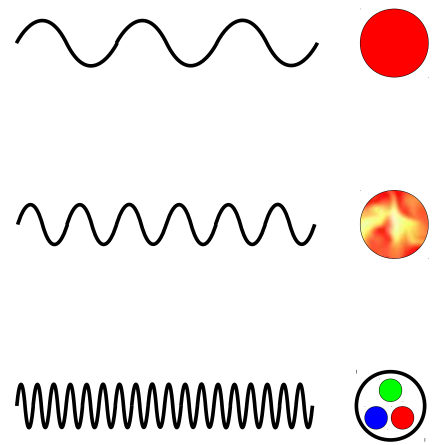
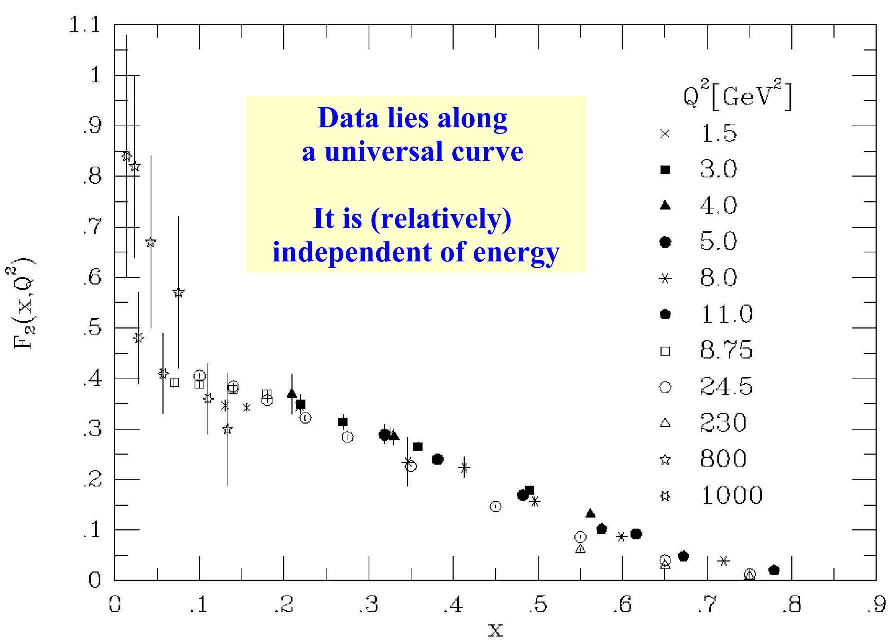


$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times 1$$

$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times F\left(\frac{Q^2}{\Lambda^2}\right)$$

Λ of order of the proton mass scale

$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times \sum_i e_i^2$$



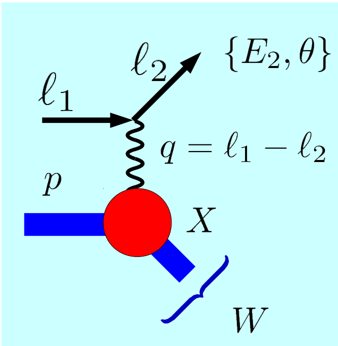
HOW TO CHARACTERIZE THE PROTON

Deeply Inelastic Scattering

(DIS)

Cf. lecture by
Simona Malace

Inclusive Deeply Inelastic Scattering (DIS)

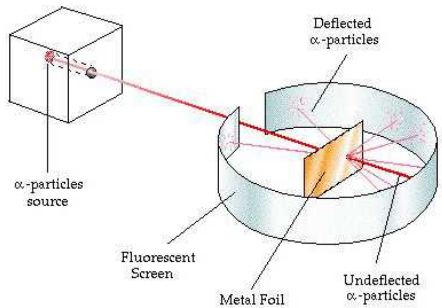


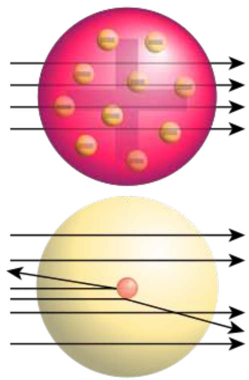
Measure $\{E_2, \theta\} \Leftrightarrow \{x, Q^2\}$ Inclusive

Deep: $Q^2 \geq 1 GeV^2$

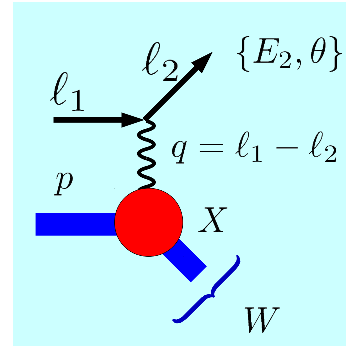
Inelastic: $W^2 \geq M_p^2$

Analogue of Rutherford scattering





Inclusive Deeply Inelastic Scattering (DIS)



Measure $\{E_2, \theta\} \Leftrightarrow \{x, Q^2\}$

$$Q^2 = -q^2 = 4E_1E_2\sin^2(\theta/2)$$

$$x = \frac{Q^2}{2p\cdot q} = \frac{2E_1E_2\sin^2(\theta/2)}{M(E_1 - E_2)}$$

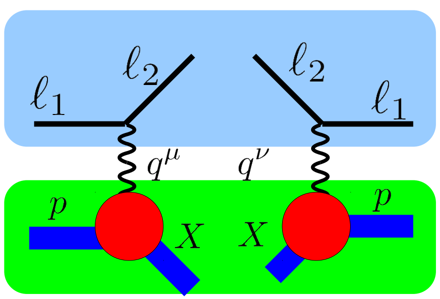
$$d\sigma \sim |A|^2$$

Other common DIS variables

$$\nu = \frac{p\cdot q}{p^2} = E_1 - E_2$$

$$y = \frac{\nu}{E_1} = \frac{Q^2}{2ME_2x}$$

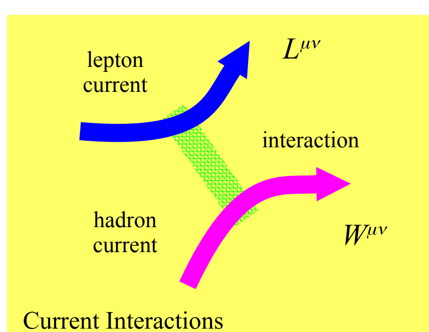
Lepton Tensor (L) and Hadronic Tensor (W)



