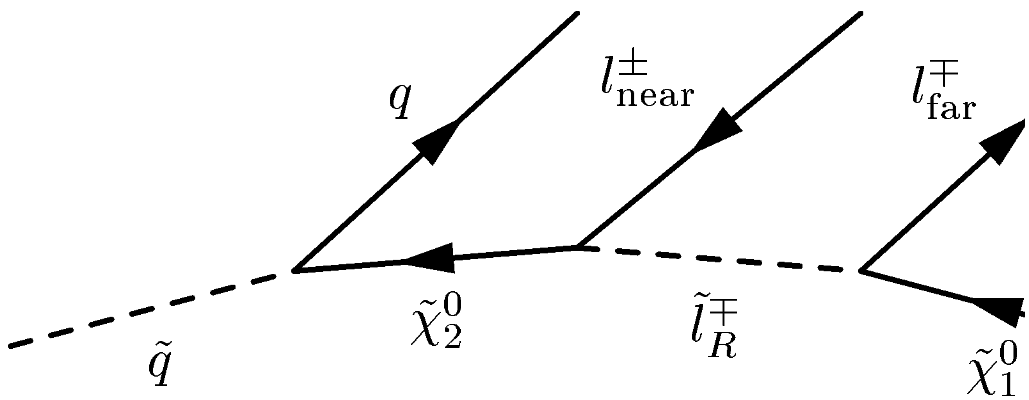
  - Proton Stability (R-Parity in SUSY)

  - Custodial Symmetry (solving Little Hierarchy Problem)

  - 5D momentum conservation (KK number conservation in UED)

  "Other Sources" for the symmetry generically means "Other Particles".

Old Standby: Edges and Endpoints



Given some assumed topology in a 4-body decay (with one missing), write down the three mass invariants among the visible particles. In addition there is (at least) one other jet, and $l^{\pm}_{\rm near}$ cannot be separated from $l^{\pm}_{\rm far}$

M_{ll}

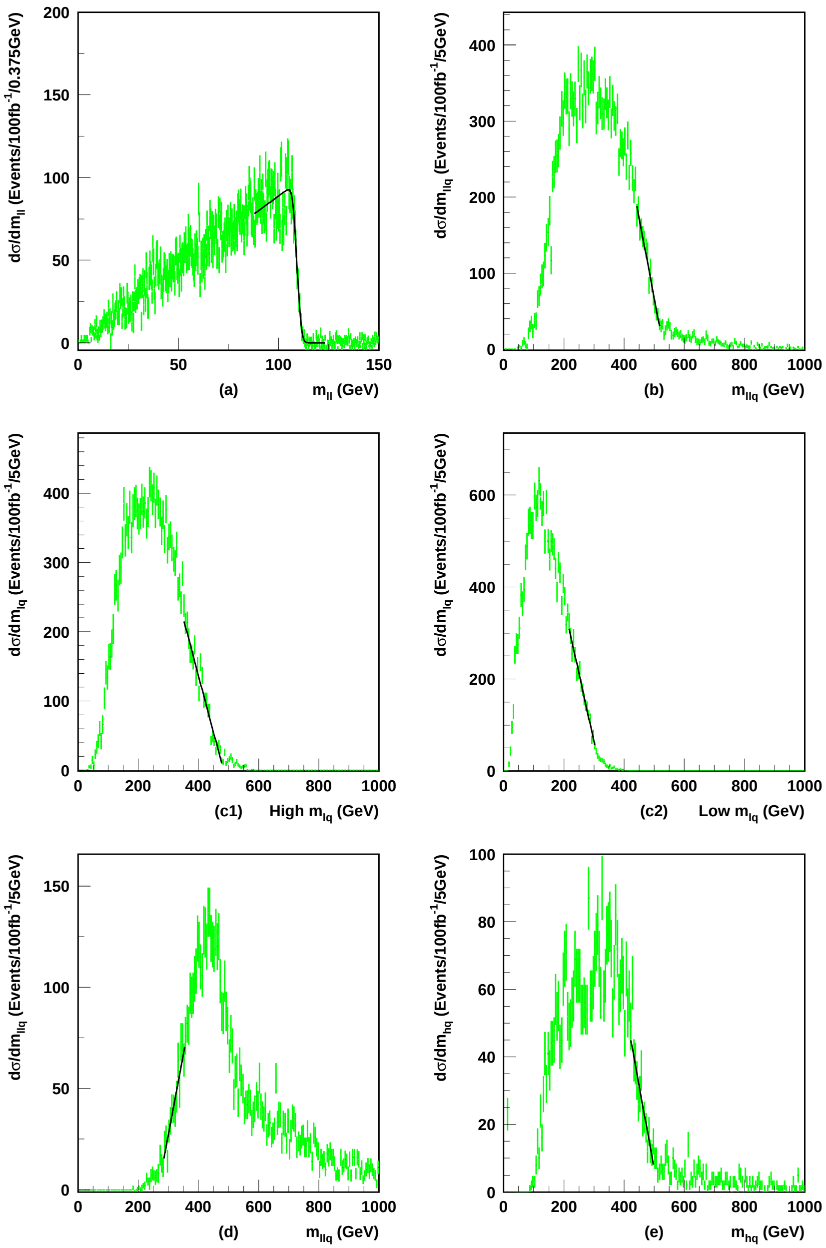
$M_{ql,\rm high}$ (larger of 2 choices)

$M_{ql,\rm low}$ (smaller of 2 choices)

$M_{qll,\rm max}$ (larger of 2 choices)

$M_{qll,\rm min}$ (smaller of 2 choices)

Old Standby: Edges and Endpoints



M_{ll}, $M_{qll,\text{min}}$

$M_{lq,\text{high}}$, $M_{lq,\text{low}}$

$M_{qll,\text{max}}$, M_{hq} $(h \longrightarrow b\bar{b})$

Existing Studies: Edges and Endpoints

$$\left(m_{ll}^{\max}\right)^2 = (m_{\tilde\chi_2^0}^2 - m_{\tilde l_R}^2)(m_{\tilde l_R}^2 - m_{\tilde\chi_1^0}^2)/m_{\tilde l_R}^2$$

$$\left(m_{qll}^{\max}\right)^2 = \begin{cases} \frac{(m_{\tilde q_L}^2 - m_{\tilde\chi_2^0}^2)(m_{\tilde\chi_2^0}^2 - m_{\tilde\chi_1^0}^2)}{m_{\tilde\chi_2^0}^2} & \text{for} \quad \frac{m_{\tilde q_L}}{m_{\tilde\chi_2^0}} > \frac{m_{\tilde\chi_2^0}}{m_{\tilde l_R}}\frac{m_{\tilde l_R}}{m_{\tilde\chi_1^0}} \quad (1) \\ \frac{(m_{\tilde q_L}^2 m_{\tilde l_R}^2 - m_{\tilde\chi_2^0}^2 m_{\tilde\chi_1^0}^2)(m_{\tilde\chi_2^0}^2 - m_{\tilde l_R}^2)}{m_{\tilde\chi_2^0}^2 m_{\tilde l_R}^2} & \text{for} \quad \frac{m_{\tilde\chi_2^0}}{m_{\tilde l_R}} > \frac{m_{\tilde l_R}}{m_{\tilde\chi_1^0}}\frac{m_{\tilde q_L}}{m_{\tilde\chi_2^0}} \quad (2) \\ \frac{(m_{\tilde q_L}^2 - m_{\tilde l_R}^2)(m_{\tilde l_R}^2 - m_{\tilde\chi_1^0}^2)}{m_{\tilde l_R}^2} & \text{for} \quad \frac{m_{\tilde l_R}}{m_{\tilde\chi_1^0}} > \frac{m_{\tilde q_L}}{m_{\tilde\chi_2^0}}\frac{m_{\tilde\chi_2^0}}{m_{\tilde l_R}} \quad (3) \\ (m_{\tilde q_L} - m_{\tilde\chi_1^0})^2 & \text{otherwise} \quad (4) \end{cases}$$

$$(m_{ql(\text{low})}^{\max}, m_{ql(\text{high})}^{\max}) = \begin{cases} (m_{ql_n}^{\max}, m_{ql_f}^{\max}) & \text{for} \quad 2m_{\tilde l_R}^2 > m_{\tilde\chi_1^0}^2 + m_{\tilde\chi_2^0}^2 > 2m_{\tilde\chi_1^0}m_{\tilde\chi_2^0} \quad (1) \\ (m_{ql(\text{eq})}^{\max}, m_{ql_f}^{\max}) & \text{for} \quad m_{\tilde\chi_1^0}^2 + m_{\tilde\chi_2^0}^2 > 2m_{\tilde l_R}^2 > 2m_{\tilde\chi_1^0}m_{\tilde\chi_2^0} \quad (2) \\ (m_{ql(\text{eq})}^{\max}, m_{ql_n}^{\max}) & \text{for} \quad m_{\tilde\chi_1^0}^2 + m_{\tilde\chi_2^0}^2 > 2m_{\tilde\chi_1^0}m_{\tilde\chi_2^0} > 2m_{\tilde l_R}^2 \quad (3) \end{cases}$$

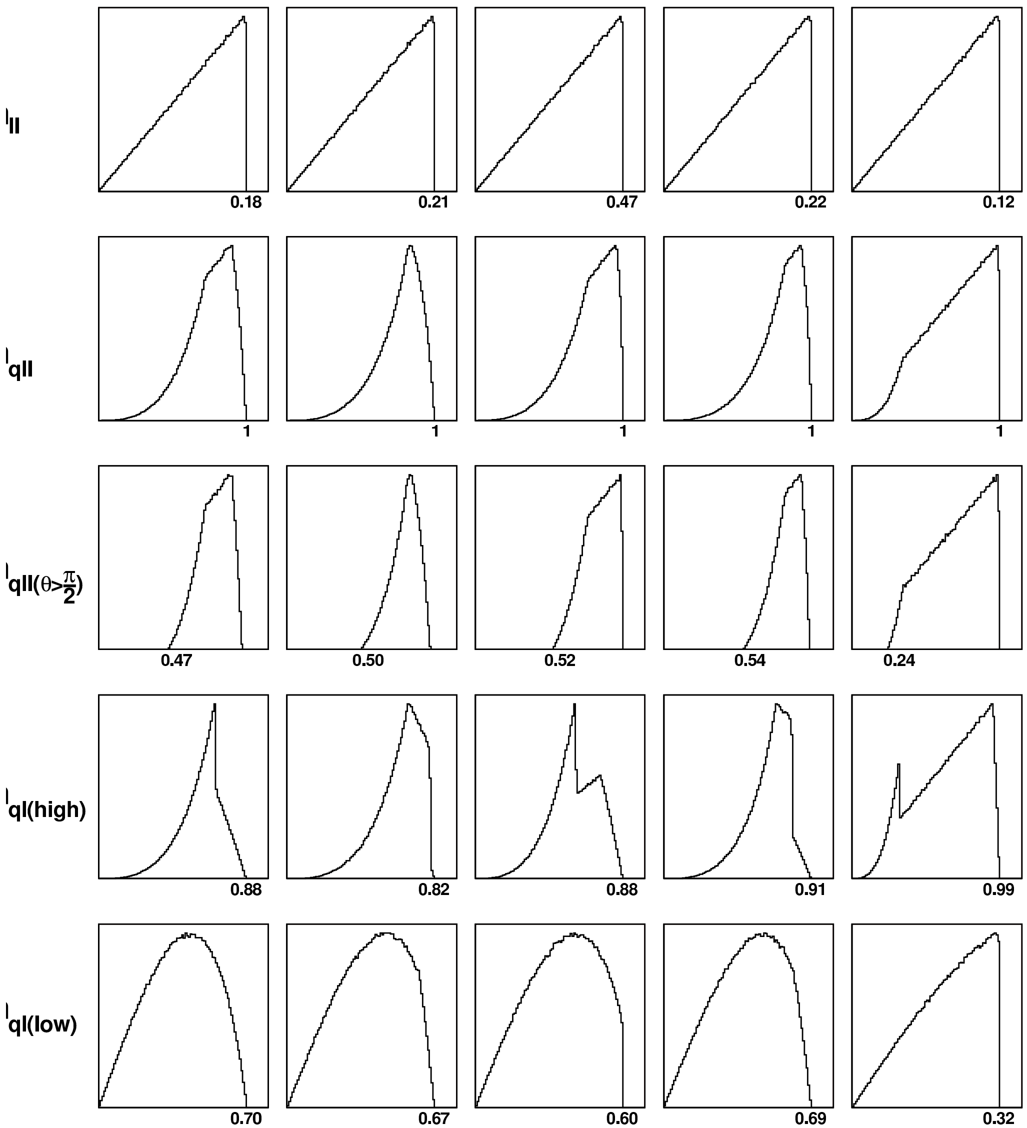
$$(m_{ql_n}^{\max})^2 = (m_{\tilde q_L}^2 - m_{\tilde\chi_2^0}^2)(m_{\tilde\chi_2^0}^2 - m_{\tilde l_R}^2)/m_{\tilde\chi_2^0}^2$$

$$\left(m_{ql_f}^{\max}\right)^2 = (m_{\tilde q_L}^2 - m_{\tilde\chi_2^0}^2)(m_{\tilde l_R}^2 - m_{\tilde\chi_1^0}^2)/m_{\tilde l_R}^2$$

$$(m_{ql(\text{eq})}^{\max})^2 = (m_{\tilde q_L}^2 - m_{\tilde\chi_2^0}^2)(m_{\tilde l_R}^2 - m_{\tilde\chi_1^0}^2)/(2m_{\tilde l_R}^2 - m_{\tilde\chi_1^0}^2)$$

$$\begin{aligned}(m_{qll(\theta>\frac{\pi}{2})}^{\min})^2 = & \Big[(m_{\tilde q_L}^2 + m_{\tilde\chi_2^0}^2)(m_{\tilde\chi_2^0}^2 - m_{\tilde l_R}^2)(m_{\tilde l_R}^2 - m_{\tilde\chi_1^0}^2) \\ & -(m_{\tilde q_L}^2 - m_{\tilde\chi_2^0}^2)\sqrt{(m_{\tilde\chi_2^0}^2 + m_{\tilde l_R}^2)^2(m_{\tilde l_R}^2 + m_{\tilde\chi_1^0}^2)^2 - 16m_{\tilde\chi_2^0}^2 m_{\tilde l_R}^4 m_{\tilde\chi_1^0}^2} \\ & +2m_{\tilde l_R}^2(m_{\tilde q_L}^2 - m_{\tilde\chi_2^0}^2)(m_{\tilde\chi_2^0}^2 - m_{\tilde\chi_1^0}^2)\Big]/(4m_{\tilde l_R}^2 m_{\tilde\chi_2^0}^2)\end{aligned}$$

Old Standby: Edges and Endpoints



As the mass spectrum changes, these curves take on various shapes.

