PHYS 4392
Fall 2014
TE Coan
Due: 10 Oct '14 6:00 pm

Homework 6

1. From lecture we know the Legendre polynomials are "complete" functions over the interval $0 \leq \theta \leq \pi$, or equivalently, $-1 \leq x \leq 1$, where $x \equiv \cos\theta$. This means we can express a function $f(\theta)$ as a linear superposition of Legendre polynomials:

$$\begin{aligned} f(\theta) &= \sum_{l=0}^{\infty} a_l P_l(\cos\theta) \\ &= \sum_{l=0}^{\infty} a_l P_l(x) \end{aligned}$$

Use this feature of Legendre polynomials to express the function $f(\theta) = \cos^4\theta + \sin^2\theta + \sin\theta$ as a sum of Legendre polynomials. (For brevity of expression, you can refer to a Legendre polynomial by its order. E.g, write $P_6(x)$ if you are talking about the 6th order Legendre polynomial. There is no need to explicitly write out your series solution using using powers of $\cos\theta$.) You will need the normalization condition for Legendre polynomials explained in lecture and in Griffiths:

$$\begin{aligned} \int_{-1}^{1} P_l(x) P_{l'}(x)\, dx &= \int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta\, d\theta \\ &= \begin{cases} 0, & \text{if } l' \neq l, \\ \dfrac{2}{2l+1} & \text{if } l' = l. \end{cases} \end{aligned}$$

To reduce tedium, use the online integrator from Wolfram: `www.integrals.wolfram.com`. A Legendre polynomial is entered into the integrator box by using the notation LegendreP[4,x], if you want to enter, say, the 4th order Legendre polynomial. See the Wolfram help utility or me if you get confused. Finally, if your answer cannot be expressed in closed form, just write down a few terms of a series expansion. Do not bother to write the Legendre polynomials as an explicit function of $\cos\theta$. Instead, use the notation $P_l(x)$. See me if you are confused.

2. Express $f(\theta) = \cos 3\theta$ in terms of Legendre polynomials. A clever way to do this is to use De Moivre's theorem and the binomial expansion. Recall that $e^{i\theta} = \cos\theta + i\sin\theta$ or $\cos\theta = \Re(e^{i\theta})$. Hence, $\cos 3\theta = \Re(e^{3i\theta}) = \Re((\cos\theta + i\sin\theta)^3)$. Expand the cubic expression by using the binomial theorem. You should then be able to determine by inspection the relevant Legendre polynomials. You still need to compute the appropriate coefficients.

3. We will use the results from problem 2. Suppose the potential V on the surface of a spherical shell of radius R is given by $V(R, \theta) = k \cos 3\theta$, where k is some constant.

(a.) Determine the potential $V(r, \theta)$ inside and outside the spherical shell. There are no charges either inside or outside the shell.

(b.) Determine the surface charge density $\sigma(\theta)$ on the shell.

4a. Determine the monopole moment and dipole moment of the charge distribution shown in the figure. The separation distance between the charges is a.

(b.) Determine the approximate potential $V(r, \theta)$ (in spherical coordinates) for large r. Be sure to indicate **explicitly** what you mean by "large" r.