#1. $f(\theta) = \cos^4\theta + \sin^2\theta + \sin\theta$

$$f(x) = x^4 + (1-x^2) + \sqrt{1-x^2}, \quad x = \cos\theta$$

$$f(x) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(x)$$

NOTE THAT $f(x)$ IS <u>EVEN</u> $(f(x) = f(-x))$
THEREFORE $a_\ell = 0$ FOR ODD ℓ.

USING ORTHOGONALITY, WE HAVE

$$\int_{-1}^{1} f(x) P_\ell(x)\,dx = \frac{2}{2\ell+1} a_\ell$$

$$a_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} f(x) P_\ell(x)\,dx$$

USE WOLFRAM INTEGRATOR

$$a_0 = \tfrac{1}{2}(3.304) = 1.65$$
$$a_2 = \tfrac{5}{2}(-0.234) = -0.585$$
$$a_4 = \tfrac{9}{2}(0.026) = 0.117$$
$$a_6 = \tfrac{13}{2}(-0.008) = -0.052$$

$$\boxed{f(x) \simeq 1.65\,P_0(x) - 0.585\,P_2(x) + 0.117\,P_4(x) - 0.052\,P_6(x)}$$

SEE PLOT NEXT PAGE.

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Plot[{f[x], lpseries[f, 6, x]}, {x, -1, 1}, PlotStyle → Directive[Thick]]
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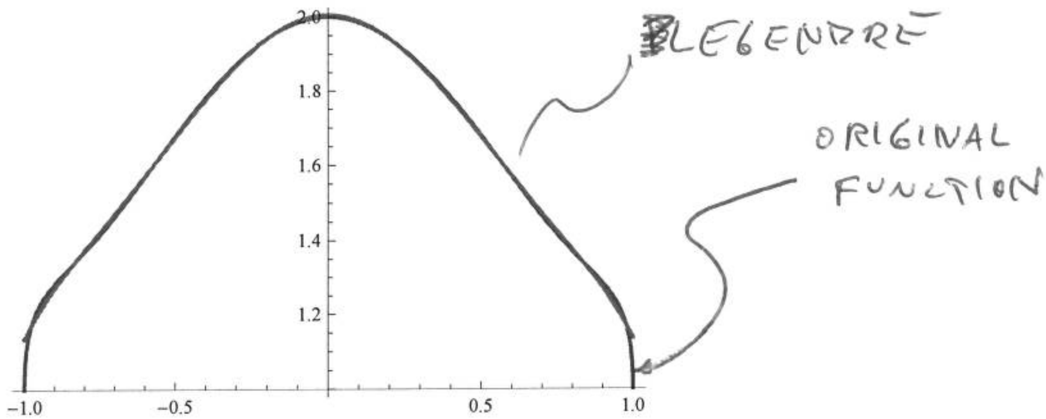

#2 FOLLOW THE HINT.

$$\cos 3\theta = \mathcal{R}\left(e^{i3\theta}\right)$$

$$= \mathcal{R}\left((\cos\theta + i\sin\theta)^3\right)$$

$$= \mathcal{R}\Big(\cos^3\theta + 3\cos^2\theta\,(i\sin\theta) + 3\cos\theta\,(i\sin^2\theta) + (i\sin\theta)^3\Big)$$

$$= \mathcal{R}\Big(\cos^3\theta - 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta\Big)$$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$= x^3 - 3x\,(1-x^2)$$

$$\boxed{\cos 3\theta = 4x^3 - 3x} \quad w/\ x = \cos\theta$$

HENCE, $\cos 3\theta$ IS A LINEAR COMBO OF $P_3(x)$ AND $P_1(x)$.

$$f(x) = \sum A_\ell P_\ell(x)$$

$$A_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} f(x)\,P_\ell(x)\,dx$$

#2.

$$A_1 = -0.6$$

$$A_3 = 1.6$$

$$\boxed{\therefore \cos 3\theta = -\frac{3}{5} P_1(\cancel{x}\cos\theta) + \frac{8}{5} P_3(\cos\theta)}$$

#3. $$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

FOR $r \leq R$, NEED FINITE $V(r,\theta)$, $\Rightarrow B_\ell = 0$

@ $r = R$, $V(r,\theta) = V(R,\theta) = k\cos 3\theta$

$$k\cos 3\theta = \frac{k}{5}\left[-3\,P_1(\cos\theta) + 8\,P_3(\cos\theta)\right]$$

SO, ONLY $\ell = 1$ & $\ell = 3$ ORDERS MATTER.

HENCE,

$$k\cos 3\theta = \frac{k}{5}\left[-3\,P_1(\cos\theta) + 8\,P_3(\cos\theta)\right]$$

$$= A_1 R\, P_1(\cos\theta) + A_3 R^3\, P_3(\cos\theta)$$

$$\Rightarrow A_1 = -3k/5R$$

$$A_3 = 8k/5R^3$$

$$\boxed{\therefore V(r,\theta) = \frac{k}{5R}\left[-3r\,P_1(\cos\theta) + \frac{8r^3}{R^2}P_3(\cos\theta)\right] \quad r \leq R}$$

$r > R$, FINITE $V(r,\theta)$ @ $r \to \infty \Rightarrow A_\ell = 0$

@ $r = R$, $$V(R,\theta) = \frac{B_1}{R^2}P_1(\cos\theta) + \frac{B_3}{R^4}P_3(\cos\theta)$$

$$\Rightarrow B_1 = -3kR^2/5, \quad B_3 = 8kR^4/5$$

SO,

$$\boxed{V(r,\theta) = \frac{kR^2}{5}\left[-\frac{3}{r^2}P_1(\cos\theta) + \frac{8R^2}{r^4}P_3(\cos\theta)\right] \qquad r > R}$$

~~#3b~~ $\cancel{\sigma(\theta) = -\varepsilon_0 \frac{\partial V}{\partial n}}$

$$\cancel{= -\varepsilon_0 \frac{\partial V}{\partial r}\Big|_{r=R}}$$

(3b) PROBLEM POORLY WORDED. I MEANT "SPHERICAL SHELL" BUT WROTE "SPHERE".
ASSUMING SPHERE IS CONDUCTOR, $V =$ const INSIDE CONDUCTOR SINCE $\underset{\sim}{E} = 0$.
THEN,

$$\sigma(\theta) = -\varepsilon_0 \, \partial V/\partial r\Big|_{r=R}$$

$$\boxed{\sigma(\theta) = -\frac{\varepsilon_0 k}{5R}\left[6P_1(\cos\theta) - 32P_3(\cos\theta)\right]}$$

FOR SHELL, $\sigma(\theta) = -\varepsilon_0 \frac{\partial V}{\partial r}\Big|_{\substack{out \\ r=R}} + \varepsilon_0 \frac{\partial V}{\partial r}\Big|_{\substack{in \\ r=R}}$

(SEE GR., p90)

$$\boxed{\sigma(\theta) = \frac{\varepsilon_0 k}{5R}\left[-9P_1(\cos\theta) + 56P_3(\cos\theta)\right]}$$

(#4) $\boxed{Q = -2q + 3q = q}$ MONOPOLE MOMENT

DIPOLE MOMENT $= \sum q_i r_i = -3qa\hat{z}$

$$\boxed{DM = -3qa\hat{z}}$$

$$V(r,\theta) \simeq \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{p \cdot \hat{r}}{r^2}\right]$$

$$\boxed{V(r,\theta) \simeq \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{3a\cos\theta}{r^2}\right]}$$