- Mass determinations are very sensitive to a small number of events (those occuring at inflection points and endpoints).
- Detector resolution makes all these distributions look similar.

[Gjelsten, Miller, Osland `hep-ph/0410303`]

Old Standby: Edges and Endpoints

My criticism:

What is done is essentially to write down some arbitrary $f(p_i^\mu)$, and derive its analytic relationship to mass. There is no reason to expect that the chosen space of observables is a good one.

The missing momentum is not used.

The ease of measuring edges is deceptive due to the fact that ISR/FSR is not included, nor possible extra jets from squark/gluino decay. As a general rule, ISR/FSR in squark/gluon production has an energy scale associated with the squark/gluon mass.

New Development 1: Discontinuities in M_{T2}

Given the decay $\tilde{g} \to qq\chi_1^0$, the M_{T2} variable is: [Lester, Summers PLB 463 (1999) 99]

$$M_T^2(m_{qqT}, m_\chi, \mathbf{p}_T^{qq}, \mathbf{p}_T^\chi) = m_{qqT}^2 + m_\chi^2 + 2(E_T^{qq} E_T^\chi - \mathbf{p}_T^{qq} \cdot \mathbf{p}_T^\chi)$$

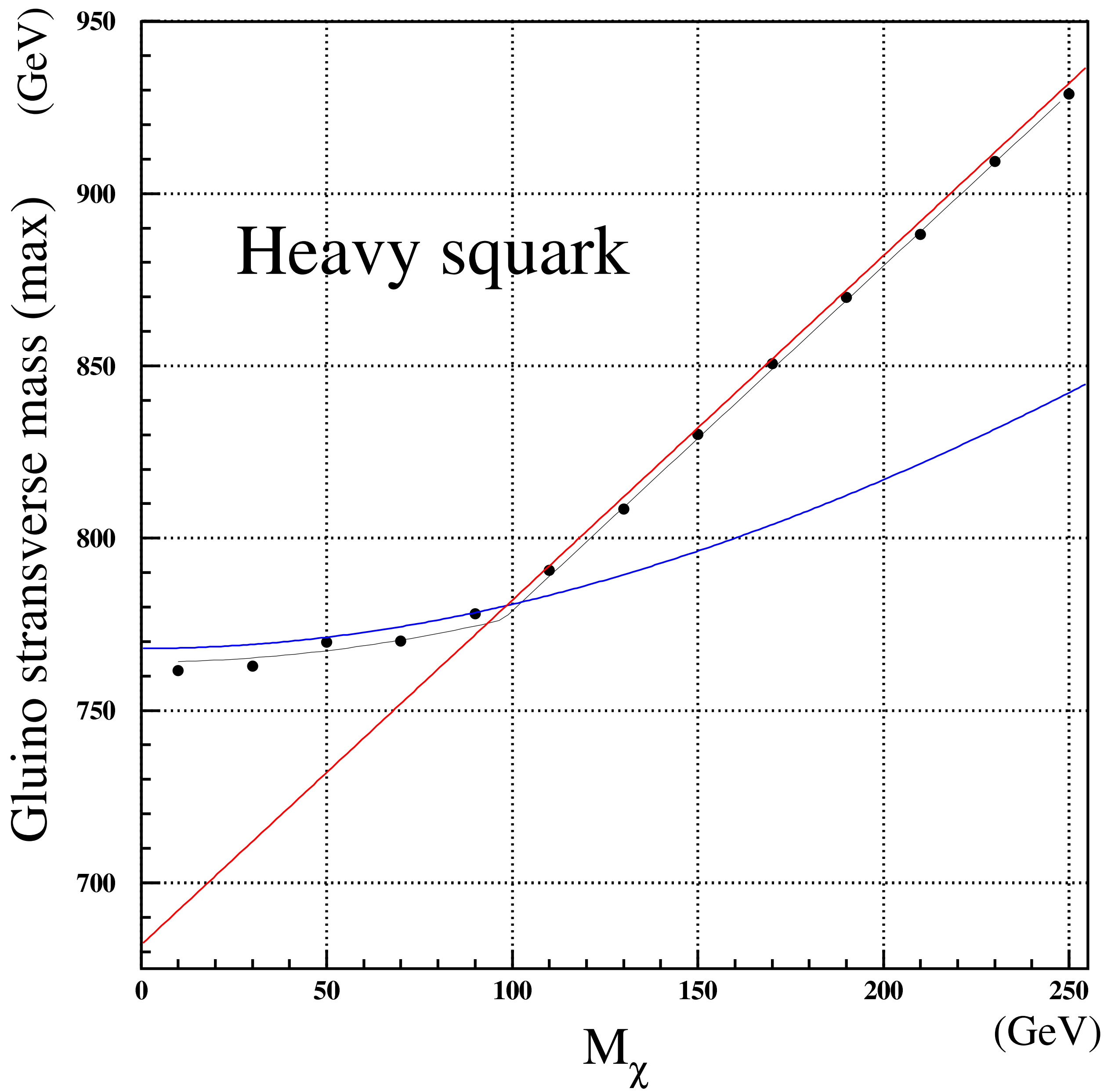
$$M_{T2}^2(m_\chi) = \min_{\mathbf{p}_T^{\chi(1)} + \mathbf{p}_T^{\chi(2)} = \not{\mathbf{p}}_\mathbf{T}} \left[\max \left\{ \left(m_T^2\right)_1, \left(m_T^2\right)_2 \right\} \right]$$

This variable has the property that if $m_\chi = m_{\chi_1^0}$, then $M_{T2}^2 \leq m_{\tilde{g}}^2$.

Cho et. al. [arXiv:0709.0288] observed that as a function of m_χ, M_{T2} contains a discontinuity at the point $m_\chi = m_{\chi_1^0}$.

Gripaios [arXiv:0709.2740] proved this for the 1 missing particle case (leptonic $W^\pm$ decay).

New Development 1: Discontinuities in M_{T2}



$$M_{T2}^{\max}(m_\chi)$$

$$= \frac{m_{\tilde{g}}^2 - m_{\tilde{\chi}_1^0}^2}{2m_{\tilde{g}}} + \sqrt{\left(\frac{m_{\tilde{g}}^2 - m_{\tilde{\chi}_1^0}^2}{2m_{\tilde{g}}}\right)^2 + m_\chi^2}$$

$$= \left(m_{\tilde{g}} - m_{\tilde{\chi}_1^0}\right) + m_\chi$$

Two expressions coincide at $m_{\tilde{\chi}_1^0} = m_\chi$.

Expressions are derived in specific kinematic limits.

General proof not published yet.

What is the *BEST* thing to do?

Most studies approach the problem as: Given that I have these 4-vectors $\{p_i^\mu\}$ that came out of my Monte Carlo or Experiment, what function $f(\{p_i^\mu\}|\lambda)$ can I write down which will tell me some hypothesis (parameter) λ?

The answer is that *every* function $f(\{p_i^\mu\}|\lambda)$ depends on the parameters λ, and I'm left with the question: Which $f(\{p_i^\mu\}|\lambda)$ is "best" for my purpose?

I approach this from the other side: the most powerful statistic for differentiating two hypotheses λ and λ' is the ratio of two Likelihoods (Neyman-Pearson Lemma). Our Likelihood for $n = 1..N$ events is

$$L(\lambda|\{\{p_j^\mu\}_n\}) = \prod_{n=1}^{N} P_n(\{p_j^\mu\}_n|\lambda).$$

$$P_n(\{p_j^\mu\}|\lambda) = \frac{1}{\sigma}\frac{d\sigma}{\prod_i d^3\vec{p}_i} = \frac{(2\pi)^{4-3N}}{2^N F\sigma \prod_i E_i}\left|\mathcal{M}(p_0^\mu, p_i^\mu|\lambda)\right|^2 \delta^4\left(p_0^\mu - \sum_i p_i^\mu\right).$$

Now let me make systematic approximations to this ideal situation.

Polynomial Systems

In a hadron collider with missing energy, the PDF is defined as

$$P(p_i^\mu|\lambda) = \int \frac{(2\pi)^{4-3N}}{2^N F\sigma \prod_i E_i} \left|\mathcal{M}(p_0^\mu, p_i^\mu|\lambda)\right|^2 \delta^4\left(p_0^\mu - \sum_i p_i^\mu\right) dx_1 dx_2 d^3p_1 d^3p_2$$

Now let us go into the narrow width approximation by replacing

$$\frac{1}{(q^2-M^2)^2 - M^2\Gamma^2/4} \to \frac{\pi}{M\Gamma}\delta(q^2-M^2)$$

in $\left|\mathcal{M}(p_0^\mu, p_i^\mu|\lambda)\right|^2$, for some hypothesis diagram (valid for $\Gamma \ll M$).

Alternatively, one can simply insert the appropriate delta functions corresponding to a diagram, and view this as a variable change.

Note that this integral is 4 dimensional at a hadron collider. Therefore, *by specifying 4 masses, the integral is reduced to a discrete set of solutions for the missing momenta*.

A pair of simultaneous quadratics is not guaranteed to have a solution!

The General Recipe for using Polynomial Systems

- Write down a hypothesis diagram describing the visible final state particles and missing energy you see.
- Combine resonances with entirely visible decay products and call it a single final state particle.
- Count the missing particles N and the intermediate, *on-shell* particles M with missing particles "down-stream".
- $M < 3N - 2$: ("underconstrained") use kinks or edges.
- $M = 3N - 2$: ("exactly constrained") one can change variables from the missing momenta into these masses. Each event defines a volume in mass space. See JHEP 0712:076,2007 and arXiv:0811.2138
- $M > 3N - 2$: ("overconstrained") it is possible to solve for discrete values of the masses, by constructing a larger polynomial system from multiple events, *under the assumption that they contain the same physics*. See: Phys.Rev.Lett.100:252001,2008.

The General Recipe for using Polynomial Systems

Once your polynomial system is constructed, one can ask the question if $M = 3N - 2$ (exactly constrained):

Is the Probability Density P zero or nonzero
for a given set of hypothesis masses?

The nonzero answer defines a volume in mass space, which one must then devise an algorithm to extract the true mass from by combining events.

If $M > 3N - 2$ (overconstrained):

Is the n-event likelihood $L_n = \prod_{i=1}^{N} P_i$ zero or nonzero?

If these are nonzero, in the narrow width approximation, you have just solved for a set of 4-momenta consistent with the event (and therefore, all the intermediate masses too).

The General Recipe for using Polynomial Systems

These are systematic approximation to the "best" Likelihood method, accurate to $\mathcal{O}\left(\frac{\Gamma}{M}\right)$ and ignoring spin. The only thing better is to use a true Matrix Element Method, which also includes off-shell effects.

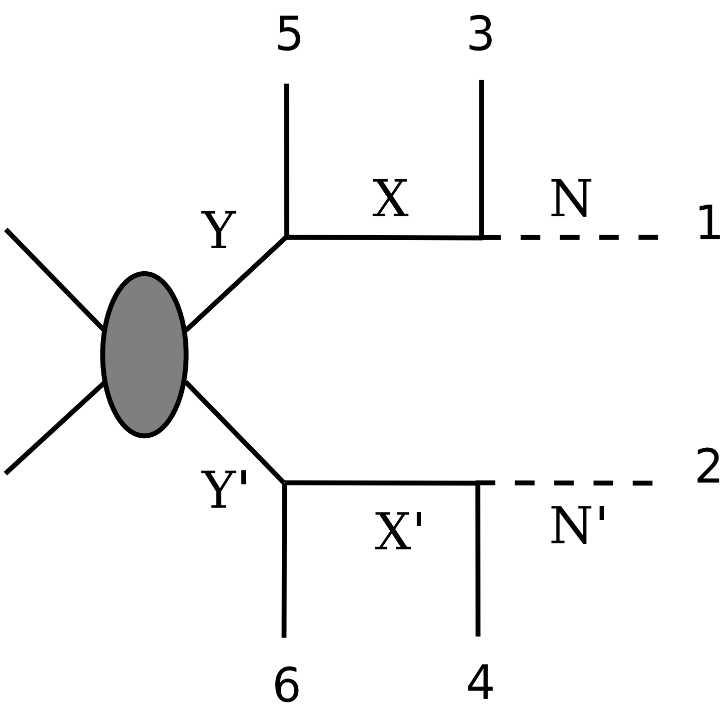
The overconstrained case is the *best* option for small data. In principle it works for as few as two events in SPS1a.

These methods need long chains: at least 5 on-shell intermediate particles is overconstrained, 4 is exactly constrained.

These methods are probably not useful with 3 or more missing particles: this needs 7 on-shell intermediate resonances.

The intermediate particles must be *on-shell*.

Exactly constrained example



This topology can be applied to many processes with 4 visible and 2 invisible particles.

For simplicity in analysis we will further assume $M_Y = M_{Y'}$, $M_X = M'_X$, and $M_N = M'_N$.