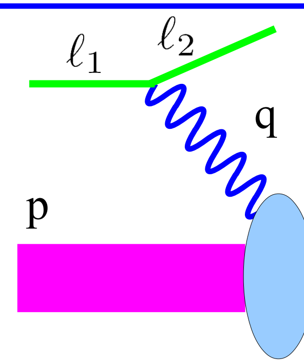
$L^{\mu\nu}$ Leptonic Tensor

$W_{\mu\nu}$ Hadronic Tensor

$$d\sigma \sim |A|^2 \sim L^{\mu\nu}\, W_{\mu\nu}$$



W and F Structure Functions



$$d\sigma \sim |A|^2 \sim L^{\mu\nu}\, W_{\mu\nu}$$

$$L^{\mu\nu} = L^{\mu\nu}(\ell_1, \ell_2)$$

$$W^{\mu\nu} = W^{\mu\nu}(p, q)$$

Details:
There are also $W_{4,5,6}$
but we neglect these

$$W^{\mu\nu} = -g^{\mu\nu}\, W_1 + \frac{p^\mu p^\nu}{M^2}\, W_2 - \frac{i\,\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma}{2M^2}\, W_3 + \ldots$$

Convert to "Scaling" Structure Functions

$$W_1 \to F_1 \qquad W_2 \to \tfrac{M}{\nu}F_2 \qquad W_3 \to \tfrac{M}{\nu}F_3$$

$$\frac{d\sigma}{dx\, dy} = N\left[xy^2F_1 + (1 - y - \tfrac{Mxy}{2E_2})F_2 \pm y(1 - y/2)xF_3\right]$$

Use Helicity Basis

$$\frac{d\sigma}{dx\,dy} = N\left[xy^2 F_1 + (1 - y - \tfrac{Mxy}{2E_2})F_2 \pm y(1-y/2)xF_3\right]$$

Taking the limit $M \to 0$ for neutrino DIS

$$\frac{d\sigma^{\nu}}{dx\,dy} = N\left[(1-y)^2 F_+ + 2(1-y)F_0 + F_-\right]$$

For $\bar{\nu}$, $F_+ \Leftrightarrow F_-$

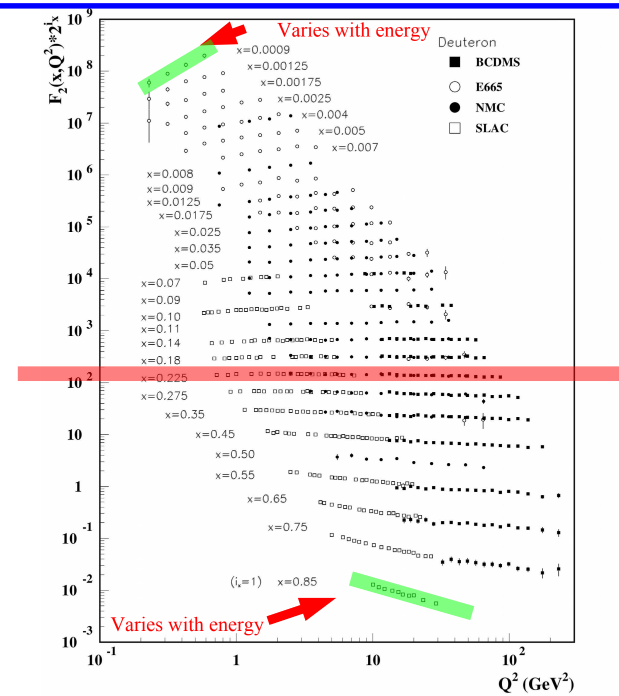
$$\begin{aligned} F_1 &= \tfrac{1}{2}(F_- + F_+) \\ F_2 &= x(F_- + F_+ + 2F_0) \\ F_3 &= (F_- - F_+) \end{aligned} \qquad \begin{aligned} F_+ &= F_1 - \tfrac{1}{2}F_3 \\ F_- &= F_1 + \tfrac{1}{2}F_3 \\ F_0 &= \tfrac{1}{2x}F_2 - F_1 \end{aligned}$$

I have not yet mentioned the parton model!!!

A Review of Target Mass Corrections.
Ingo Schienbein et al.
J.Phys.G35:053101,2008.

The Scaling of the Proton Structure Function

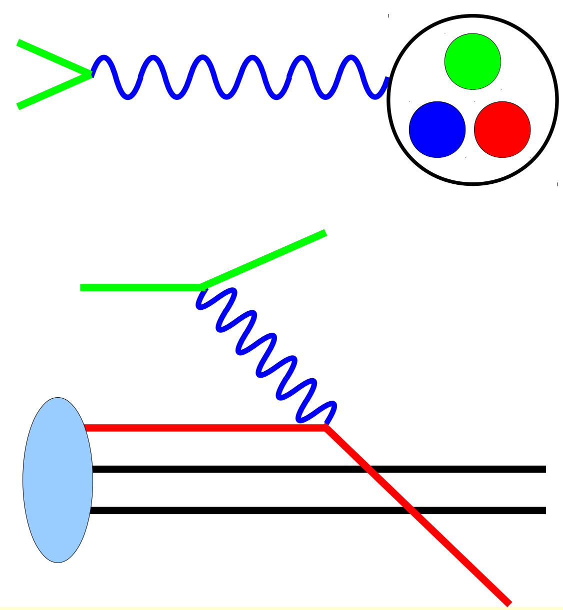
Data is (relatively) independent of energy



