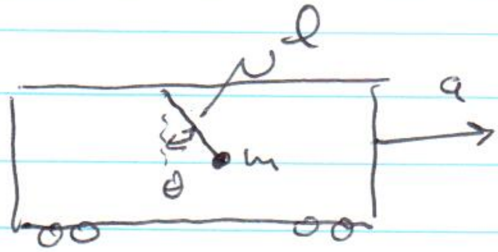
LAGRANGIAN EXAMPLE

ACCELERATING TRAIN PROBLEM



CONST ACCELERATION a

Q: WHAT IS FREQUENCY ω OF SMALL OSC?

A: TRY TO EXTRACT THIS FROM EQN OF MOTION

$L = T - U$

STEP 1: FIND T IN INERTIAL REF FRAME

INITIAL CONDITIONS $x(t=0) = 0$

$\dot{x}(t=0) = v_0$

$$x = \underbrace{v_0 t + \tfrac{1}{2} a t^2}_{\text{TRAIN MOTION}} + \ell \sin\theta$$

$y = -\ell \cos\theta$ $\quad +\hat{y}$ POINTS UPWARD

$\dot{x} = v_0 + at + \ell\dot{\theta}\cos\theta$

$\dot{y} = +\ell\dot{\theta}\sin\theta$

$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$ & $U = -mg\ell\cos\theta$

$$T = \frac{m}{2}(v_0 + at + l\dot\theta\cos\theta)^2 + \frac{m}{2}l^2\dot\theta^2\sin^2\theta$$

$$= \frac{m}{2}(v_0 + at)^2 + m(v_0+at)l\dot\theta\cos\theta + \frac{ml^2}{2}\dot\theta^2\cos^2\theta$$

$$+ \frac{m}{2}l^2\dot\theta^2\sin^2\theta$$

$$T = \frac{m}{2}(V_0^2 + 2v_0at + a^2t^2) + m(v_0+at)l\dot\theta\cos\theta$$

$$+ \frac{m}{2}l^2\dot\theta^2$$

$$U = -mgl\cos\theta$$

$$\frac{\partial L}{\partial\dot\theta} = m(v_0+at)l\cos\theta + ml^2\dot\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = \begin{matrix}-mv_0 l\dot\theta\sin\theta\\ +mal\cos\theta\end{matrix} - matl\dot\theta\sin\theta + ml^2\ddot\theta$$

$$\frac{\partial L}{\partial\theta} = -m(v_0+at)l\dot\theta\sin\theta - mgl\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = \frac{\partial L}{\partial\theta}$$

$$mal\cos\theta + ml^2\ddot\theta = -mgl\sin\theta$$

$$\boxed{\ddot\theta = -\frac{g}{l}\sin\theta - \frac{a}{l}\cos\theta}$$

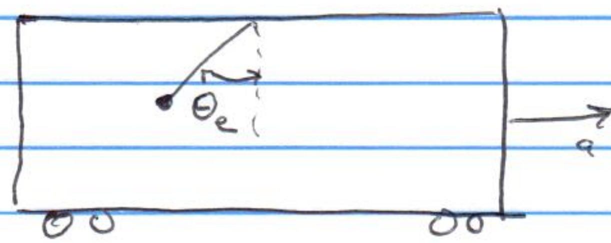
FIND EQUILIBRIUM $\measuredangle$ θ_e

$$\ddot{\theta} = -\frac{g}{l}\sin\theta_e - \frac{a}{l}\cos\theta_e$$

$$g/a \cdot \sin\theta_e = -\cos\theta_e$$

$$\cancel{g/a \sin(-\theta_e) = \cos(-\theta_e)} \qquad \cos\ \cancel{(\text{EVEN})}$$

$$\Rightarrow \boxed{\tan\theta_e = -a/g}$$



INTERESTED IN <u>SMALL</u> OSCILLATIONS ABOUT θ_e. WRITE

$$\theta = \theta_e + \eta \quad , \eta << 1$$

$$\ddot{\theta} = \ddot{\eta} = -\frac{g}{l}\sin(\theta_e + \eta) - \frac{a}{l}\cos(\theta_e + \eta)$$

EXPAND SIN & COS

$$\sin(A+B) = \sin A\cos B + \sin B\cos A$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\ddot{\eta} = -\frac{g}{\ell}\left(\sin\theta_e \cos\eta + \cos\theta_e \sin\eta\right)$$

$$- \frac{a}{\ell}\left(\cos\theta_e \cos\eta - \sin\theta_e \sin\eta\right)$$

$$\simeq -\frac{g}{\ell}\left(\sin\theta_e + \eta \cos\theta_e\right)$$

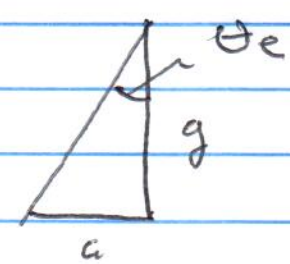
$$- \frac{a}{\ell}\left(\cos\theta_e - \eta \sin\theta_e\right)$$

$$\simeq -\frac{1}{\ell}\Big[\underbrace{\left(g \sin\theta_e + a \cos\theta_e\right)}_{\equiv 0 \text{ SINCE } \tan\theta_e = -a/g} + \eta\left(g\cos\theta_e - a\sin\theta_e\right)\Big]$$

So,

$$\ddot{\eta} = -\frac{1}{\ell}\left(g\cos\theta_e - a\sin\theta_e\right)\eta$$

RECALL $\tan\theta_e = -a/g$



$$\Rightarrow \cos\theta_e = g/\sqrt{g^2+a^2}$$

$$\sin\theta_e = a/\sqrt{g^2+a^2}$$ USING $|\theta_e|$

$$\ddot{\eta} = -\frac{1}{\ell}\left[g^2/\sqrt{g^2+a^2} + a \sin(-\theta_e) \right] \eta$$

$$= -\frac{1}{\ell}\left[g^2/\sqrt{g^2+a^2} + a^2/\sqrt{g^3+a^2} \right] \eta$$

$$\ddot{\eta} + \frac{\sqrt{g^3+a^2}}{\ell}\,\eta = 0$$

$$\Rightarrow \boxed{\omega^2 = \frac{\sqrt{g^2+a^2}}{\ell}}$$

NOTE THAT WHEN $a=0$, WE RECOVER THE RESULT FOR A STD SIMPLE PENDULUM.