Q1(a).

$$L_+ Y_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m+1)}\; Y_\ell^{m+1} \qquad (1)$$

$$\Rightarrow Y_4^2 = \frac{1}{\hbar 3\sqrt{2}} L_+ Y_{\ell=4}^{m=1} \qquad (2)$$

Now,

$$L_+ = \hbar e^{i\varphi}\left( \partial/\partial\theta + i\cot\theta\, \partial/\partial\varphi \right) \qquad (3)$$

$$\frac{\partial}{\partial\theta} Y_4^1 = -\frac{3}{8} e^{i\varphi}\sqrt{5/\pi}\Big[-\sin\theta(-3+7\cos^2\theta)\sin\theta$$

$$+ \cos\theta(-14\cos\theta\sin\theta)\sin\theta$$

$$+ \cos\theta(-3+7\cos^2\theta)\cos\theta\Big] \qquad (4)$$

$$i\cot\theta\, \partial/\partial\varphi\, Y_4^1 = i\cot\theta\left[-\frac{3}{8}\sqrt{5/\pi}\,\cos\theta(-3+7\cos^2\theta)\sin\theta\, i e^{i\varphi}\right]$$

$$= -\cos\theta\left[-\frac{3}{8}\sqrt{5/\pi}\,\cos\theta(-3+7\cos^2\theta)e^{i\varphi}\right]$$

$$= +\frac{3}{8}\sqrt{5/\pi}\, e^{i\varphi}\cos^2\theta(-3+7\cos^2\theta) \qquad (5)$$

So,

$$\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) Y_4^1 = -\frac{3}{8} e^{i\varphi}\sqrt{5/\pi}\Big[$$

$$-\sin\theta(-3+7\cos^2\theta)\sin\theta$$

$$+\cos\theta(-14\cos\theta\sin\theta)\sin\theta\Big]$$

$$= -3/8\, e^{i\varphi}\sqrt{5/\pi}\left[-21\cos^2\theta\sin^2\theta + 3\sin^2\theta\right]$$

$$= -\frac{3}{8} e^{i\varphi} \sqrt{5/\pi} \left[ -3 \left( 7\cos^2\theta - 1 \right) \sin^2\theta \right]$$

$$= +\frac{9}{8} \sqrt{5/\pi}\, e^{i\varphi} \left[ -1 + 7\cos^2\theta \right] \sin^2\theta \qquad (8)$$

$$\Rightarrow L_+ Y_4^1 = \hbar e^{i\varphi}\, \frac{9}{8} \sqrt{5/\pi}\, e^{i\varphi} \left[ -1 + 7\cos^2\theta \right] \sin^2\theta \qquad (9)$$

SUBSTITUTE (9) INTO (2):

$$\boxed{Y_4^2 = \frac{3}{8} e^{2i\varphi} \sqrt{5/2\pi} \left[ -1 + 7\cos^2\theta \right] \sin^2\theta}$$

(b) $m$ SPANS FROM $-l$ TO $+l$ IN STEPS OF 1. $L_z$'S EIGENVALUES ARE EIGENVALUES OF ANGULAR MOMENTUM. HENCE

$$\boxed{\text{EIGENVALUES OF } \hat{L}_z = 2\hbar, 1\hbar, 0, -1\hbar, -2\hbar.}$$

(1C) A LITTLE BIT TRICKY/SUBTLE.

NOTE $\frac{3}{5}\hat{L}_x - \frac{4}{5}\hat{L}_y = \underset{\sim}{L} \cdot \hat{n}$, w/ $\hat{n} = \frac{3}{5}\hat{x} - \frac{4}{5}\hat{y}$,

i.e., $\hat{n}$ POINTS IN SOME CRAZY DIRECTION.

LET THIS DIRECTION BE YOUR NEW $z$ DIRECTION. THE EIGENVALUES ARE JUST THE POSSIBLE VALUES OF ANGULAR MOMENTUM YOU CAN MEASURE ALONG THIS DIRECTION. SO, THIS IS NOW JUST LIKE PROBLEM 1 IN pt (b).

$$\boxed{\text{EIGENVALUES} = \{-2\hbar, \hbar, 0, \hbar, 2\hbar\}}$$

Q2 $[L^2, L_z] = 0 \Rightarrow$ COMMON SET OF EIGENSTATES.