Scaling Violations observed at extreme x values

Parton Model

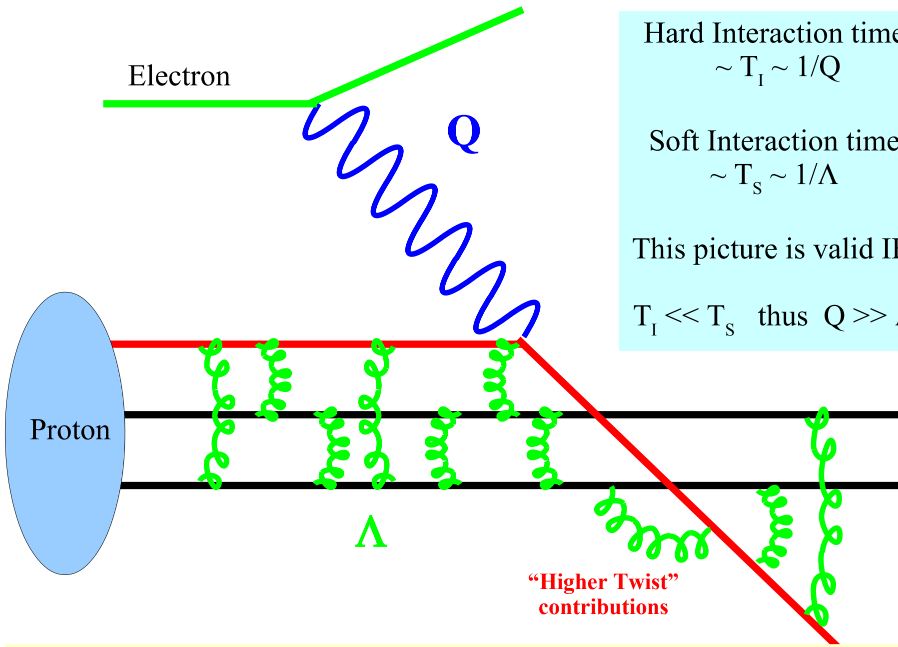
Proton as a bag of free Quarks



Fred's PDFs

$$f(x,Q) = u(x,Q) + d(x,Q) = 2\,\delta(x - \tfrac{1}{3}) + 1\,\delta(x - \tfrac{1}{3})$$

Quarks are not quite free



Hard Interaction time
$\sim T_I \sim 1/Q$

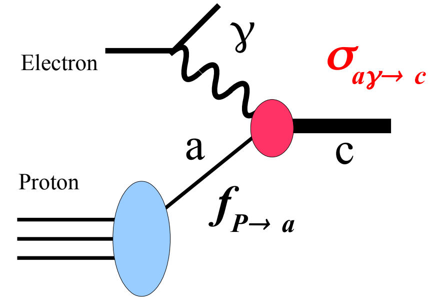
Soft Interaction time
$\sim T_S \sim 1/\Lambda$

This picture is valid IF:

$T_I << T_S$ thus $Q >> \Lambda$

Corrections to this picture (non-factorizable/ higher twist) terms are suppressed by powers of Λ/Q

The Parton Model and Factorization



Parton Distribution Functions

(PDFs) $f_{P\to a}$

are the key to calculations

involving hadrons!!!

$$\sigma_{P\gamma\to c} = f_{P\to a} \otimes \hat{\sigma}_{a\gamma\to c}$$

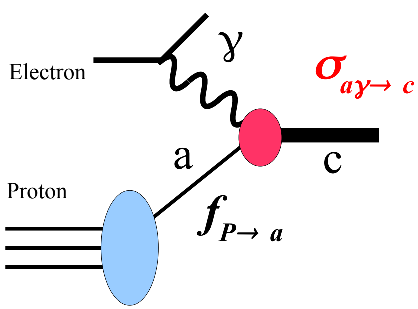
must extract from experiment

calculable from theoretical model

Corrections of order (Λ^2/Q^2)

Cross section is product of independent probabilities!!! *(Homework Assignment)*

The Parton Model and Factorization



Parton Distribution Functions

(PDFs) $f_{P\to a}$

are the key to calculations

involving hadrons!!!

$$\sigma_{P\gamma\to c} = f_{P\to a} \otimes \hat{\sigma}_{a\gamma\to c}$$

Scalar

$$f(x) = \sum q(x) + \bar{q}(x) + \phi(x) + \ldots = u(x) + d(x) + \ldots$$

Homework Problem: Convolutions

Part 1) Show these 3 definitions are equivalent; work out the limits of integration.

$$f\otimes g=\int_0^1\int_0^1 f(x)g(y)\delta(z-x*y)\,dx\,dy$$
$$f\otimes g=\int f(x)g\left(\frac{z}{x}\right)\frac{dx}{x}$$
$$f\otimes g=\int f\left(\frac{z}{y}\right)g(y)\frac{dy}{y}$$

Part 2) Show convolutions are the "natural" way to multiply probabilities.

If f represents the heads/tails probability distribution for a single coin flip, show that the distribution of 2 coins is $f \oplus f$ and 3 coins is: $f \oplus f \oplus f$.

$$f\oplus g=\int f(x)g(y)\delta(z-(x+y))\,dx\,dy$$
$$f(x)=\frac{1}{2}(\delta(1-x)+\delta(1+x))$$

Careful: convolutions involve + and *

***BONUS**: How many processes can you think of that don't factorize?*

Structure Function & PDF Correspondence at Leading Order

$$\frac{d\sigma^\nu}{dx\,dy} = N\left[(1-y)^2 F_+ + 2(1-y)F_0 + F_-\right]$$

Compute with Hadronic Tensor

$$\frac{d\sigma^\nu}{dx\,dy} = N\left[(1-y)^2(2\bar{q}) + 2(1-y)(\phi) + (2q)\right]$$

Compute in Parton Model

Scalar

$$\begin{aligned} F_+ &= 2\bar{q} & F_+ &= F_1 - \tfrac{1}{2}F_3 \\ F_- &= 2q & F_- &= F_1 + \tfrac{1}{2}F_3 \\ F_0 &= \phi & F_0 &= \tfrac{1}{2x}F_2 - F_1 \end{aligned}$$

Scalar

$$F_L = 0 = F_0$$

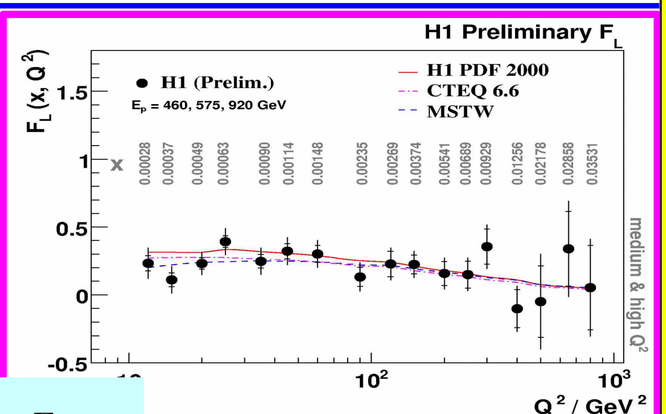
$$F_2 = 2xF_1$$

Callan-Gross Relation

$F_L = 2xF_0$

F_L

Why is F_L special ???