Examples that fit this:

$$
\begin{aligned}
t\bar{t} &\rightarrow bW^+bW^- \rightarrow bl^+\nu bl^-\bar{\nu} \\
\tilde{\chi}_2^0\tilde{\chi}_2^0 &\rightarrow l\tilde{l}l\tilde{l} \rightarrow ll\tilde{\chi}_1^0 ll\tilde{\chi}_1^0 \\
\tilde{q}\tilde{q} &\rightarrow q\tilde{\chi}_2^0 q\tilde{\chi}_2^0 \rightarrow ql\tilde{l}ql\tilde{l} \rightarrow qll\tilde{\chi}_1^0 qll\tilde{\chi}_1^0 \\
\tilde{t}\bar{\tilde{t}} &\rightarrow b\tilde{\chi}^+\bar{b}\tilde{\chi}^- \rightarrow bW^+\tilde{\chi}_1^0\bar{b}W^-\tilde{\chi}_1^0
\end{aligned}
$$

Changing Variables

If we want to talk about masses, the first thing we had better do is change variables.

The $t\bar{t}$ di-lepton topology at the LHC contains 4 kinematic unknowns, which is nice because it also has 4 unknown masses.

$$\begin{aligned}
a &= (p_2 + p_4 + p_6)^2 \\
b &= (p_2 + p_4)^2 \\
c &= (p_1 + p_3 + p_5)^2 \\
d &= (p_1 + p_3)^2 \\
0 &= \not{p}_x - p_{1x} - p_{2x} \\
0 &= \not{p}_y - p_{1y} - p_{2y} \\
0 &= \sqrt{s}\sigma - p_{vz} - p_{1z} - p_{2z} \\
0 &= \sqrt{s}\tau - E_v - E_1 - E_2 \\
M_1^2 &= E_1^2 - \vec{p}_1^{\,2} \\
M_2^2 &= E_2^2 - \vec{p}_2^{\,2}
\end{aligned}$$

This variable change is non-linear, and incurs a Jacobian J (important if you want to integrate your Probability Density in the mass basis!)

New Development 2: Polynomial Systems

The polynomial system of interest is contained *entirely in the delta functions* and is a system of two quadratics in six variables. Take these to be four masses, and the energies of the missing particles:

$$
\begin{aligned}
P(\{p_i^\mu\}|\lambda) \;=\; & f(\{p_i^\mu\},\lambda)\int |\mathcal{M}(\lambda, p_0^\mu,\ldots,p_N^\mu)|^2 \\
& \times\delta^4(p_0^\mu - \textstyle\sum_i p_i^\mu) \\
& \times 2E_1\,\delta(E_1^2 - m_1^2 - |\vec{p}_1|^2)\,2E_2\,\delta(E_2^2 - m_2^2 - |\vec{p}_2|^2) \\
& \times d\tau\, d\sigma\, d^3\vec{p}_1\, d^3\vec{p}_2\, dE_1\, dE_2
\end{aligned}
$$

Next expand the dimensionality by 4 and add 4 delta functions, corresponding to the 4 propegators.

$$
\begin{aligned}
P(\{p_i^\mu\}|\lambda) \;=\; & f(\{p_i^\mu\},\lambda)\int |\mathcal{M}(\lambda, p_0^\mu,\ldots,p_N^\mu)|^2 \\
& \times\delta^4(p_0^\mu - \textstyle\sum_i p_i^\mu) \\
& \times 2E_1\,\delta(E_1^2 - m_1^2 - |\vec{p}_1|^2)\,2E_2\,\delta(E_2^2 - m_2^2 - |\vec{p}_2|^2) \\
& \times \delta((p_2+p_4+p_6)^2 - a)\,\delta((p_2+p_4)^2 - b) \\
& \times \delta((p_1+p_3+p_5)^2 - c)\,\delta((p_1+p_3)^2 - d) \\
& \times d\tau\, d\sigma\, d^3\vec{p}_1\, d^3\vec{p}_2\, dE_1\, dE_2\, da\, db\, dc\, dd
\end{aligned}
$$

New Development 2: Polynomial Systems

Now we wish to ask the question: Is $P(p_i^\mu, \lambda)$ zero or non-zero for a single event?

The system of equations can be divided into two pieces: a linear part, and a quadratic part. Any invariant containing *exactly one* missing particle is linear in each component of the missing particle's 4-vector. The δ^4 is linear. This linear system of equations can be written in matrix notation:

$$MV = C$$

with $V = (p_{1x}, p_{1y}, p_{1z}, E_1, \ldots, \sqrt{s}\sigma, \sqrt{s}\tau)$. M is a square matrix, and can be inverted. (if it cannot, then you have specified a perverse system!)

With n intermediate masses, this dimension of M is $n + 4$.

Thus the vector of unknowns is given by $V = M^{-1}C$.

For Instance

$$MV = C$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{s} \\ 0 & 0 & -1 & 0 & 0 & -1 & \sqrt{s} & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 2p_{3x} & 2p_{3y} & 2p_{3z} & 0 & 0 & 0 & 0 & 0 \\ 2p_{5x} & 2p_{5y} & 2p_{5z} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2p_{4x} & 2p_{4y} & 2p_{4z} & 0 & 0 \\ 0 & 0 & 0 & 2p_{6x} & 2p_{6y} & 2p_{6z} & 0 & 0 \end{pmatrix} \begin{pmatrix} p_{1x} \\ p_{1y} \\ p_{1z} \\ p_{2x} \\ p_{2y} \\ p_{2z} \\ \sigma \\ \tau \end{pmatrix} = \begin{pmatrix} E_{vis} + E_1 + E_2 \\ p_{z,vis} \\ \not{p}_y \\ \not{p}_x \\ M_{31}^2 - M_1^2 + 2E_1E_3 \\ M_{51}^2 - M_1^2 + 2E_1E_5 \\ M_{42}^2 - M_2^2 + 2E_2E_4 \\ M_{62}^2 - M_2^2 + 2E_2E_6 \end{pmatrix}$$

Solves for unknowns V in terms of E_1, E_2 and new unknown masses $M_{31}, M_{51}, M_{42}, M_{62}, M_1, M_2$ using visible 4-vectors $p_3^\mu, p_4^\mu, p_5^\mu, p_6^\mu$.

M is *numeric* so each unknown in V can be written with numeric coefficients α_i

$$V_i = \alpha \cdot u; \qquad u = (E_1, E_2, M_1^2, M_2^2, M_{31}^2, M_{51}^2, M_{42}^2, M_{62}^2)$$

The remaning unknowns E_1 and E_2 are solved using the above solutions in the mass shell constraints for p_1, p_2.

$$E_1^2 - p_{1x}^2 - p_{1y}^2 - p_{1z}^2 = M_1^2$$
$$E_2^2 - p_{2x}^2 - p_{2y}^2 - p_{2z}^2 = M_2^2$$

$\Rightarrow$ Now we have a system of 2 quadratics in 6 unknowns, where the unknowns are *masses*.

New Development 2: Polynomial Systems

Example of a perverse system: $p_3 = p_4$. One can then do row arithmetic on M to generate a zero row, guaranteeing that $\det M = 0$.

Nojiri et al. provided a nice way to understand this. Each missing vector can be expanded in terms of the visible vectors:

$$p_1 = \alpha\,\vec{p}_3 + \beta\,\vec{p}_4 + \gamma\,\vec{p}_3 \times \vec{p}_4$$

The two dot-products coming from mass constraints $(p_1 + p_3)^2$ and $(p_4 + p_1)^2$ provide a basis in 3-space in which to solve for p_1. In order to have a basis, p_1, p_3 and p_4 must be non-parallel.

New Development 2: Polynomial Systems

These quadratics are ellipses, and can also be written using the vector $x_i = (E_1, E_2, 1)$ as

$$A = x_i f^{ij} x_j = 0, \qquad B = x_i g^{ij} x_j$$

using the tensors

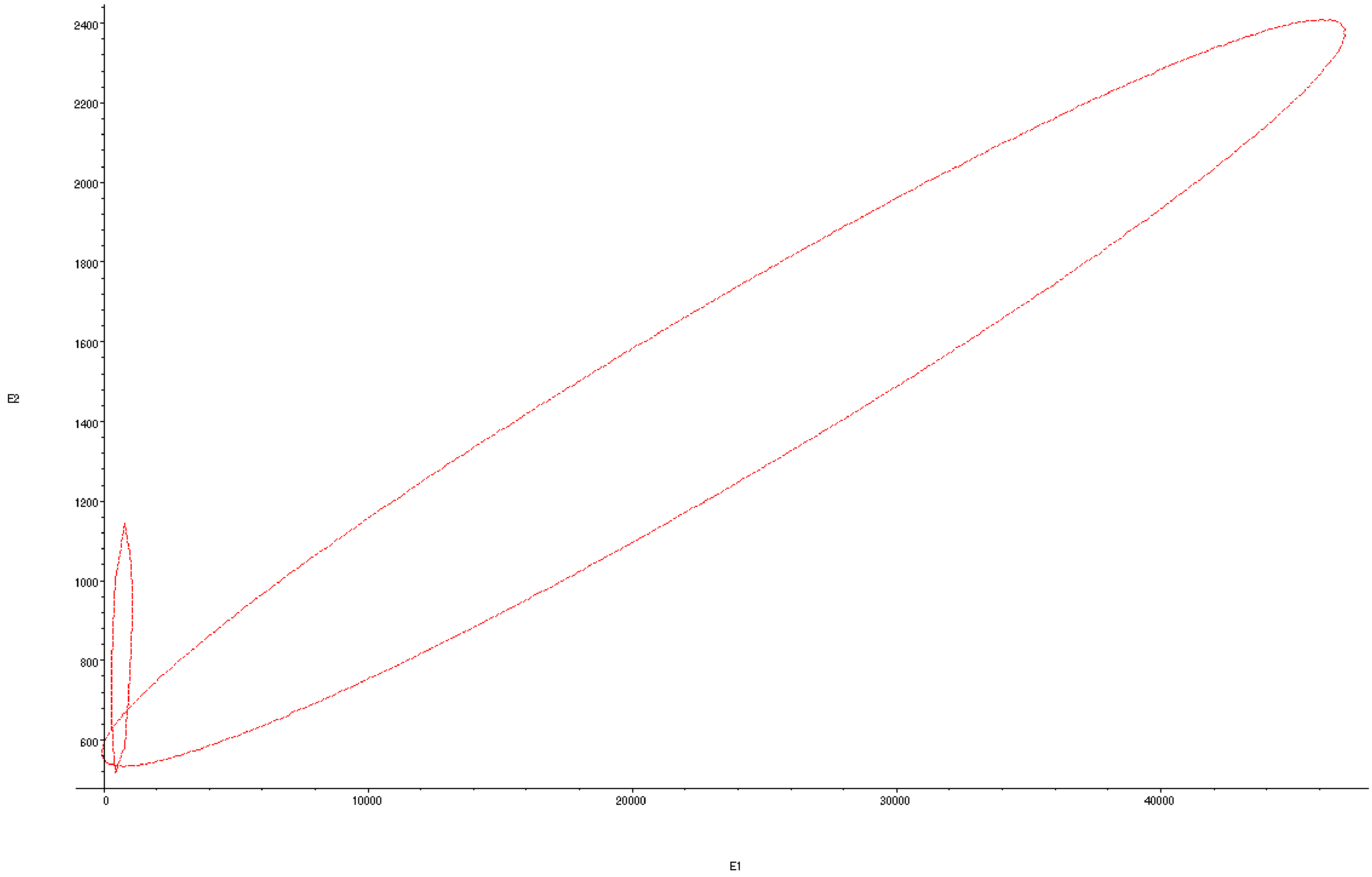
$$f^{ij} = \begin{bmatrix} a_{11} & a_{12} & a_1(m) \\ a_{12} & a_{22} & a_2(m) \\ a_1(m) & a_2(m) & a_0(m, m^2) \end{bmatrix} \qquad g^{ij} = \begin{bmatrix} b_{11} & b_{12} & b_1(m) \\ b_{12} & b_{22} & b_2(m) \\ b_1(m) & b_2(m) & b_0(m, m^2) \end{bmatrix}$$

For determining masses, we only care about *whether* a solution exists, not the actual values of E_1, E_2. Therefore, without loss of generality, we are free to rotate, translate, and scale the vector x_i.

$$f^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2(m, m^2) \end{bmatrix}, \qquad \begin{array}{c} x^2 + y^2 - r^2(m, m^2) = 0 \\ \frac{(x - x_0(m))^2}{a^2} + \frac{(y - y_0(m))^2}{b^2} - R^2(m, m^2) = 0 \end{array}$$

$$g^{ij} = \begin{bmatrix} 1/a^2 & 0 & -x_0(m)/a^2 \\ 0 & 1/b^2 & -y_0(m)/b^2 \\ -x_0(m)/a^2 & -y_0(m)/b^2 & -R^2(m, m^2) + x_0(m)^2/a^2 + y_0(m)^2/b^2 \end{bmatrix}$$

Polynomial Systems: J. Random Event



Polynomial Systems: Constructing an Algorithm

Each event defines a *volume* in mass space that is consistent with it. One can then, in principle, construct a high dimensional histogram, and fit it.

For instance, in a chain decay in which both sides are assumed to be the same, one can reduce to 3 masses, and make a 3-D histogram.

It is important to note that we have *projected into mass space*. Irrelevant angles are removed, and all information available is used. If you want to measure mass, you really want to be working in this space.

All other methods find a correlated variable, and derive its correlation to mass. Since the correlation is not $1:1$, these variables contain non-mass information (such as angles) as well!

Practically, high dimensional histograms are difficult to deal with, so let us make a series of projections.

Multi-Event Likelihood

One can ask: What is the volume of mass space allowed simultaneously by multiple events?