NOTE ALSO, $[H, L^2] = [H, L_z] = 0$ SO $\hat{H}$ & $\hat{L}^2$ & $\hat{L}_z$ HAVE SAME EIGENSTATES.

HENCE

$$H|\alpha\rangle = E|\alpha\rangle$$

$$\left(\frac{L^2}{I} + \alpha L_z\right)|\alpha\rangle = E|\alpha\rangle$$

$$\left(\frac{l(l+1)\hbar^2}{I} + \alpha m\hbar\right)|\alpha\rangle = E|\alpha\rangle$$

$$\boxed{E = \frac{l(l+1)\hbar^2}{I} + \alpha m\hbar}$$

#3 (a)

RECAST THE OPERATOR $\hat{\sigma}_1 \cdot \hat{\sigma}_2$

$$\hat{S}^2 = \left( \frac{\hbar}{2}\hat{\sigma}_1 + \frac{\hbar}{2}\hat{\sigma}_2 \right)^2$$

$$= \frac{\hbar^2}{4}\sigma_1^2 + \frac{\hbar^2}{4}\sigma_2^2 + \frac{\hbar^2}{2}\hat{\sigma}_1 \cdot \hat{\sigma}_2$$

$$= \hat{S}_1^2 + \hat{S}_2^2 + \frac{\hbar^2}{2}\hat{\sigma}_1 \cdot \hat{\sigma}_2$$

$$\Rightarrow \hat{\sigma}_1 \cdot \hat{\sigma}_2 = \frac{2}{\hbar^2}\left[ \hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2 \right]$$

$$= \frac{2}{\hbar^2}\left[ \hbar^2 s(s+1) - \frac{\hbar^2}{4} s_1(s_1+1) - \frac{\hbar^2}{4} s_2(s_2+1) \right]$$

$$\boxed{\hat{\sigma}_1 \cdot \hat{\sigma}_2 = 2s(s+1) - 3}$$

FOR SINGLET $(s=0)$ $\hat{\sigma}_1 \cdot \hat{\sigma}_2 = -3$

FOR TRIPLET $\hat{\sigma}_1 \cdot \hat{\sigma}_2 = +1$

TAKE HINT AND CHOOSE $\hat{e} \parallel \hat{z}$.

FIRST TERM IN $S_{12}$ BECOMES

$$3(\hat{\sigma}_1 \cdot \hat{e})(\hat{\sigma}_2 \cdot \hat{e}) = 3\sigma_{1z}\sigma_{2z}$$

NOW, $3\sigma_{1z}\sigma_{2z}\chi_{SINGLET} = 3\sigma_{1z}\sigma_{2z} \cdot \frac{1}{\sqrt{2}}\left[ |\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2 \right]$

$$= -\frac{3}{\sqrt{2}}\left[ |\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2 \right]$$

$$= -3\chi_{SINGLET}.$$

$$\boxed{3\sigma_{1_z}\sigma_{2_z}\chi_{\text{SINGLET}} = -3\chi_{\text{SINGLET}}}$$

WHERE I HAVE USED $\sigma_z|\uparrow\rangle = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

$$\sigma_z|\downarrow\rangle = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -\begin{pmatrix}0\\1\end{pmatrix}$$

COMBINING RESULTS:

$$S_{12}\chi_{\text{SINGLET}} = \left[3(\sigma_1\cdot\hat{e})(\sigma_2\cdot\hat{e}) - \sigma_1\cdot\sigma_2\right]\chi_{\text{SINGLET}}$$

$$= -3\chi_{\text{SINGLET}} + 3\chi_{\text{SINGLET}}$$

$$\boxed{S_{12}\chi_{\text{SINGLET}} = 0}$$

(b) FOR TRIPLET $S = 1\hbar$

$S_z = \pm\hbar, \phi.$

| $S_z$ | $\hat{\sigma}_1\cdot\hat{\sigma}_2 = 2S(S+1)-3$ | $3\sigma_{1_z}\sigma_{2_z}$ |
|---|---|---|
| $+\hbar$ | $4-3=1$ | $3$ |
| $-\hbar$ | $4-3=1$ | $3$ |
| $\phi$ | $1$ | $-3$ |