H1 Collaboration and ZEUS Collaboration (S. Glazov for the collaboration). Nucl.Phys.Proc.Suppl.191:16-24,2009.

$$F_L = 2xF_0 = F_2 - 2xF_1$$

$$F_L = 0 \implies F_2 = 2xF_1$$

Callan-Gross

$$F_L \sim \frac{m^2}{Q^2}\,q(x) + \alpha_S\left\{c_g \otimes g(x) + c_q \otimes q(x)\right\}$$

Masses are important

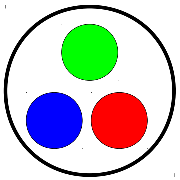
Higher orders are important

TOY

PDFs

Proton as a bag of free Quarks: Part 2

$$f(x,Q) = u(x,Q) + d(x,Q) = 2\ \delta(x - \tfrac{1}{3}) + 1\ \delta(x - \tfrac{1}{3})$$



$$u(x,Q) = 2\ \delta(x - \tfrac{1}{3})$$

$$d(x,Q) = 1\ \delta(x - \tfrac{1}{3})$$

Perfect Scaling PDFs
Q independent

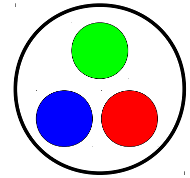
Quark Number Sum Rule

$$\langle q \rangle = \int_0^1 dx\, q(x) \qquad \langle u \rangle = 2 \quad \langle d \rangle = 1 \quad \langle s \rangle = 0$$

Quark Momentum Sum Rule

$$\langle x\, q \rangle = \int_0^1 dx\, x\, q(x) \qquad \langle x\, u \rangle = \tfrac{2}{3} \quad \langle x\, d \rangle = \tfrac{1}{3}$$

Problem #1: The proton does not add up???



$$F_+ = 2\bar{q}$$
$$F_- = 2q$$
$$F_L = \phi$$

$$q + \bar{q} = \frac{F_+ + F_-}{2}$$

Momentum Sum Rule

$$\sum_i \langle x\, q_i \rangle = \int_0^1 dx \sum x\, [q_i(x) + \bar{q}_i(x)] = 50\% \neq 100\%$$

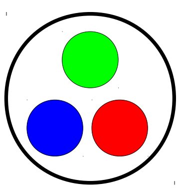
Substitute F

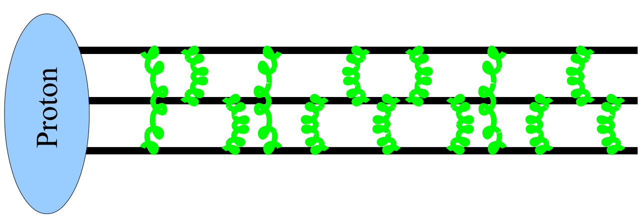
SOLUTION:

Gluons carry half the momentum,

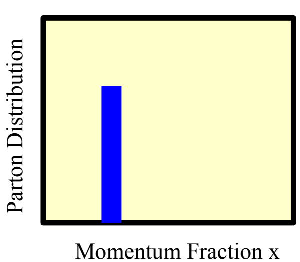
but don't couple to the photons

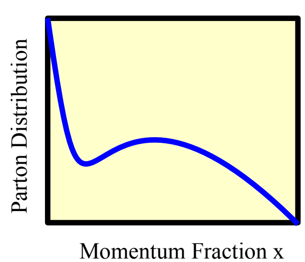
Gluons smear out PDF momentum





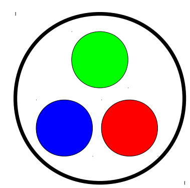
Gluons allow partons to exchange momentum fraction





α_s is large at low Q, so it is easy to emit soft gluons

Problem #2: Infinitely many quarks

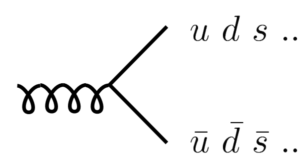


Reconsider the Quark Number Sum Rule

$$\langle u, d \rangle = \infty \qquad \langle q \rangle = \int_0^1 dx\, q(x)$$

Quark Number Sum Rule:
More Precisely

$$q(x) \sim 1/x^{1.5}$$



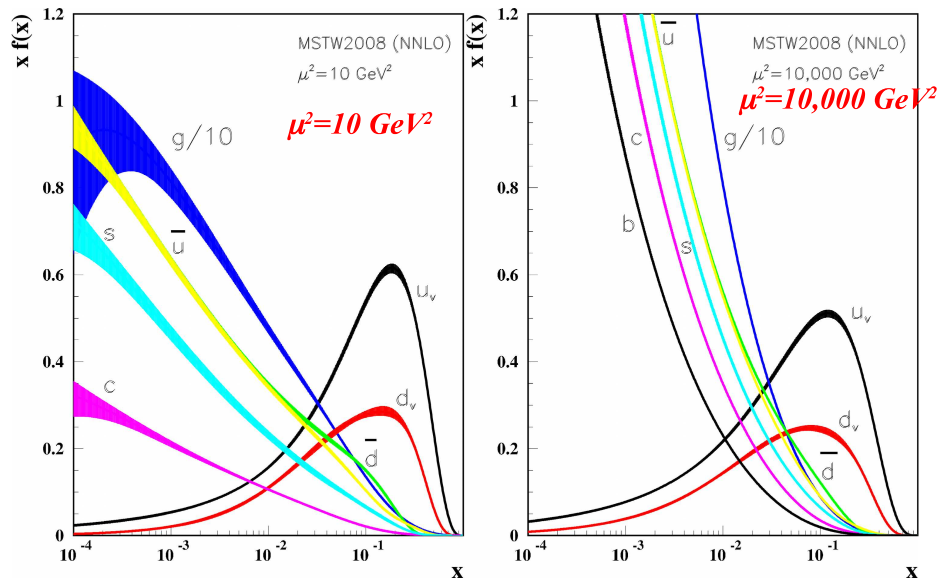
$$\langle u - \bar{u} \rangle = 2 \qquad \langle d - \bar{d} \rangle = 1 \qquad \langle s - \bar{s} \rangle = 0$$

SOLUTION: Infinite number of u quarks in proton, because they can be pair produced:
(We neglect saturation ....)