This is essentially to ask if the narrow width, n-event likelihood:

$$L_n = \prod_i^n P_i(\{p_j^\mu\}_i|\lambda)$$

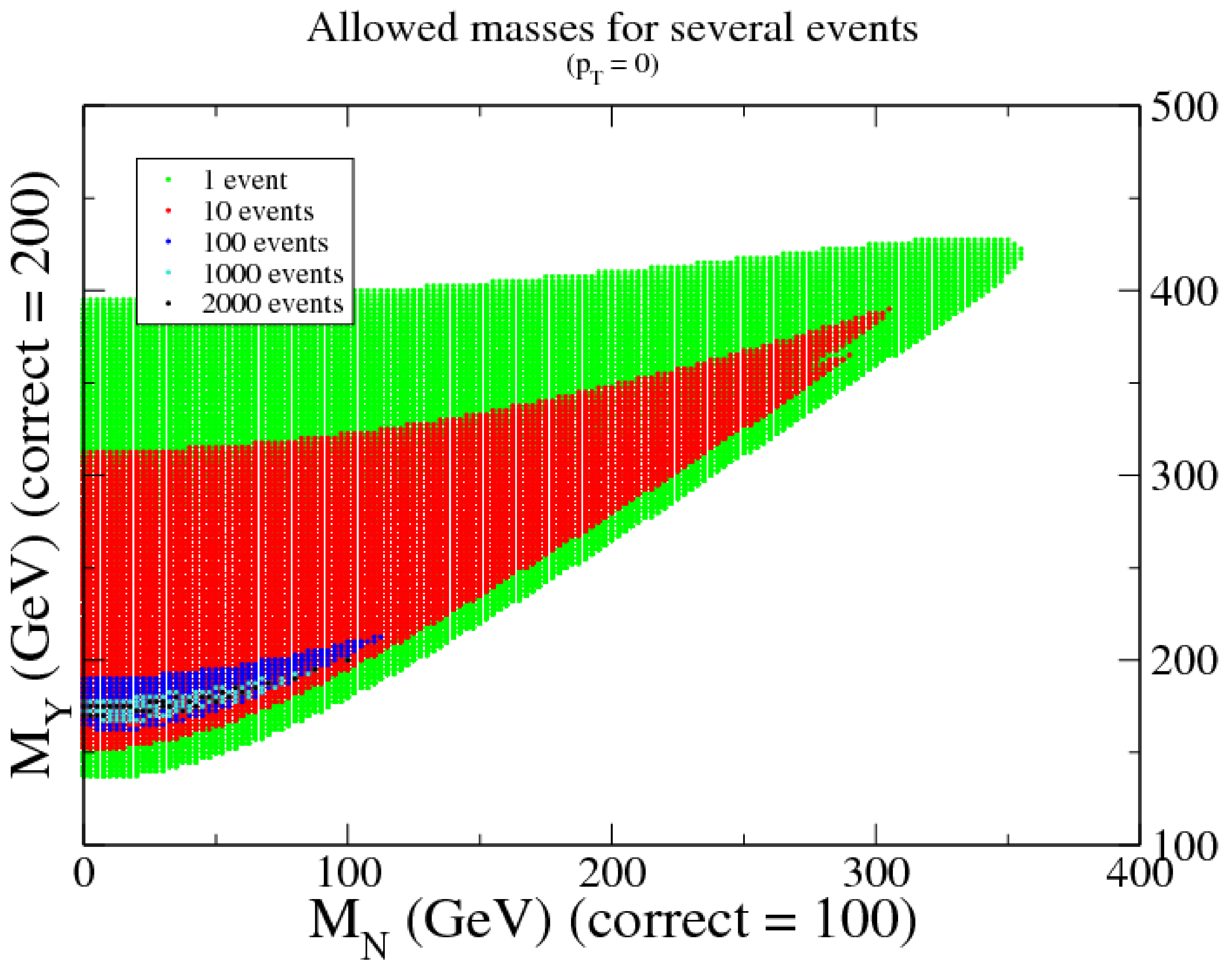
is zero or non-zero.

This volume decreases in size as you increase the number of events. But, it is always a volume.

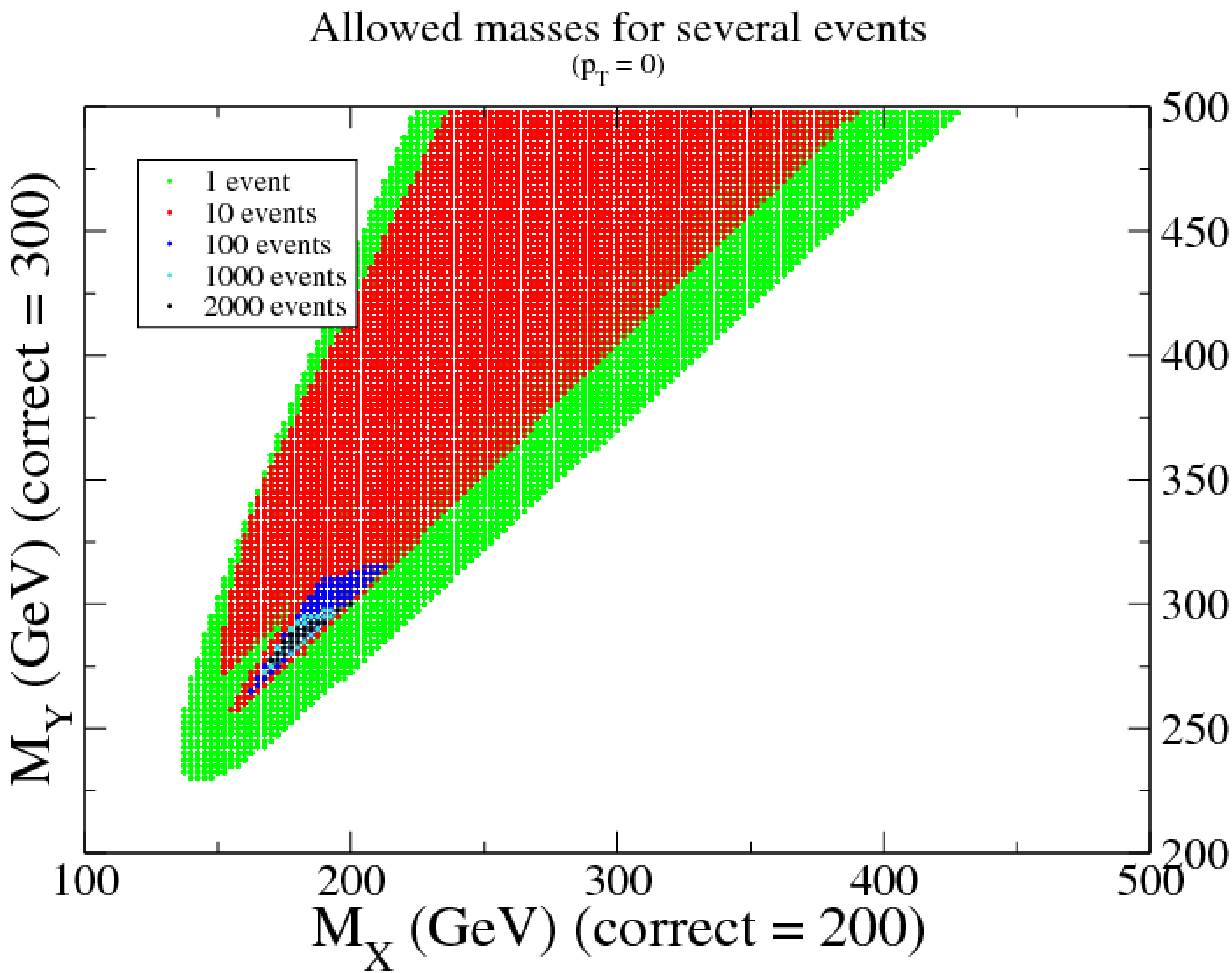
The following plots are for $p_T = 0$, no smearing, and

$$M_X = M_{X'} = 300\text{GeV}$$
$$M_Y = M_{Y'} = 200\text{GeV}$$
$$M_N = M_{N'} = 100\text{GeV}$$

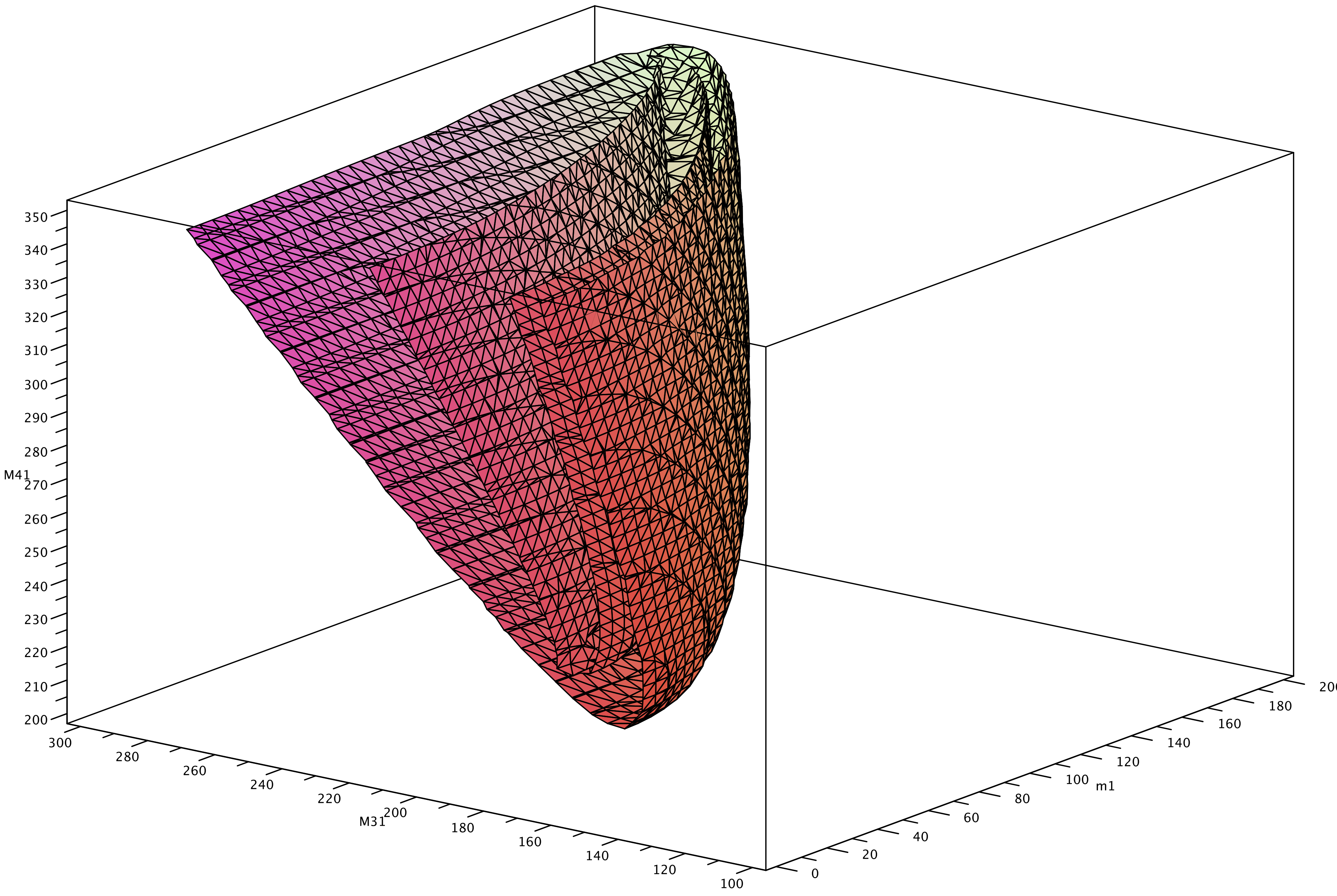
(so, not realistic, but to give you an idea).



Allowed masses for several events
($p_T = 0$)
1 event
10 events
100 events
1000 events
2000 events
0
100
200
300
400
100
200
300
400
500
M_N (GeV) (correct = 100)
M_Y (GeV) (correct = 200)

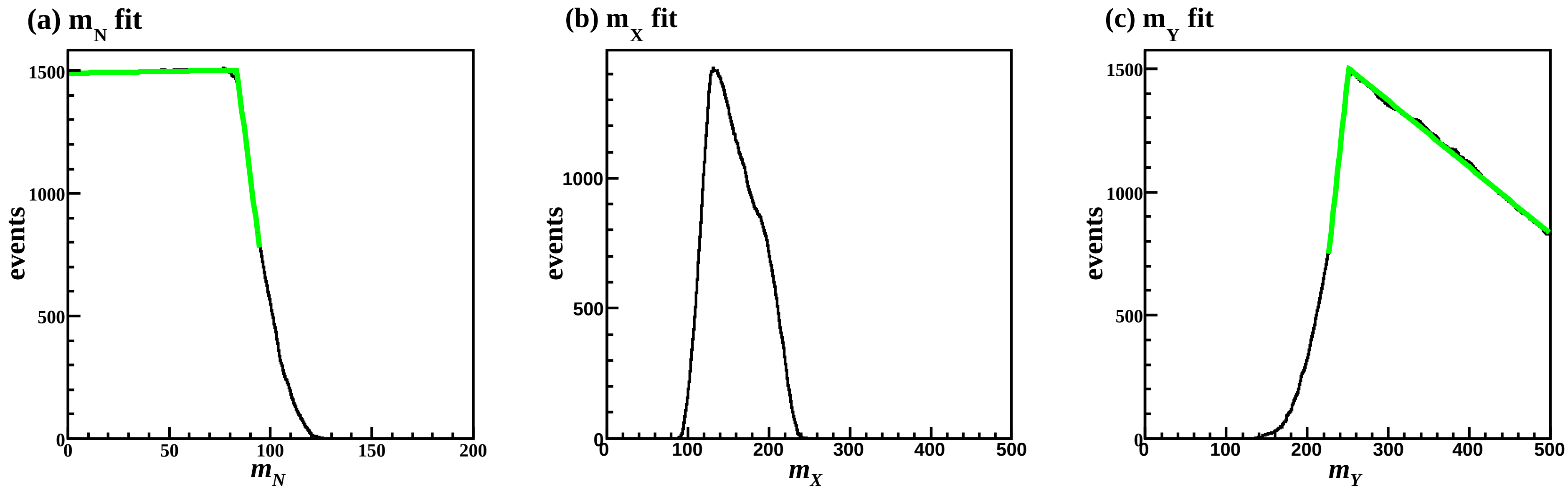


Allowed masses for several events
($p_T = 0$)
1 event
10 events
100 events
1000 events
2000 events
500
450
400
350
300
250
200
100
200
300
400
500
M_Y (GeV) (correct = 300)
M_X (GeV) (correct = 200)