AGAIN, $S_{12} = 3\sigma_{1z}\sigma_{2z} - \sigma_1\cdot\hat{\sigma}_2$

FOR $S_z = \pm\hbar$

$$S_{12}\,\chi_{TRIPLET}(m=\pm\hbar) = 3\,\chi_{TRIP}(m=\pm\hbar) - \chi_{TRIP}(m=\pm\hbar)$$

$$\boxed{S_{12}\,\chi_{TRIP}(m=\pm\hbar) = 2\,\chi_{TRIP}(m=\pm\hbar)}$$

FOR $S_z = 0\hbar$

$$S_{12}\,\chi_{TRIP}(m=0) = -3\,\chi_{TRIP}(m=0) - \chi_{TRIP}(m=0)$$

$$\boxed{S_{12}\,\chi_{TRIP}(m=0) = -4\,\chi_{TRIPLET}(m=0)}$$

COMBINING

$$\boxed{(S_{12}-2)(S_{12}+4)\,\chi_{TRIPLET} = 0}$$

#4(a)

IN A SINGLET STATE THE SPINS ARE ANTI-PARALLEL SO

$\boxed{\text{PROB OF BOTH SPINS "UP"} = 0}$

(b). THE KEY HERE IS TO EXPRESS SINGLET STATE IN TERMS OF EIGEN SPINORS OF $S_x$ & $S_y$.

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$ FROM TEXT SPIN "UP" ALONG X

$$\chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$ SPIN DOWN ALONG X.

FOR EIGENSPINORS FOR $S_y$.

$$S_y\,\chi_+^{(y)} = +\frac{\hbar}{2}\chi_+^{(y)}$$

$$\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}a\\b\end{pmatrix}$$

$$-ib = a$$

$$ia = b$$

$$\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$

SIMILARLY,

$$\chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

$$\chi_{\text{SINGLET}} = \frac{1}{\sqrt{2}}\left[\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2\right]$$

$$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]$$

$$\begin{pmatrix}1\\0\end{pmatrix}_1 = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}\right]$$

$$\boxed{\begin{pmatrix}1\\0\end{pmatrix}_1 = \frac{1}{\sqrt{2}}\left[\xi_+ + \xi_-\right]}$$

w/ $\xi_+$ SPIN UP ALONG Y FOR PART (1)

$\xi_-$ = SPIN DOWN ALONG Y

$$\begin{pmatrix}0\\1\end{pmatrix}_1 = \frac{-i}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}\right]$$

$$\boxed{\begin{pmatrix}0\\1\end{pmatrix}_1 = \frac{-i}{\sqrt{2}}\left[\xi_+ - \xi_-\right]}$$

EXPRESS $\begin{pmatrix}1\\0\end{pmatrix}$ & $\begin{pmatrix}0\\1\end{pmatrix}$ IN TERMS OF EIGENSPINORS ALONG X-DIRECTION

$$\begin{pmatrix}1\\0\end{pmatrix}_2 = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right]$$

$$\boxed{\begin{pmatrix}1\\0\end{pmatrix}_2 = \frac{1}{\sqrt{2}}\left[\eta_+ + \eta_-\right]}$$

$\eta_\pm$ = SPINOR ALONG $\pm X$.

$$\boxed{\begin{pmatrix}0\\1\end{pmatrix}_2 = \frac{1}{\sqrt{2}}\left[\eta_+ - \eta_-\right]}$$

(b) NOW WE WRITE SINGLET STATE

$$\chi_{\text{SINGLET}} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2\right]$$

$$= \frac{1}{\sqrt{2}}\Big[\frac{1}{\sqrt{2}}(\xi_+ + \xi_-)\cdot\frac{1}{\sqrt{2}}(\eta_+ - \eta_-)$$
$$- \frac{-i}{\sqrt{2}}(\xi_+ - \xi_-)\cdot\frac{1}{\sqrt{2}}(\eta_+ + \eta_-)\Big]$$

THIS IS AN EXPANSION IN TERMS OF EIGENSPINORS ALONG $y$ & $x$. LOOK FOR TERM $\xi_+\eta_+$ — THIS IS SPIN "UP" ALONG $y$ AND SPIN "UP" ALONG $x$. FIND ~~THE~~ ITS COEFFICIENT AND SQUARE IT!