PDFs

cf., lectures by Pavel Nadolsky

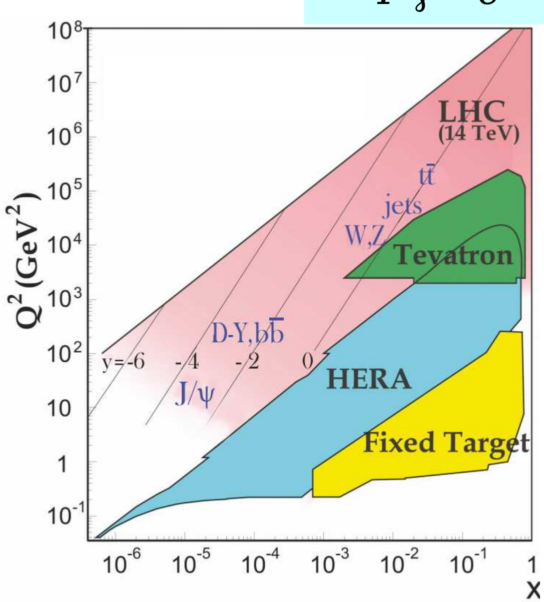
Sample PDFs: The rich structure of the proton



Scaling violations are essential feature of PDFs

Where do PDFs come from???? Universality!!!

$$\sigma_{P\gamma\to c} = f_{P\to a} \otimes \sigma_{a\gamma\to c}$$



Calculable from theoretical model

Must extract from experiment

Note we can combine different experiments.
FACTORIZATION!!!

HOMEWORK

Sum Rules
&
Structure Functions

$$
\begin{aligned}
F_2^{ep} &= \tfrac{4}{9}x\left[u+\bar{u}+c+\bar{c}\right] \\
&+ \tfrac{1}{9}x\left[d+\bar{d}+s+\bar{s}\right] \\
F_2^{en} &= \tfrac{4}{9}x\left[d+\bar{d}+c+\bar{c}\right] \\
&+ \tfrac{1}{9}x\left[u+\bar{u}+s+\bar{s}\right] \\
F_2^{\nu p} &= 2x\left[d+s+\bar{u}+\bar{c}\right] \\
F_2^{\nu n} &= 2x\left[u+s+\bar{d}+\bar{c}\right] \\
F_2^{\bar{\nu} p} &= 2x\left[u+c+\bar{d}+\bar{s}\right] \\
F_2^{\bar{\nu} n} &= 2x\left[d+c+\bar{u}+\bar{s}\right] \\
F_3^{\nu p} &= 2\left[d+s-\bar{u}-\bar{c}\right] \\
F_3^{\nu n} &= 2\left[u+s-\bar{d}-\bar{c}\right] \\
F_3^{\bar{\nu} p} &= 2\left[u+c-\bar{d}-\bar{s}\right] \\
F_3^{\bar{\nu} n} &= 2\left[d+c-\bar{u}-\bar{s}\right]
\end{aligned}
$$

Verify:

i.e., Check for typos ...

We use these different observables to dis-entangle the flavor structure of the PDFs

See talks by Stephen Parke & Jonathan Paley (Neutrinos) & Pavel Nadolsky (PDFs)

In the limit

$\theta_{Cabibbo} = 0$

$m_c = 0$

Verify:

i.e., Check for typos ...

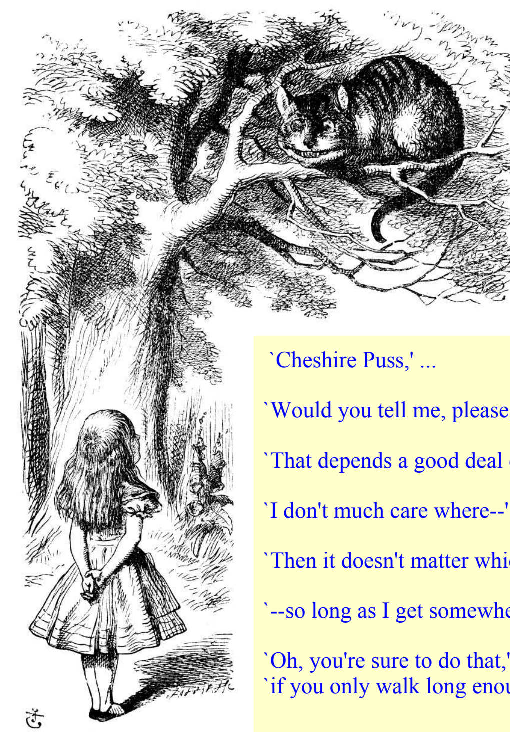
Adler (1966)		$\int_0^1 \frac{dx}{2x}\left[F_2^{\nu n} - F_2^{\nu p}\right] = 1$
Bjorken (1967)		$\int_0^1 \frac{dx}{2x}\left[F_2^{\bar{\nu} p} - F_2^{\nu p}\right] = 1$
Gross Llewellyn-Smith (1969)		$\int_0^1 dx\left[F_3^{\nu p} + F_3^{\bar{\nu} p}\right] = 6$
Gottfried (1967)	if $\bar{u} = \bar{d}$	$\int_0^1 dx\left[F_2^{ep} - F_2^{en}\right] = \frac{1}{3}$
Homework (19??)		$\frac{5}{18}F_2^{\nu N} - F_2^{eN} = ?$

Before the parton model was invented, these relations were observed. Can you understand them in the context of the parton model?

This one has been particularly important/controversial

Evolution

What does the proton look like???



The answer is dependent upon the question

\`Cheshire Puss,' ...

\`Would you tell me, please, which way I ought to go from here?'

\`That depends a good deal on where you want to get to,' said the Cat.

\`I don't much care where--' said Alice.

\`Then it doesn't matter which way you go,' said the Cat.

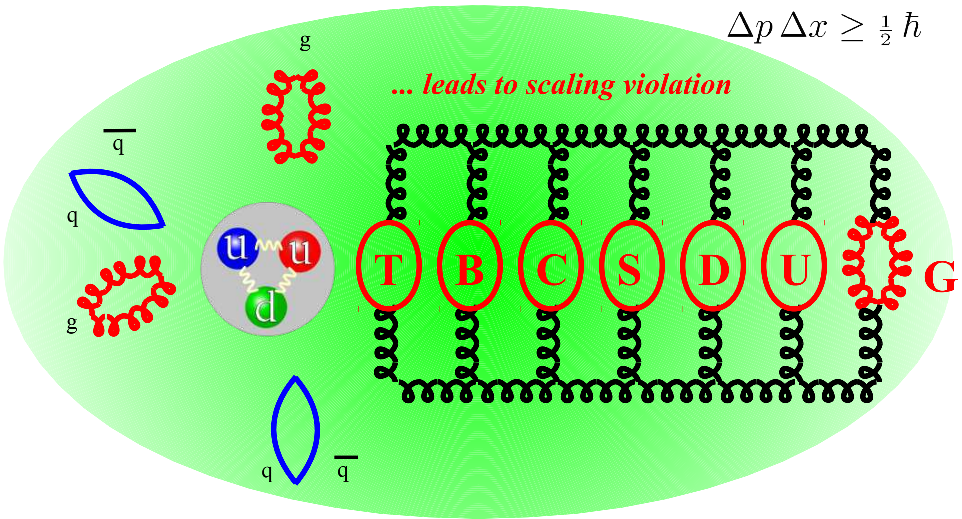
\`--so long as I get somewhere,' Alice added as an explanation.