M41
350
340
330
320
310
300
290
280
270
260
250
240
230
220
210
200
M31
300
280
260
240
220
200
180
160
140
120
100
m1
0
20
40
60
80
100
120
140
160
180
200

Graphical Algorithm

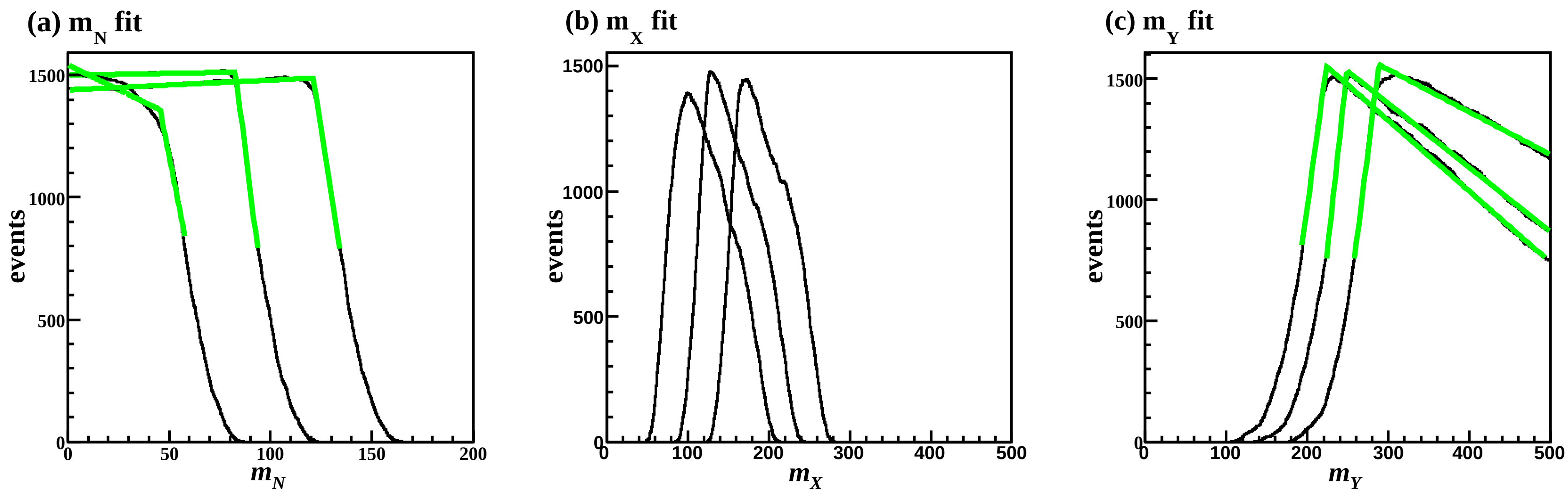


Fixing two of the masses, we scan in the third mass. Unfortunately an analytic expression for these curves is probably intractable to derive.

For a large number of events, we want the *largest* M_N compatible with the event. Large p_T cuts off the zero mass solution, but the high mass solution converges to the correct value faster, and our understanding of p_T in hadron colliders is poor. (e.g. M_T used to measure M_W is designed to be p_T insensitive)

But! Features are simple. We fit a line to the "corner" to determine its location.

Iterate

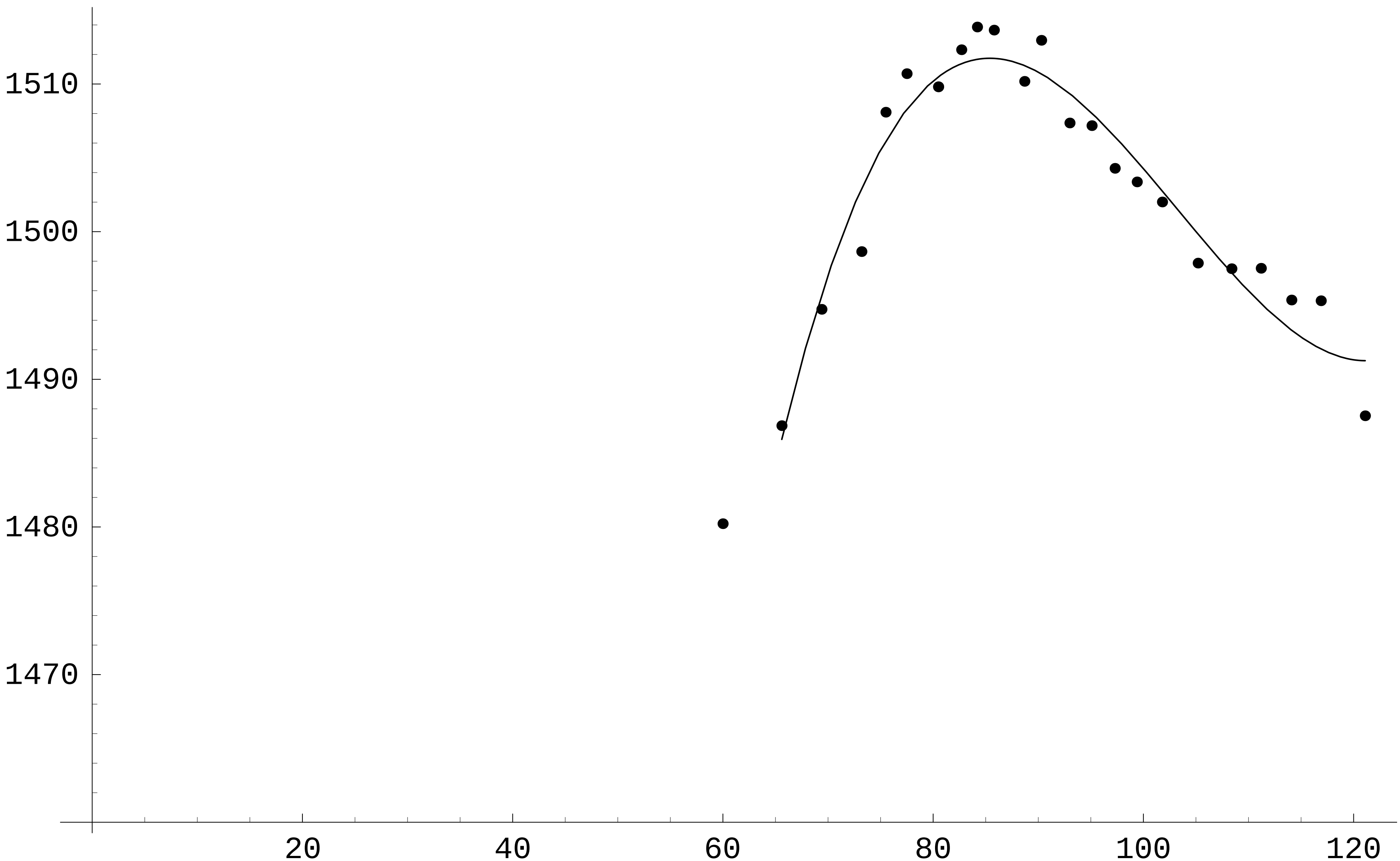


Iterate in each mass, fitting for each mass successively.

This procedure "walks up" the mass space, increasing the over mass scale, and is not convergent. (e.g. there still exists a solution at $M_N = \infty$ for most events)

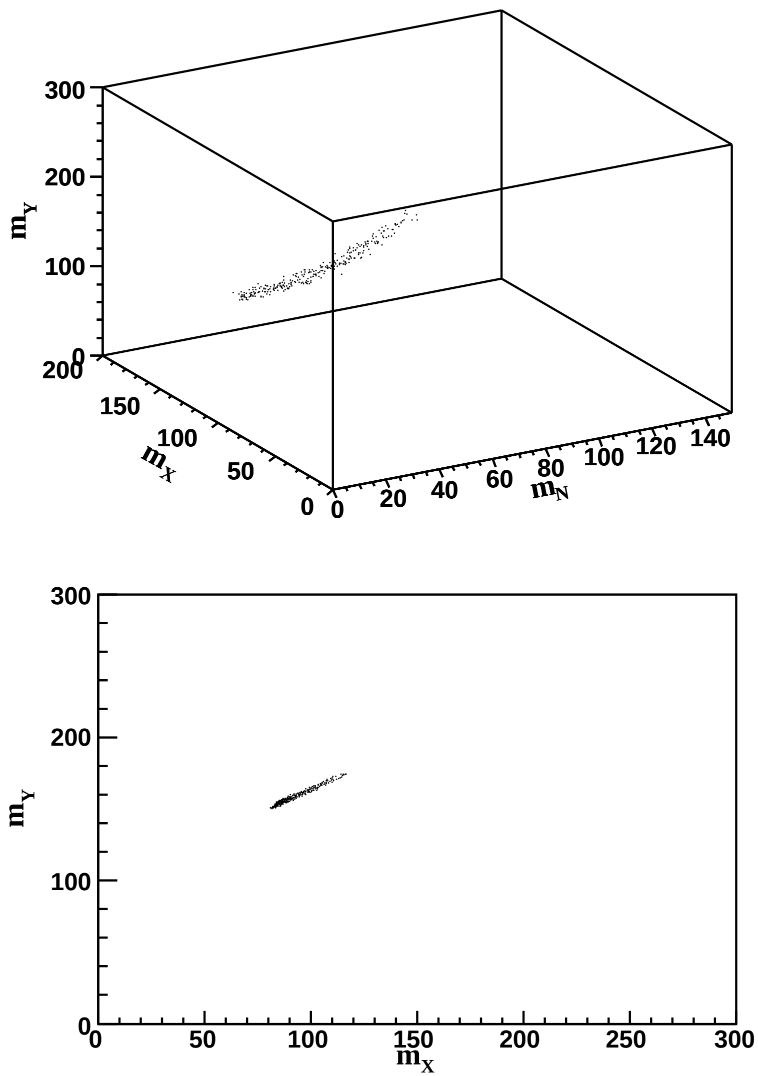
But! We have not yet used the total number of events fit.

