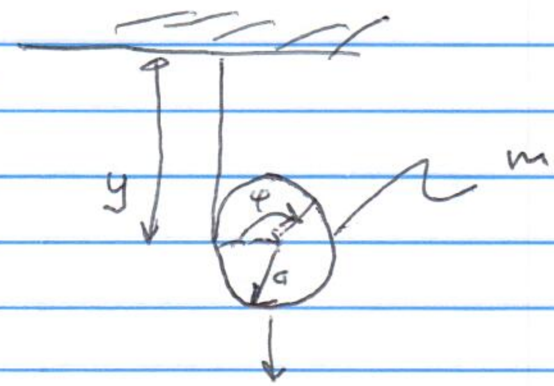
LAGRANGIAN EXAMPLE w/ CONSTRAINT

EXAMPLE OF HOW TO USE THE EULER-LAGRANGE EQN w/ CONSTRAINTS.

$$\frac{\partial L}{\partial \theta_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} + \sum \lambda_j \, \partial g_j / \partial \theta_i = 0$$

A YO-YO HAS A STRING AROUND IT ATTACHED TO A FIXED SUPPORT. YO-YO IS DROPPED AND STRING UNWINDS. FIND EQUATIONS OF MOTION AND FORCES OF CONSTRAINT.

$\cancel{T = \frac{1}{2} M}$



$$T = \frac{m}{2}\dot{y}^2 + \frac{1}{2} I \dot{\varphi}^2 \qquad I = \frac{1}{2} m a^2 \text{ FOR DISC}$$

$$= \frac{m}{2}\dot{y}^2 + \frac{1}{4} \cdot m a^2 \dot{\varphi}^2$$

$$U = -mgy \qquad (+\hat{y} \text{ DIRECTION DOWNWARD})$$

$$L = T - U = \frac{1}{2} m\dot{y}^2 + \frac{1}{4} m a^2 \dot{\varphi}^2 + mgy \qquad (0)$$

YO - YO II

EQN OF CONSTRAINT

$$g(y, \varphi) = y - a\varphi = 0 \qquad (0.5)$$

Y & φ ARE GENERALIZED COORDINATES
AWAY WE GO:

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} + \lambda \frac{\partial g}{\partial y} = 0 \qquad (1)$$

$$\partial L/\partial \varphi - \frac{d}{dt}\,\partial L/\partial \dot{\varphi} + \lambda\,\partial g/\partial \varphi = 0 \qquad (2)$$

FROM EQN (0) ABOVE

$$\partial L/\partial y = mg$$

$$d/dt\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{d}{dt}\left[m\dot{y}\right] = m\ddot{y}$$

$$\lambda\,\partial g/\partial y = \lambda$$

SO, (1) BECOMES

$$mg - m\ddot{y} + \lambda = 0 \qquad (3)$$

FOR EQN (2) WE HAVE

$$\partial L/\partial \varphi = 0$$

$$d/dt(\partial L/\partial \dot{\varphi}) = \frac{d}{dt}\left(\frac{1}{2}ma^2\dot{\varphi}\right) = \frac{1}{2}ma^2\ddot{\varphi}$$

$$\partial g/\partial \varphi = -a$$

YO-YO III

EQN (2) BECOMES

$$-\frac{1}{2} m a^2 \ddot{\varphi} - \lambda a = 0 \qquad (4)$$

SO, WE HAVE 3 KEY EQUATIONS,
2 FROM EULER-LAGRANGE AND 1
EQN OF CONSTRAINT

SOLVE (4) BY DIFFERENTIATING EQN (0.5):

$$\ddot{\varphi} = \ddot{y}/a \qquad (5)$$

SUB INTO (4)

$$-\frac{1}{2} m a \ddot{y} = \lambda a$$

$$\lambda = -\frac{1}{2} m \ddot{y}$$

$$\ddot{y} = -2\lambda/m$$

SUB LATTER INTO (3)

$$mg - m\ddot{y} + \lambda = 0$$

$$mg + 2\lambda + \lambda = 0$$

$$\boxed{\lambda = -\frac{1}{3} mg}$$

AND

SINCE $mg - m\ddot{y} + \lambda = 0$

$$\boxed{\ddot{y} = \frac{2}{3} g}$$ ACCELERATION OF YO-YO CM

YO-YO IV

THE ANGULAR ACCELERATION $\ddot{\varphi}$ OF YO-YO IS GIVEN BY (5)

$$\ddot{\varphi} = \ddot{y}/a$$

$$\boxed{\ddot{\varphi} = \frac{2}{3}\, g/a}$$

GENERALIZED FORCES:

$$\boxed{\begin{aligned} Q_y &= \lambda\, \partial g/\partial y = \lambda = -\tfrac{1}{3} mg \\ Q_\varphi &= \lambda\, \partial g/\partial \varphi = -\lambda a = \tfrac{1}{3} mga \end{aligned}}$$

Q_y IS TENSION IN STRING PULLING UP ON YO-YO. OTHERWISE IT WOULD FALL w/ ACCELERATION g.

Q_φ IS TORQUE ON YO-YO CAUSING IT TO ROTATE ABOUT ITS COM.