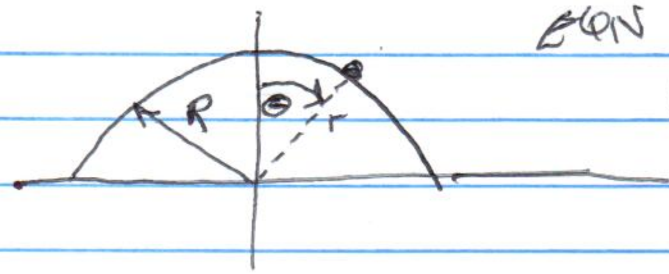
LAGRANGIAN EXAMPLE w/ EQN OF CONSTRAINT

PARTICLE SLIDES OFF DOME

- DOME IS HEMISPHERE w/ RADIUS
- PARTICLE STARTS @ TOP & FROM REST.

Q: FIND FORCE OF CONSTRAINT

FIND ANGLE θ_0 WHEN PARTICLE LEAVES DOME.



EQN OF CONSTRAINT:

$$g(r,\theta) = r - R = 0$$

USE POLAR COORDINATES TO SKIP WRITING THINGS (POSITION & TIME DERIVATIVE OF POSITION) IN TERMS OF X & Y.

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$U = mgr\cos\theta \qquad (U=0 \text{ @ DOME BASE})$$

$$L = T - U$$

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta$$

$$g(r,\theta) = r - R = 0$$

NOTICE THAT WE DO NOT USE THE INFO THAT $r = \text{const} = R$ ~~with~~ JUST YET WHEN WRITING L. WE WILL HANDLE THE FIXED R BEHAVIOR WHEN WE ADD THE $\sum \lambda \, \partial g / \partial q$ TERM TO THE EULER - LAGRANGE EQNS. IF WE DIDN'T CARE ABOUT THE FORCE OF CONSTRAINT, WE WOULD IMMEDIATELY USE THE FACT THAT $\dot{r} = 0$.

NOW WE SOLVE

$$\partial L / \partial r - \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} + \lambda \, \partial g / \partial r = 0 \qquad (1)$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt} \, \partial L / \partial \dot{\theta} + \lambda \, \partial g / \partial \theta = 0 \qquad (2)$$

FROM THE EQN OF CONSTRAINT $g(r, \theta) = 0$

$$\partial g / \partial r = 1 \quad \& \quad \partial g / \partial \theta = 0$$

(1) & (2) BECOME

$$mr\dot{\theta}^2 - mg\cos\theta - m\ddot{r} + \lambda = 0 \qquad (3)$$

$$mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} = 0 \qquad (4)$$

NOW WE CAN INVOKE EQN OF CONSTRAINT

$g(r,\theta) = r - R = 0 \Rightarrow r = R$ & SO $\dot{r} = 0$

HENCE, EQ (4) BECOMES

$$mgR\sin\theta - mR^2\ddot{\theta} = 0$$

$$\Rightarrow \boxed{\ddot{\theta} = g/R \cdot \sin\theta} \quad \text{EQN OF MOTION.}$$

INTEGRATE TO FIND $\dot{\theta}^2$. USE TRICK SEEN BEFORE:

$$\ddot{\theta} = \frac{d}{dt}\,d\theta/dt = d\dot{\theta}/dt = \frac{d\dot{\theta}}{d\theta}\cdot\frac{d\theta}{dt} = \dot{\theta}\,d\dot{\theta}/d\theta$$

SO,

$$\dot{\theta}\,d\dot{\theta}/d\theta = g/R \cdot \sin\theta$$

$$\int \dot{\theta}\,d\dot{\theta} = g/R \int \sin\theta\,d\theta$$

$$\dot{\theta}^2/2 = -g/R\cos\theta + g/R, \quad \dot{\theta} = 0 \ @ \ \theta = 0$$

BACK TO (3) TO FIND λ.

$$mR\dot{\theta}^2/2 - \tfrac{1}{2}mg\cos\theta + \tfrac{1}{2}\lambda = 0 \qquad \text{SINCE } r = R$$

$$\lambda/2 = \tfrac{1}{2}mg\cos\theta - mR\left(\frac{-g}{R}\cos\theta + g/R\right)$$

$$= \tfrac{3}{2}mg\cos\theta - mg$$

$$\lambda = 3mg\cos\theta - 2mg$$

$$\boxed{\lambda = mg(3\cos\theta - 2)}$$

RECALL THAT SINCE $\partial g/\partial r = 1$,

$$\text{FORCE OF CONSTRAINT} = \lambda \, \partial g/r$$

$$\boxed{\text{FORCE OF CONSTRAINT} = mg(3\cos\theta - 2)}$$

NOTE THAT WHEN PARTICLE IS A TOP, $\theta = 0$ & FORCE OF CONSTRAINT $= mg$. NO SURPRISES HERE.

TO FIND WHEN PARTICLE LEAVES DOME AND GOES AIRBORNE, FIND WHEN FORCE OF CONSTRAINT GOES TO ZERO.

$$FOC = 0 = mg(3\cos\theta_0 - 2)$$

$$\boxed{\theta_0 = \cos^{-1}(2/3)}$$