\`Oh, you're sure to do that,' said the Cat,
\`if you only walk long enough.'

Evolution: What you see depends upon what you ask

Proton is a complex object

$$\Delta E\,\Delta t \geq \tfrac{1}{2}\hbar$$
$$\Delta p\,\Delta x \geq \tfrac{1}{2}\hbar$$



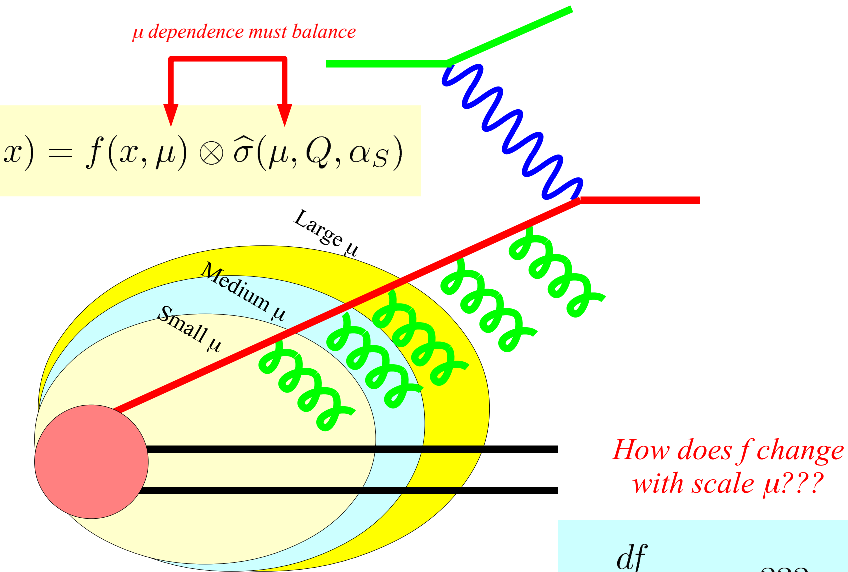
$\Lambda_{QCD} \sim 200\,\text{MeV}$

m_t	m_b	m_c	m_s	m_d	m_u	m_g
175	4.5	1.3	0.3	0.00?	0.00?	0

Evolution of the PDFs

μ dependence must balance

$$\sigma(Q,x) = f(x,\mu) \otimes \hat{\sigma}(\mu, Q, \alpha_S)$$



How does f change with scale μ???

$$\frac{df}{d\,\ln[\mu]} = ???$$

Homework: Mellin Transform

$$\tilde{f}(n) = \int_0^1 dx\, x^{n-1}\, f(x)$$

$$\sigma = f \otimes \omega$$

$$f(x) = \frac{1}{2\pi i}\int_C dn\, x^{-n}\, \tilde{f}(n)$$

$$\tilde{\sigma} = \tilde{f}\,\tilde{\omega}$$

C is parallel to the imaginary axis, and to the right of all singularities

1) Take the Mellin transform of $f(x) = \sum_{m=1}^{\infty} a_m x^m$, and verify the inverse transform of $\tilde{f}$ regenerates $f(x)$

2) Take the Mellin transform of $\sigma = f \otimes \omega$ to demonstrate that the Mellin transform separates a convolution yields $\tilde{\sigma} = \tilde{f}\,\tilde{\omega}$.

A useful reference:
Courant, Richard and Hilbert, David. Methods of Mathematical Physics, Vol. 1. New York: Wiley, 1989. 561 p.

Renormalization Group Equation

ω or $\hat{\sigma}$
Not physical!
Poor notation

Parton Model
$$\sigma = f \otimes \omega$$

Renormalization Group Equation
$$\frac{d\sigma}{d\mu} = 0 = \frac{d\tilde{f}}{d\mu}\,\tilde{\omega} + \tilde{f}\,\frac{d\tilde{\omega}}{d\mu}$$

← Take Mellin Transform

Separation of variables
$$\frac{1}{\tilde{f}}\,\frac{d\tilde{f}}{d\ln[\mu]} = -\gamma = -\frac{1}{\tilde{\omega}}\,\frac{d\tilde{\omega}}{d\ln[\mu]}$$

DGLAP
$$\frac{d\tilde{f}}{d\ln[\mu]} = -\tilde{f}\,\gamma \qquad \frac{df}{d\ln[\mu]} = P \otimes f$$

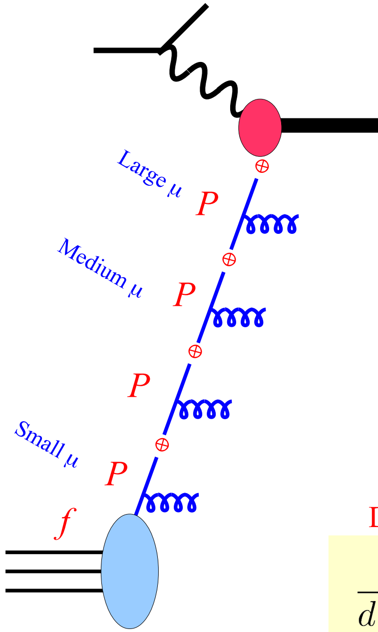
← DGLAP Equation

$$\tilde{f} \sim \mu^{-\gamma}$$

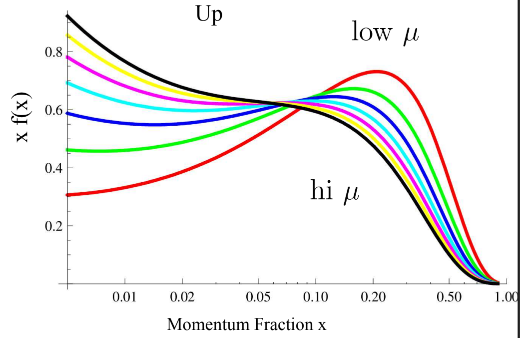
← Anomalous Dimension

If "f" scaled, γ would vanish

It is the dimension of the mass scaling

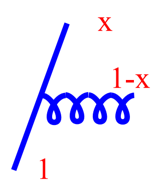


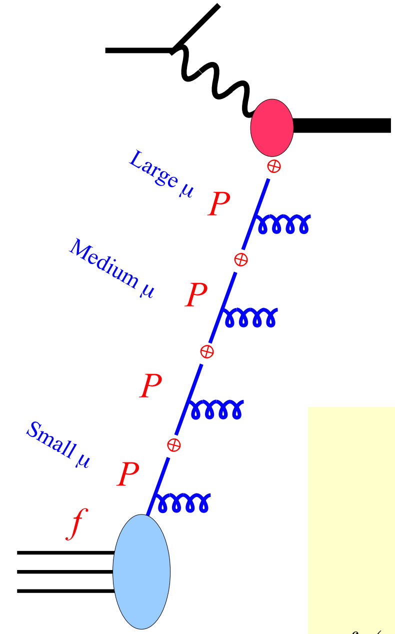
Evolution (generally) shifts partons from hi-x to low-x