M_N vs. Number of Events



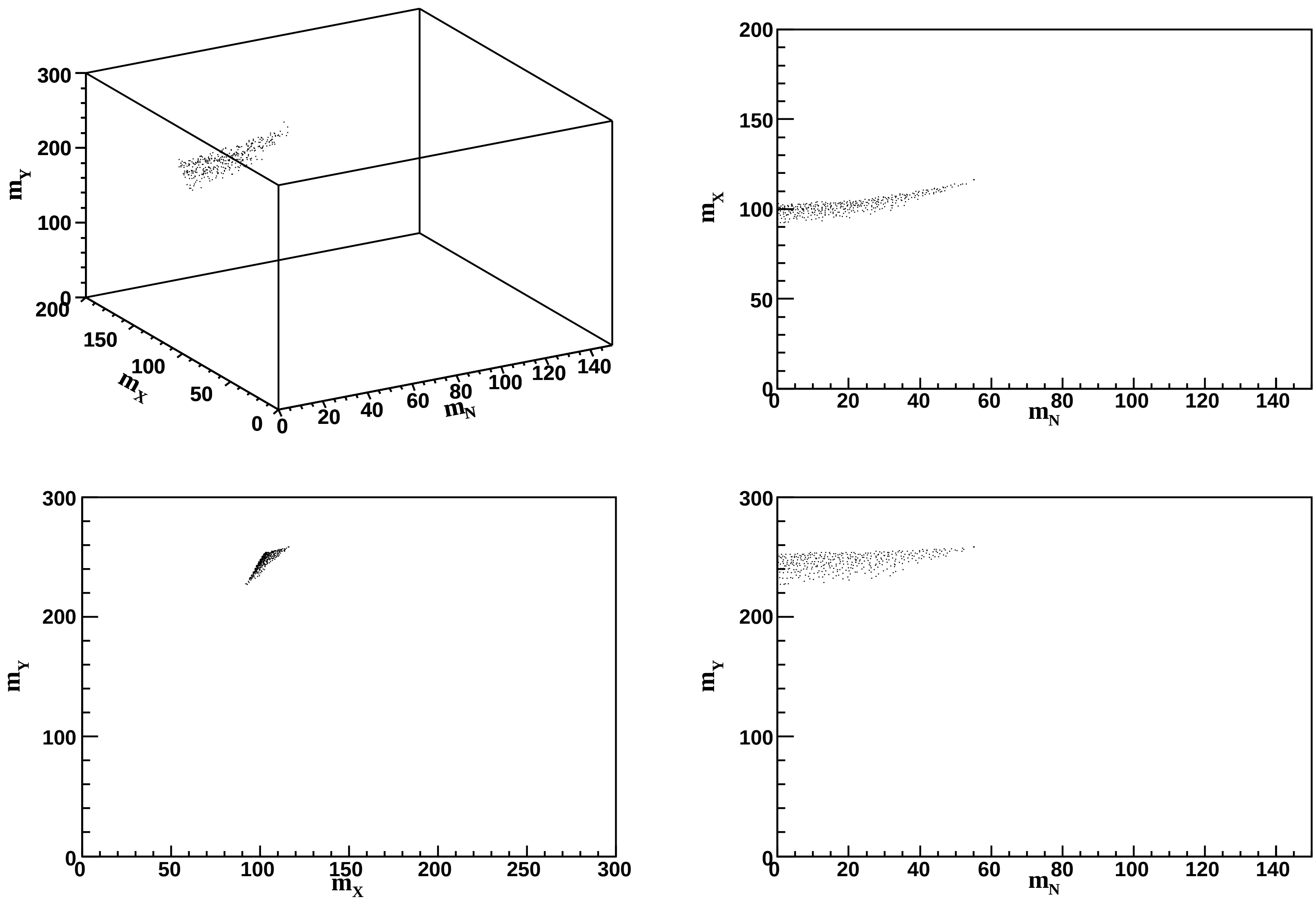
3D projections with no smearing

$$(m_Y, m_X, m_N) = (180.8, 147.1, 85.2)$$

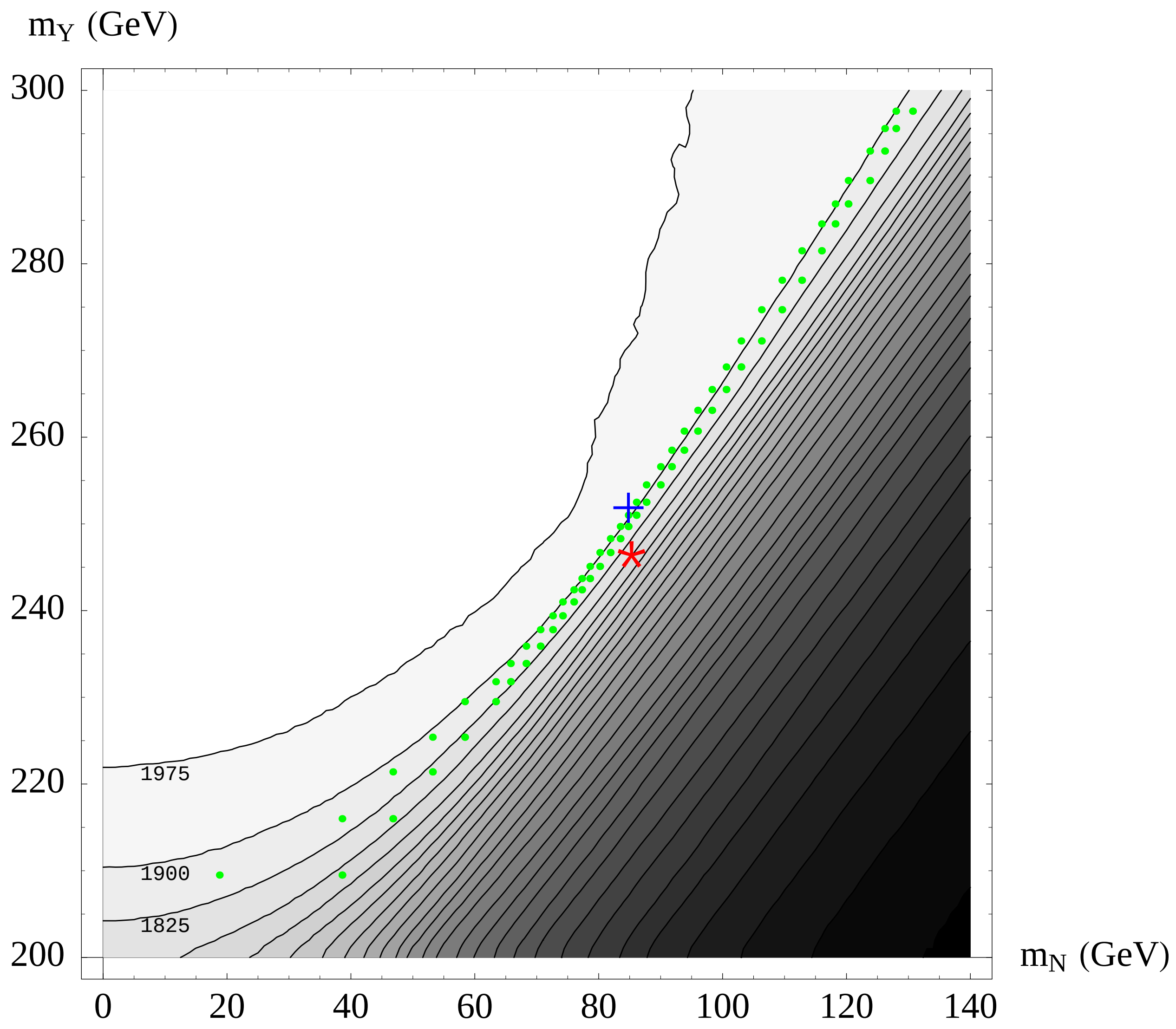


3D projections with smearing

$$(m_Y, m_X, m_N) = (246.6, 128.4, 85.3)$$



Contour Plot of Procedure



Fit Results

For the point

$$(m_Y, m_X, m_N) = (246.6, 128.4, 85.3)$$

we reconstruct

$m_Y = 252.2 \pm 4.3$ GeV, $m_X = 130.4 \pm 4.3$ GeV, $m_N = 86.2 \pm 4.3$ GeV.

The statistical variations for the mass differences are much smaller:

$$m_Y - m_X = 119.8 \pm 1.0 \text{ GeV}, \quad m_X - m_N = 46.4 \pm 0.7 \text{ GeV}.$$

We smear momenta using ATLFAST's muon resolution, and a missing momentum resolution given by a gaussian with width 18 GeV. This uses 2900 events (1900 after cuts), which corresponds to 90 fb^{-1} at the LHC for our chosen model point

$$\mu = +300\text{GeV}, \quad \tan\beta = 10, \quad (\widetilde{M}_1, \widetilde{M}_2, \widetilde{M}_3) = (90, 300, 500)\text{GeV}$$
$$\widetilde{m}_L^{(1,2,3)} = \widetilde{m}_E^{(3)} = 1000\text{GeV}, \quad \widetilde{m}_E^{(1,2)} = 120\text{GeV}$$
$$\widetilde{m}_Q^{(1,2)} = 400\text{GeV}, \quad \widetilde{m}_{U,D}^{(1,2)} = 300\text{GeV}, \quad \widetilde{m}_Q^{(3)} = \widetilde{m}_{U,D}^{(3)} = 1000\text{GeV}$$

New Development 3: Combining Events

Kawagoe et al [hep-ph/0410160] suggested that if we have a long decay chain such as

$$\tilde{g} \to \tilde{b} b_2 \to \tilde{\chi}_2^0 b_1 b_2 \to \tilde{\ell} b_1 b_2 \ell_2 \to \tilde{\chi}_1^0 b_1 b_2 \ell_1 \ell_2,$$

one can solve for the intermediate masses by combining several events.

This decay chain has 4 missing kinematic quantities (from the neutralino 4-vector – initial state is given by the δ^4), and 5 masses. Thus each event describes a 4-dimensional hypersurface in a 5-dimensional space. They demonstrate this by assuming the $\tilde{\chi}_1^0, \tilde{\chi}_2^0$, and $\tilde{\ell}$ masses are *known* (thus giving them a one dimensional hypersurface in a 2 dimensional space to solve).

Again the neutralino mass shell condition gives a quadratic (in m^2) for each event. They then combine all possible pairs of events.

$$Q_{11} m_{\tilde{g}}^4 + 2 Q_{12} m_{\tilde{g}}^2 m_{\tilde{b}}^2 + Q_{22} m_{\tilde{b}}^4 + 2 Q_1 m_{\tilde{g}}^2 + 2 Q_2 m_{\tilde{b}}^2 + Q = 0,$$

Combining Events: our method

Given the decay $\tilde{q}\tilde{q} \to q\chi_2^0 q\chi_2^0 \to ql\tilde{l}ql\tilde{l} \to qll\chi_1^0 qll\chi_1^0$ (as occurs in SPS 1a):

This process is underconstrained by 2. There are 4 kinematic unknowns and 6 unknown intermediate masses. So, not enough constraints to solve simultaneously for the masses and the kinematic unknowns in one event.

But, under the assumption that the masses are the same on both sides of the event, and the same between two events, one can solve for the masses using a *pair* of events.

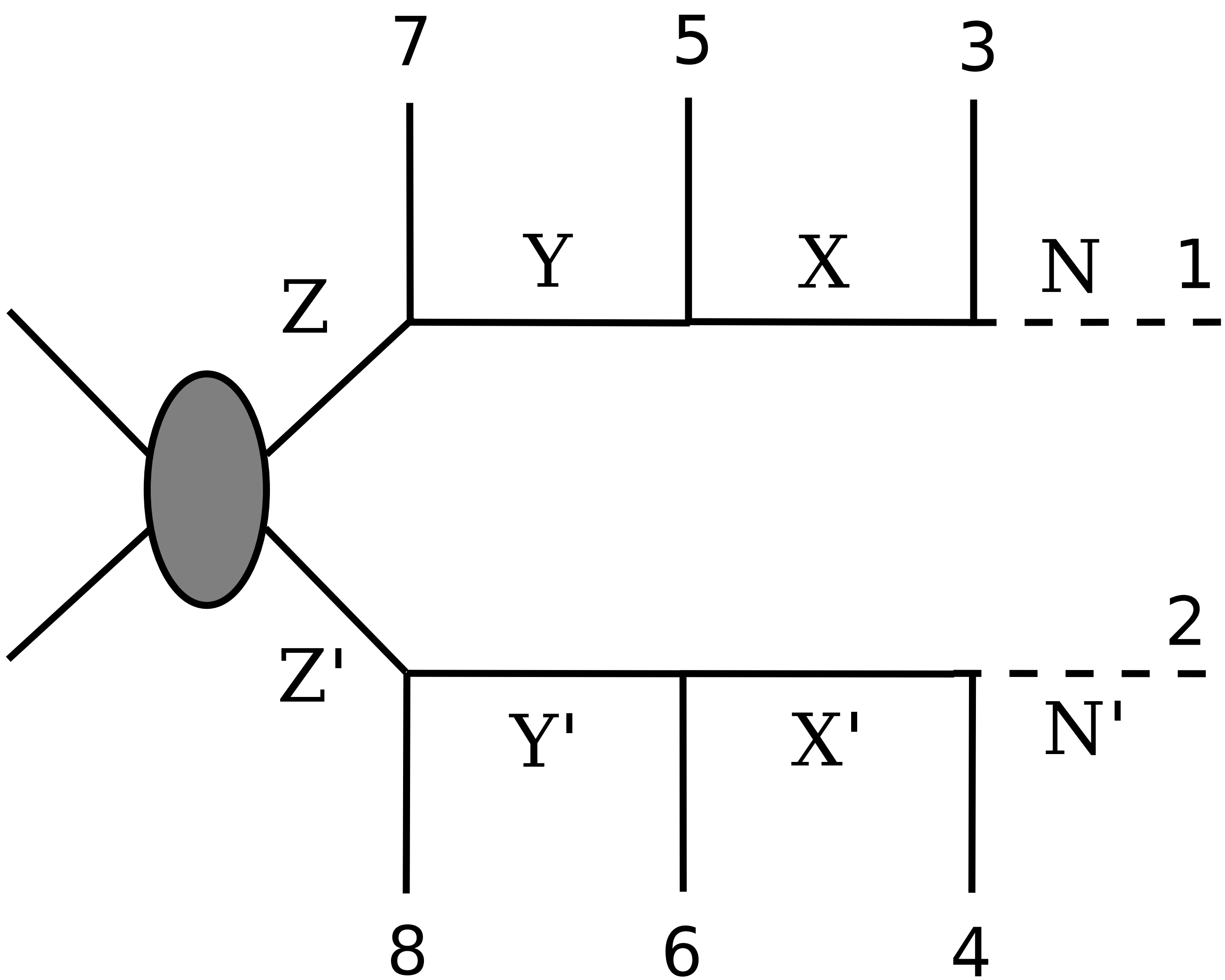
This is equivalent to asking: Is the 2-particle likelihood, in the narrow-width approximation zero or non-zero?

$$L_2 = P_1(\{p_i\}_1|\{M_j\})P_2(\{p_i\}_2|\{M_j\})$$

Naively this gives 4 quadratic equations. However one can use instead three quadratics by relating momenta $p_1^2 = p_2^2$.

Another nice way to think of this is doing OSET's *backwards*.