$\xi_+\eta_+$ TERM : $\frac{1}{\sqrt{2}}\Big[\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\xi_+\eta_+ \quad \frac{+i}{\sqrt{2}}\xi_+\eta_+ = \frac{1}{2\sqrt{2}}(1+i)\xi_+\eta_+$

$$\text{PROB} = \left|\frac{1}{2\sqrt{2}}(1+i)\right|^2$$

$$= \frac{2}{4\cdot 2}$$

$$\boxed{\text{PROB} = 1/4}$$

#5

(a) $V(r) = V_1(r) + V_2(r)\left[\frac{3\, \sigma_1 \cdot \underset{\sim}{r}\; \sigma_2 \cdot \underset{\sim}{r}}{r^2} - \hat{\sigma}_1 \cdot \hat{\sigma}_2\right]$

$- V_3(r)\, \hat{\sigma}_1 \cdot \hat{\sigma}_2$

$= V_1(r) + V_2(r)\left(S_{12}\right) + V_3\, \hat{\sigma}_1 \cdot \hat{\sigma}_2$

$= V_1(r) + V_2(r)\, S_{12} + V_3(r)\underbrace{\left[2 s(s+1) - 3\right]}_{\text{From Prob \#1, pt. B}}$

For $\underline{\underline{\text{SINGLET}}}$ $S_{12} = 0$, Prob #1.

$$\boxed{V(r) = V_1(r) - 3 V_3(r)}$$

Also from Prob #1

(b)

For TRIPLET STATE

$V(r) = V_1(r) + V_2(r)\left[3\sigma_{1z}\sigma_{2z} - 2s(s+1) + 3\right]$

$+ V_3(r)\left(2s(s+1) - 3\right)$

$V(r) = V_1(r) + V_2(r)\left[3\sigma_{1z}\sigma_{2z} - 1\right] + V_3(r)$

Assuming EQUAL ~~PARTS~~ PROBABILITY
FOR $m = 0, \pm 1\, \hbar$,

$$\boxed{V(r) = V_1(r) - V_2(r) + V_3(r)}$$

#6 (a)

A measurement of $S_x$ will yield same eigenvalues as a measurement of $S_z$.

$$\boxed{\text{POSSIBLE VALUES OF } S_x = 0, \pm \hbar}$$

(b). FIND EIGENSPINOR FOR CASE $m_x = -\hbar$.

$$S_x \chi_-^{(x)} = -\hbar \chi_-^{(x)}$$

FROM HW 4, $S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

LET $\chi_-^{(x)} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$S_x \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\hbar \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\hbar \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$\Rightarrow \frac{1}{\sqrt{2}} b = -a$

$\frac{1}{\sqrt{2}} (a+c) = -b$

$\frac{1}{\sqrt{2}} b = -c$

So,

$$\left.\begin{array}{l} a = c \\ \frac{1}{\sqrt{2}} 2a = -b \end{array}\right\} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -b/\sqrt{2} \\ b \\ -b/\sqrt{2} \end{pmatrix}$$

(b) NORMALIZING, i.e., $|a|^2 + |b|^2 + |c|^2 = 1$

$$\boxed{\chi_-^{(x)} = \frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}}$$

$$\langle S_z \rangle = \chi_-^{(x)\dagger} S_z \chi_-^{(x)}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & \sqrt{2} & -1 \end{pmatrix} \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix} -1 & \sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{\langle S_z \rangle = 0}$$

$$\langle S_z^2 \rangle = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{2} & -1 \end{pmatrix} \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} -1 & \sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} -1 & \sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

$$\boxed{\langle S_z^2 \rangle = \frac{\hbar^2}{2}}$$

#6 (c)

$$\langle S_y \rangle = \chi_-^{(x)\dagger} \; S_y \; \chi_-^{(x)}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & \sqrt{2} & -1 \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}$$

w/ $S_y = (S_+ - S_-)/2i$, see HW 4,

$$\langle S_y \rangle = \frac{\hbar}{4} \begin{pmatrix} -1 & \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} -i\sqrt{2} \\ 0 \\ i\sqrt{2} \end{pmatrix}$$

$$\boxed{\langle S_y \rangle = 0}$$

$$\langle S_y^2 \rangle = \frac{\hbar^2}{8} \begin{pmatrix} -1 & \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}$$

$$\langle S_y^2 \rangle = \frac{\hbar^2}{8} \begin{pmatrix} -1 & \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 2\sqrt{2} \\ 0 \end{pmatrix}$$

$$\boxed{\langle S_y^2 \rangle = \hbar^2/2}$$