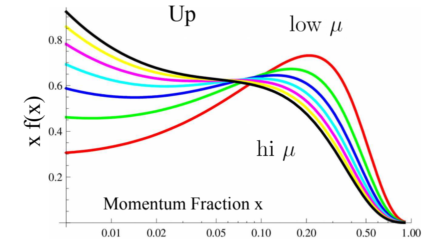


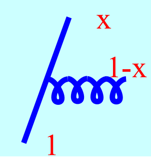
DGLAP Equation

$$\frac{d\tilde{f}}{d\ln[\mu]} = P \otimes f$$









$$P_{qq}^{(1)}(x) = C_F \left[\frac{1+x^2}{1-x}\right]_+$$

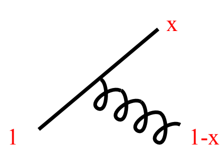
$$\frac{df}{d\ \ln[\mu]} = P \otimes f \ \simeq\ \frac{\alpha_S}{2\pi} P^{(1)} \otimes f$$

$$P \simeq \delta + \tfrac{\alpha_s}{2\pi} P^{(1)} + \left(\tfrac{\alpha_s}{2\pi}\right)^2 P^{(2)} + \ldots$$

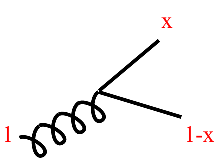
$$f_a(x,\mu_1) \sim f_a(x,\mu_0) + \frac{\alpha_S}{2\pi} P_{ab}^{(1)} \otimes f_b \ln\left(\frac{\mu_1^2}{\mu_2^0}\right)$$

Read backwards

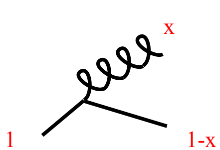
Note singularities



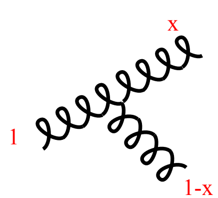
$$P_{qq}^{(1)}(x) = C_F \left[\frac{1+x^2}{1-x}\right]_+$$



$$P_{qg}^{(1)}(x) = T_F \left[(1-x)^2 + x^2\right]$$



$$P_{gq}^{(1)}(x) = C_F \left[\frac{(1-x)^2+1}{x}\right]$$



$$P_{gg}^{(1)}(x) = 2C_A \left[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x)\right] + \left[\frac{11}{6}C_A - \frac{2}{3}T_F N_F\right] \ \delta(1-x)$$

Definition of the Plus prescription:

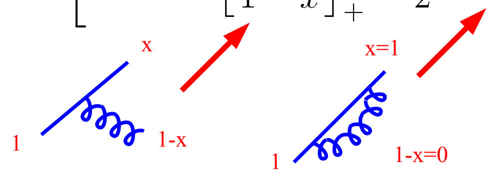
$$\int_0^1 dx\ \frac{f(x)}{(1-x)_+} = \int_0^1 dx\ \frac{f(x)-f(1)}{(1-x)}$$

1) Compute: $\int_a^1 dx\ \frac{f(x)}{(1-x)_+} = ???$

Homework: Part 2 P(q←q)

2) Verify:

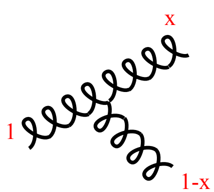
$$P_{qq}^{(1)}(x) = C_F \left[\frac{1+x^2}{1-x}\right]_+ \equiv C_F \left[(1+x^2)\left[\frac{1}{1-x}\right]_+ + \frac{3}{2}\,\delta(1-x)\right]$$

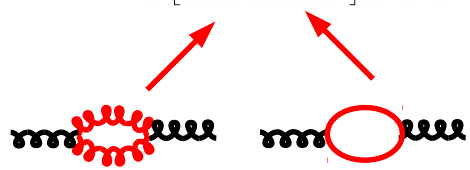


Observe

$$P_{gg}^{(1)}(x) = 2C_F \left[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x)\right]$$

$$+ \left[\frac{11}{6} C_A - \frac{2}{3} T_F N_F\right] \; \delta(1-x)$$





HOMEWORK: Part 3: Symmetries & Limits

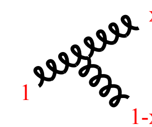
Verify the following relation among the regular parts (from the real graphs)

For the regular part show:

$$P_{gq}^{(1)}(x) = P_{qq}^{(1)}(1-x)$$

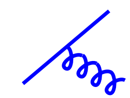
For the regular part show:

$$P_{gg}^{(1)}(x) = P_{gg}^{(1)}(1-x)$$

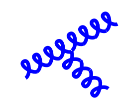


Verify, in the soft limit:

$$P_{qq}^{(1)}(x) \underset{x\to 1}{\longrightarrow} 2C_F \frac{1}{(1-x)_+}$$



$$P_{gg}^{(1)}(x) \underset{x\to 1}{\longrightarrow} 2C_F \frac{1}{(1-x)_+}$$

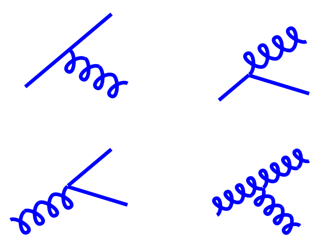


HOMEWORK: Part 4: Conservation Rules

Verify conservation of momentum fraction

$$\int_0^1 dx\, x\; [P_{qq}(x) + P_{gq}(x)] = 0$$

$$\int_0^1 dx\, x\; [P_{qg}(x) + P_{gg}(x)] = 0$$



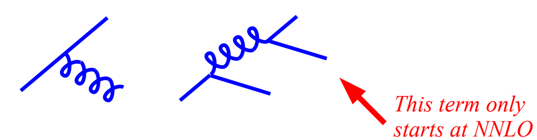
Verify conservation of fermion number

$$\int_0^1 dx \quad [P_{qq}(x) - P_{q\bar{q}}(x)] = 0$$

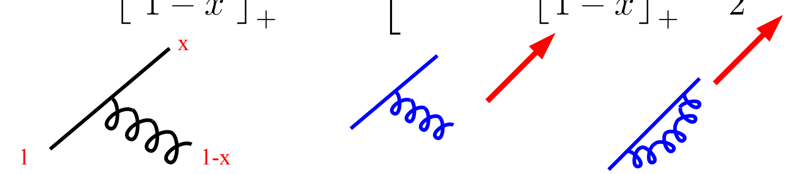
Homework: Part 5: Using the Real to guess the Virtual