Example Two: Overconstrained



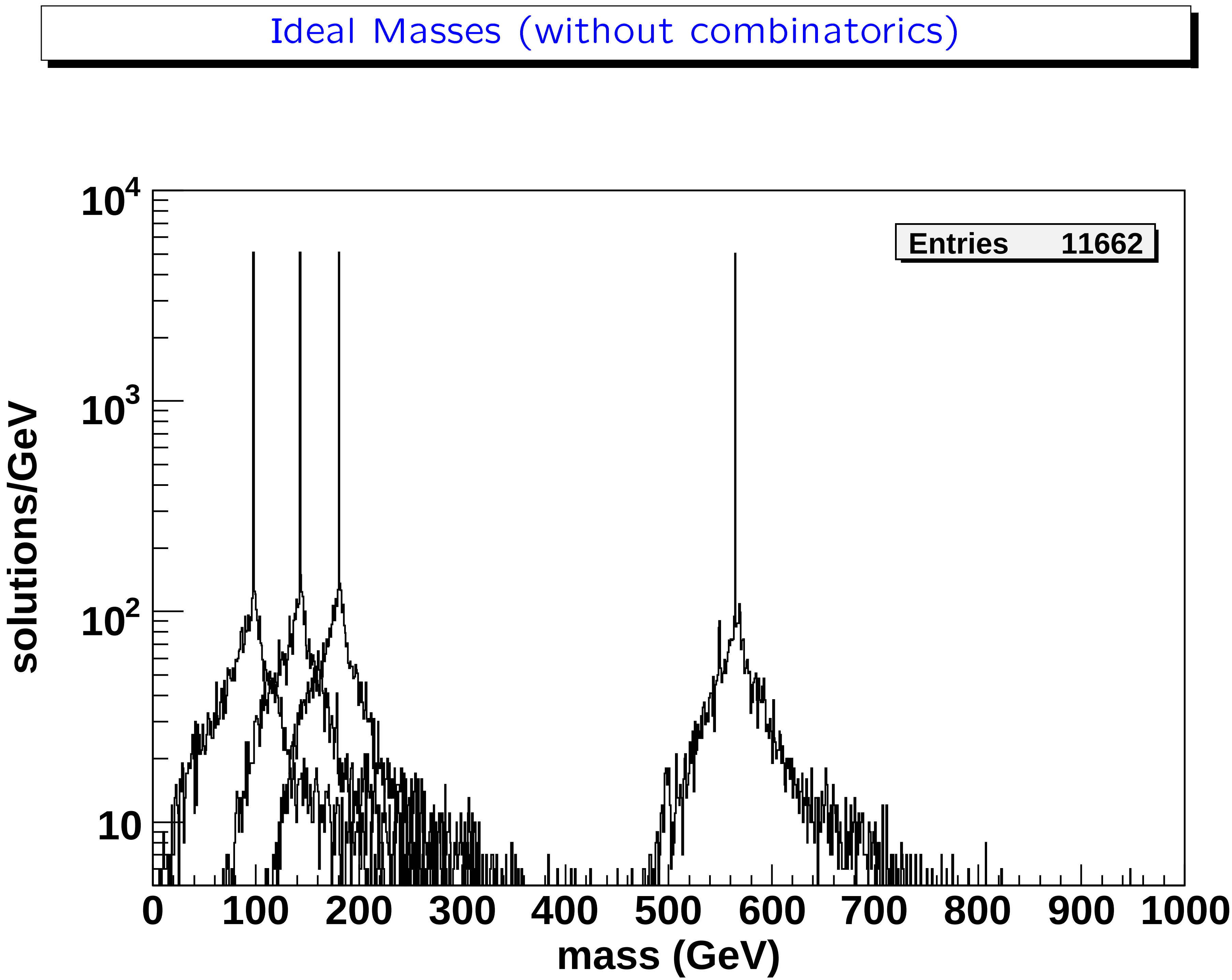
Constraint Equations

$$
\begin{array}{rrcl}
(M_Z^2 =) & (p_1+p_3+p_5+p_7)^2 & = & (p_2+p_4+p_6+p_8)^2, \\
(M_Y^2 =) & (p_1+p_3+p_5)^2 & = & (p_2+p_4+p_6)^2, \\
(M_X^2 =) & (p_1+p_3)^2 & = & (p_2+p_4)^2, \\
(M_N^2 =) & p_1^2 & = & p_2^2.
\end{array}
\qquad (1)
$$

$$p_1^x + p_2^x = p_{miss}^x, \quad p_1^y + p_2^y = p_{miss}^y.$$

$$
\begin{array}{rclcl}
q_1^2 & = & q_2^2 & = & p_2^2, \\
(q_1+q_3)^2 & = & (q_2+q_4)^2 & = & (p_2+p_4)^2, \\
(q_1+q_3+q_5)^2 & = & (q_2+q_4+q_6)^2 & = & (p_2+p_4+p_6)^2, \\
(q_1+q_3+q_5+q_7)^2 & = & (q_2+q_4+q_6+q_8)^2 & = & (p_2+p_4+p_6+p_8)^2,
\end{array}
$$

$$q_1^x + q_2^x = q_{miss}^x, \qquad q_1^y + q_2^y = q_{miss}^y.$$



Ideal Masses (without combinatorics)
Entries 11662
solutions/GeV
10^4
10^3
10^2
10
0
100
200
300
400
500
600
700
800
900
1000
mass (GeV)

Application of Realism

- Combinatorics: There are 16 choices of where to assign the leptons/jets per event for 4μ or $4e$, or 8 for $2\mu 2e$. Combinatorics are fundamental and *must* be taken into account. There is no magic cut which gets rid of them. Combinatorics also *carry information about mass*.

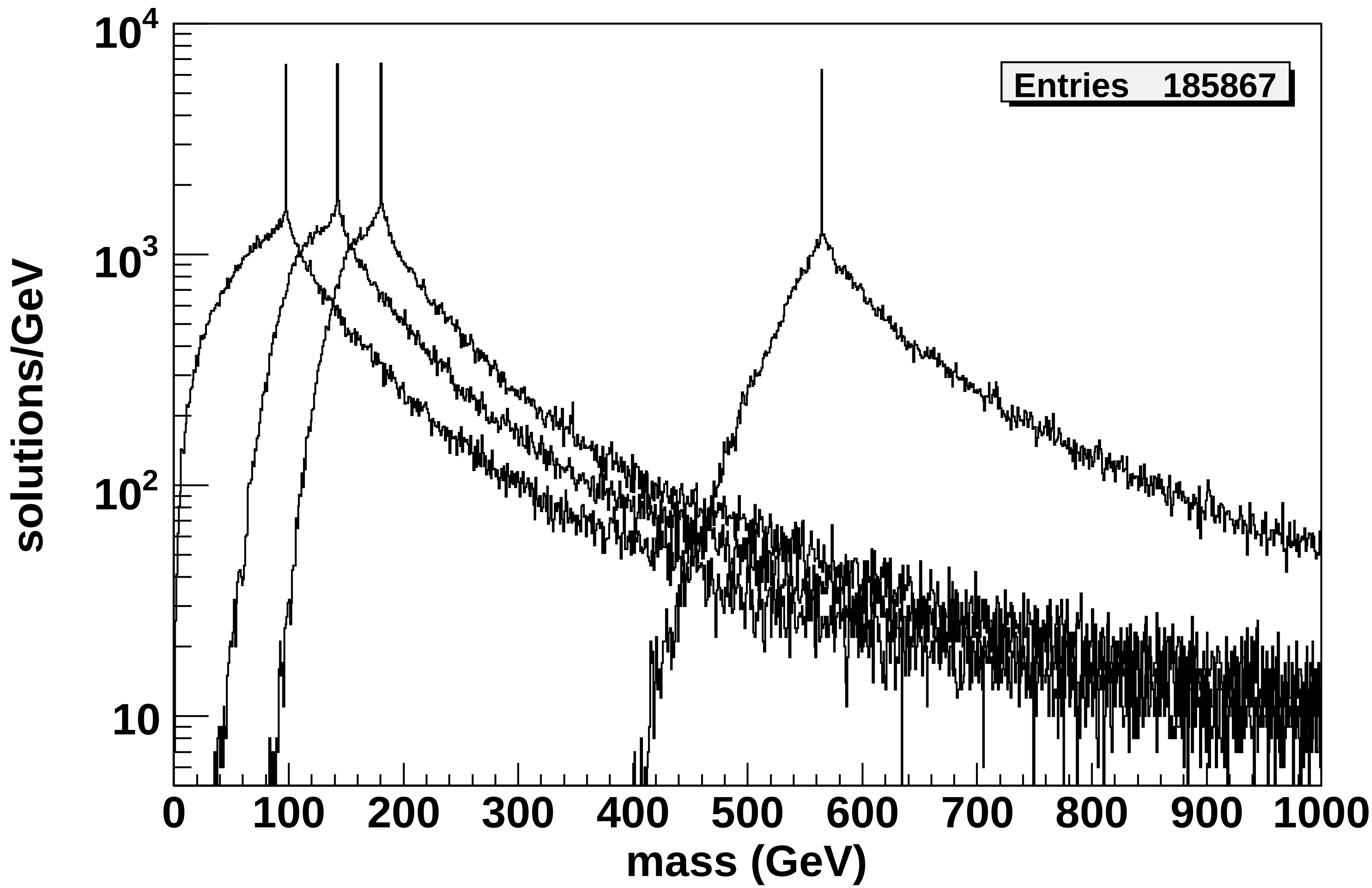
- Backgrounds: This signal has no real SM background. We include all SUSY backgrounds including $\widetilde{\tau}$ decays and $\widetilde{\chi}_2^0$ not from squark decay, and $\widetilde{g}$ events (which have extra hard jets).

- Finite widths: $\Gamma_{\widetilde{q}} = 5$ GeV, $\Gamma_{\widetilde{\chi}_2^0} = 20$ MeV, $\Gamma_{\widetilde{\ell}_R} = 200$ MeV.

- Mass splitting: Different flavor squarks have different masses by 6 GeV. Therefore, our squark mass result is an average of these signals.

Note that these techniques work with *very* few events (e.g. ten).

Ideal Masses (with combinatorics)



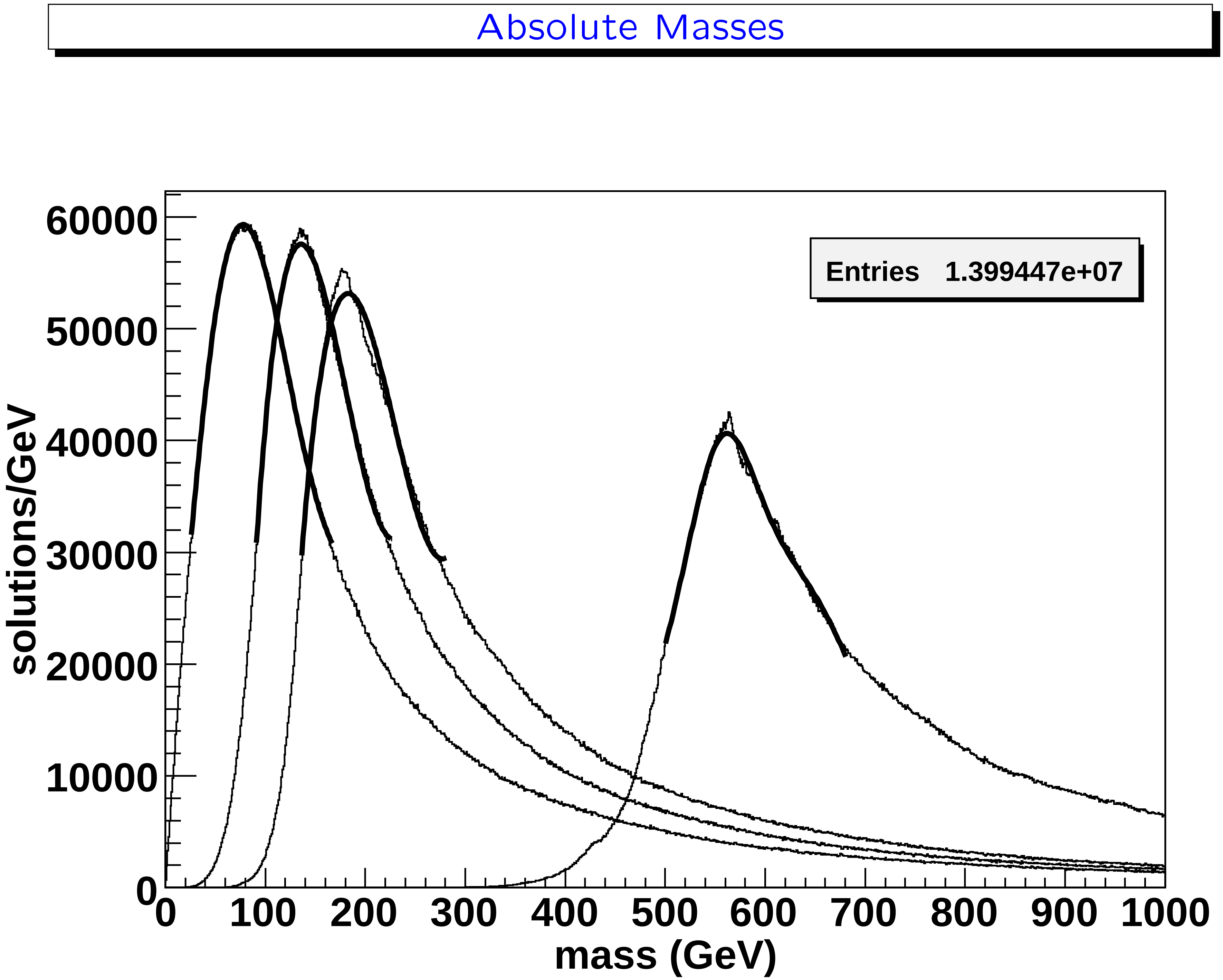
Application of Realism

We simulate all events with ATLFAST running in high-luminosity mode. We assume 300 fb^{-1} of luminosity. We require

- 4 isolated ($\Delta R < 0.4$) leptons with $p_T > 10$ GeV, $|\eta| < 2.5$. (flavors, charges chosen to match our $\tilde{\chi}_2^0 \to \tilde{\ell} \to \tilde{\chi}_1^0$ topology.

- no b-jets and ≥ 2 jets with $p_T > 100$ GeV, $|\eta| < 2.5$. The highest p_T jets are taken to be particles 7,8 (extra jets from parton shower/reconstruction are present).

- Missing $p_T > 50$ GeV.