Use conservation of fermion number to compute the delta function term in P(q←q)

$$\int_0^1 dx \quad [P_{qq}(x) - P_{q\bar{q}}(x)] = 0$$

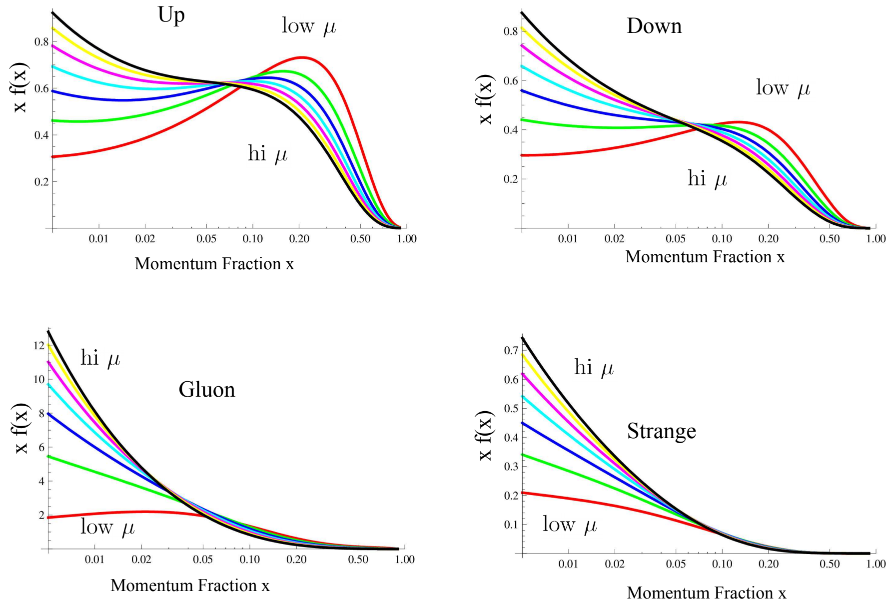


$$P_{qq}^{(1)}(x) = C_F \left[\frac{1+x^2}{1-x}\right]_+ \equiv C_F \left[(1+x^2)\left[\frac{1}{1-x}\right]_+ + \frac{3}{2}\,\delta(1-x)\right]$$

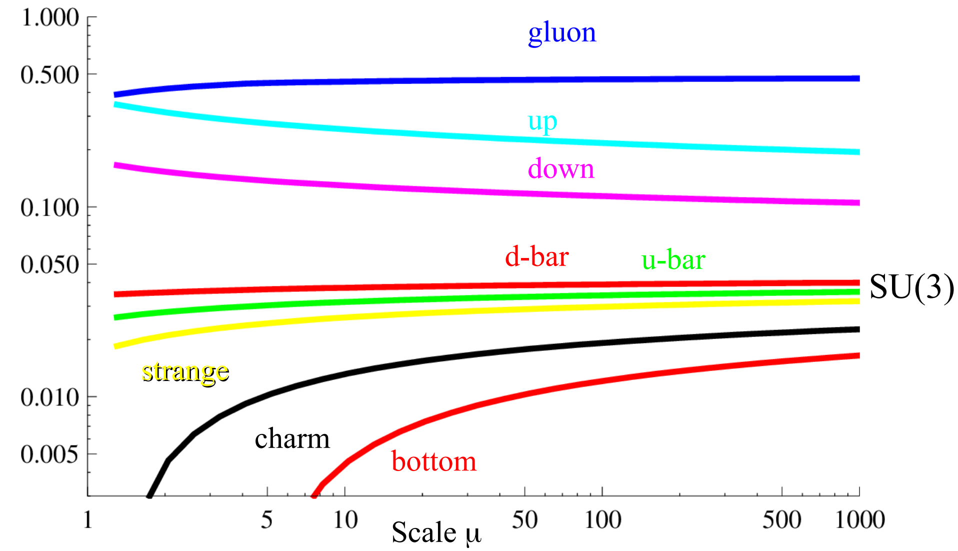


Powerful tool: Since we know real and virtual must balance, we can use to our advantage!!!

Evolution of the PDFs



PDF Momentum Fractions vs. scale μ



Scaling violations are essential feature of PDFs

End of lecture 2: Recap

Rutherford Scattering ⇒ Deeply Inelastic Scattering (DIS)

- Works for protons as well as nuclei

Compute Lepton-Hadron Scattering 2 ways

- Use Leptonic/Hadronic Tensors to extract Structure Functions
- Use Parton Model; relate PDFs to F_{123}

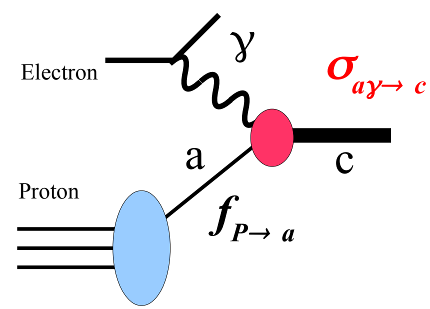
Parton Model Factorizes Problem:

- PDFs are independent of process
- Thus, we can combine different experiments. ESSENTIAL!!!

PDFs are not truly scale invariant; they evolve

- We use evolution to "resum" an important set of graphs

The Parton Model and Factorization



Parton Distribution Functions (PDFs) $f_{P\to a}$ are the key to calculations involving hadrons!!!

$$\sigma_{P\gamma\to c} = f_{P\to a} \otimes \hat{\sigma}_{a\gamma\to c}$$

Corrections of order (Λ^2/Q^2)

must extract from experiment

calculable from theoretical model

Cross section is product of independent probabilities!!! *(Homework Assignment)*

END OF LECTURE 2