Absolute Masses
Entries 1.399447e+07
solutions/GeV
60000
50000
40000
30000
20000
10000
0
0
100
200
300
400
500
600
700
800
900
1000
mass (GeV)

Extra Cuts

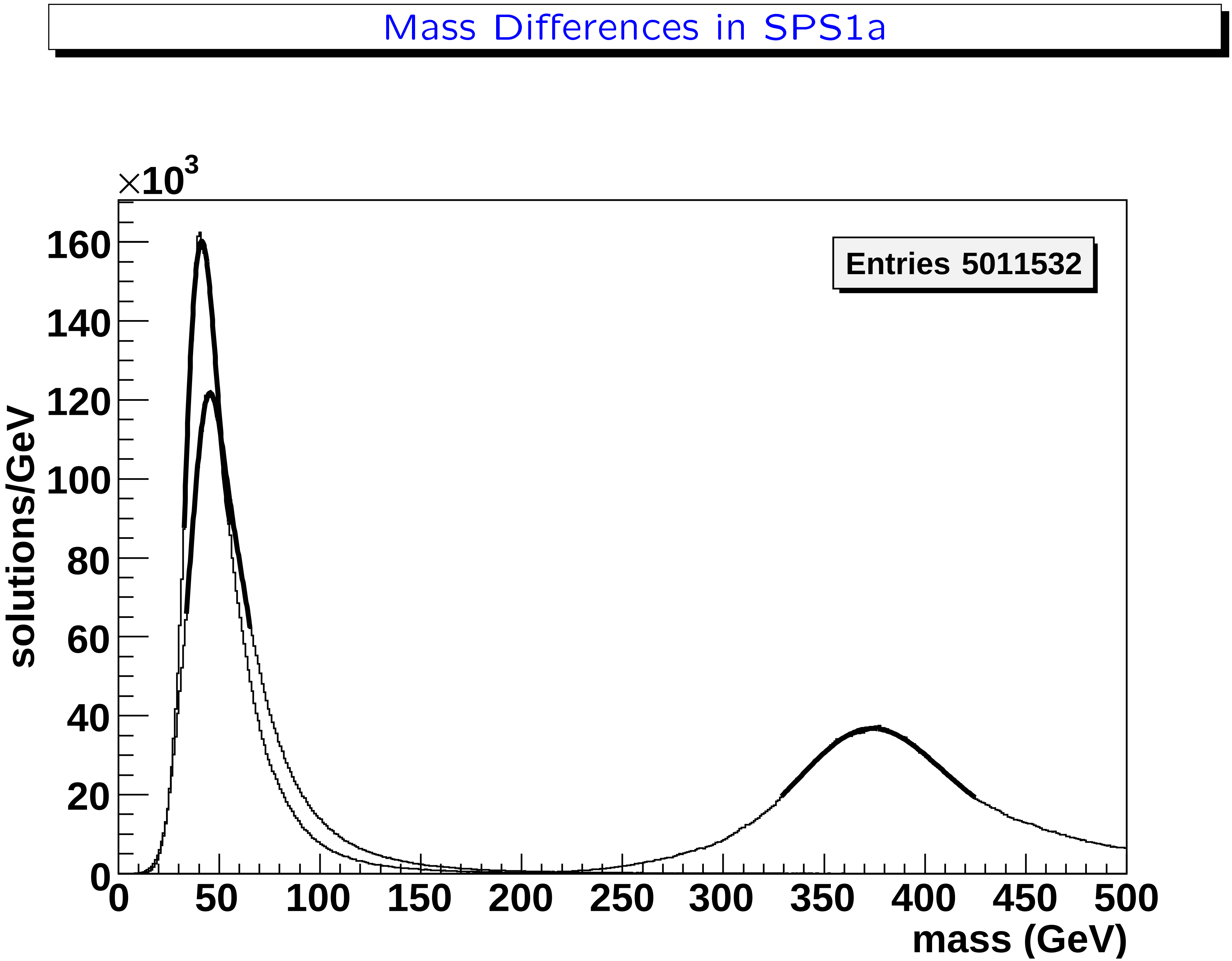
We add new cuts to improve S/B and decrease bias

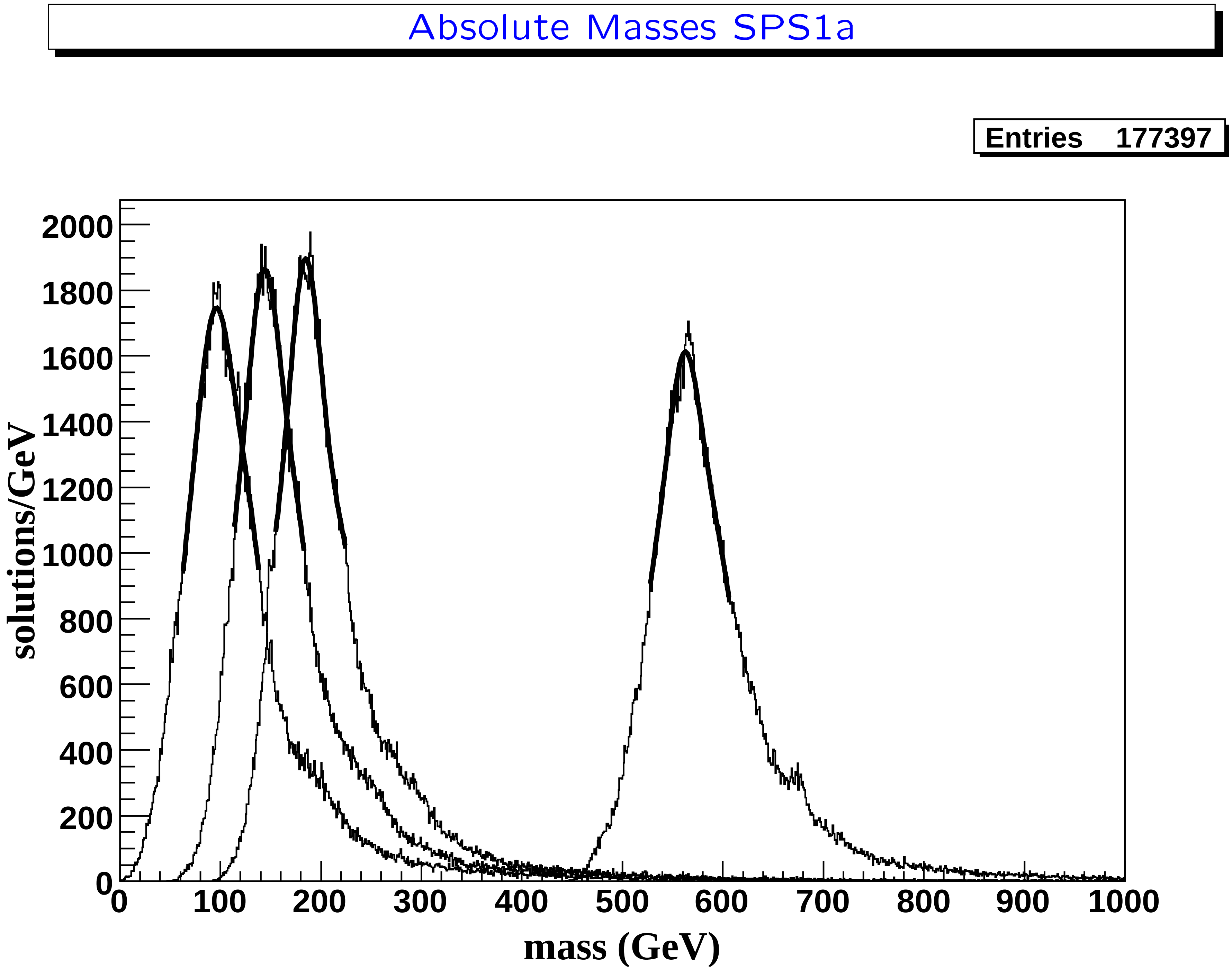
- We require that each combination c in each event i have solutions with some combination in 75% of the other events. $N_{pair}(c, i) < 0.75 N_{events}$

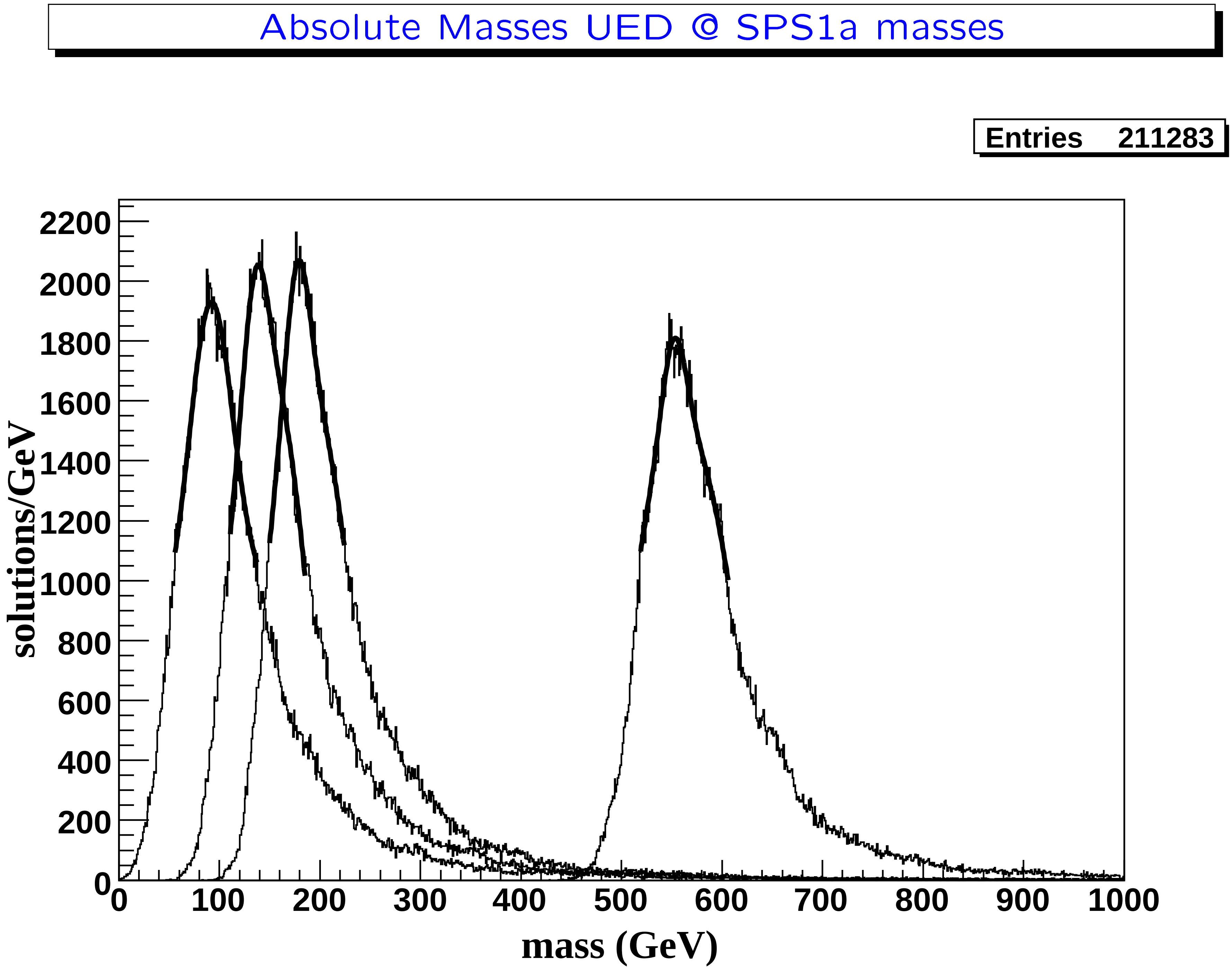
- We weight the final histogram by $1/N$ where N is the number of solutions in a given pair.

- We cut on the mass differences (window defined by 0.6 of peak height – e.g. Full Width at 0.6 Max)

There are many other interesting manipulations one can do, that are quite different from cutting on physical observables.


Mass Differences in SPS1a
Entries 5011532
×10^3
solutions/GeV
mass (GeV)
0
20
40
60
80
100
120
140
160
0
50
100
150
200
250
300
350
400
450
500


Absolute Masses SPS1a
Entries 177397
solutions/GeV
mass (GeV)
0
200
400
600
800
1000
1200
1400
1600
1800
2000
0
100
200
300
400
500
600
700
800
900
1000


Absolute Masses UED @ SPS1a masses
Entries 211283
solutions/GeV
2200
2000
1800
1600
1400
1200
1000
800
600
400
200
0
0
100
200
300
400
500
600
700
800
900
1000
mass (GeV)

Results

We fit peaks using a gaussian+quadratic polynomial, and use the maximum as our mass estimator. This is a biased estimator, but can be used to estimate our statistical error by repeating the measurement. Using 10 independent sets of Monte Carlo, for the SPS1a point with masses {91.7,135.9, 175.7 558.0}

$$\begin{aligned} m_N &= 94.1 \pm 2.8\text{GeV}, \\ m_X &= 138.8 \pm 2.8\text{GeV}, \\ m_Y &= 179.0 \pm 3.0\text{GeV}, \\ m_Z &= 561.5 \pm 4.1\text{GeV}. \end{aligned} \tag{2}$$

There are 539 signal + 195 background events in this sample after all cuts.

Precision is degraded by our “bias reduction” procedure. This is great for getting the mass within 5% very quickly (without scanning in masses), but final errors using these techniques is about a factor 2 better.

Code Availability

For the construction of the polynomial system, the problem can be divided into two stages: a linear stage and a quadratic stage. (Don't spend a lot of time with equations in Mathematica/Maple, there's an easier way to do it, and it's just a matrix) Each missing particle mass-shell constraint provides one quadratic, and any resonance with two or more invisible particles downstream provides a quadratic.

Solving a system of 2 quadratics is straightforward (it can be reduced to a quartic, and solved analytically).

Solving systems of $n > 2$ quadratics is highly nontrivial.

We have packaged up our code to solve a 2-quadratic system and 3-quadratic system, and the construction of the quadratic systems described in our paper(s).

http://particle.physics.ucdavis.edu/hefti/projects/doku.php?id=wimpmass

I have some (unfinished) C++ classes which are very general and could be used for any process with any number of quadratics. (I need collaborators)

Les Houches 2009 Project

We've made a mess. There are $\mathcal{O}(30)$ variables and techniques on the market, including the subset I've presented here.

Which are best? Which fail, under which circumstances? Which are more/less sensitive to ISR/FSR? Which are sensitive to experimental uncertanties (e.g. Jet Energy Scale)? Which areas need more attention?

We have generated Monte Carlo and background for SPS1a and UED with SPS1a mass choices, including correct spin dependence (via BRIDGE) correct matrix elements (via MadGraph), correct extra radiation (ISR/FSR) via MadGraph matching, reasonable detector simulation (via Delphes), and public availability (via MCDB). Answering as many of the above questions as possible is the goal for the Les Houches 2009 report.

This is a *public* project. Anyone can join. We also encourage reuse of the above Monte Carlo we painstakingly generated, in other people's projects.

http://www.lpthe.jussieu.fr/LesHouches09Wiki/index.php/Mass_methods