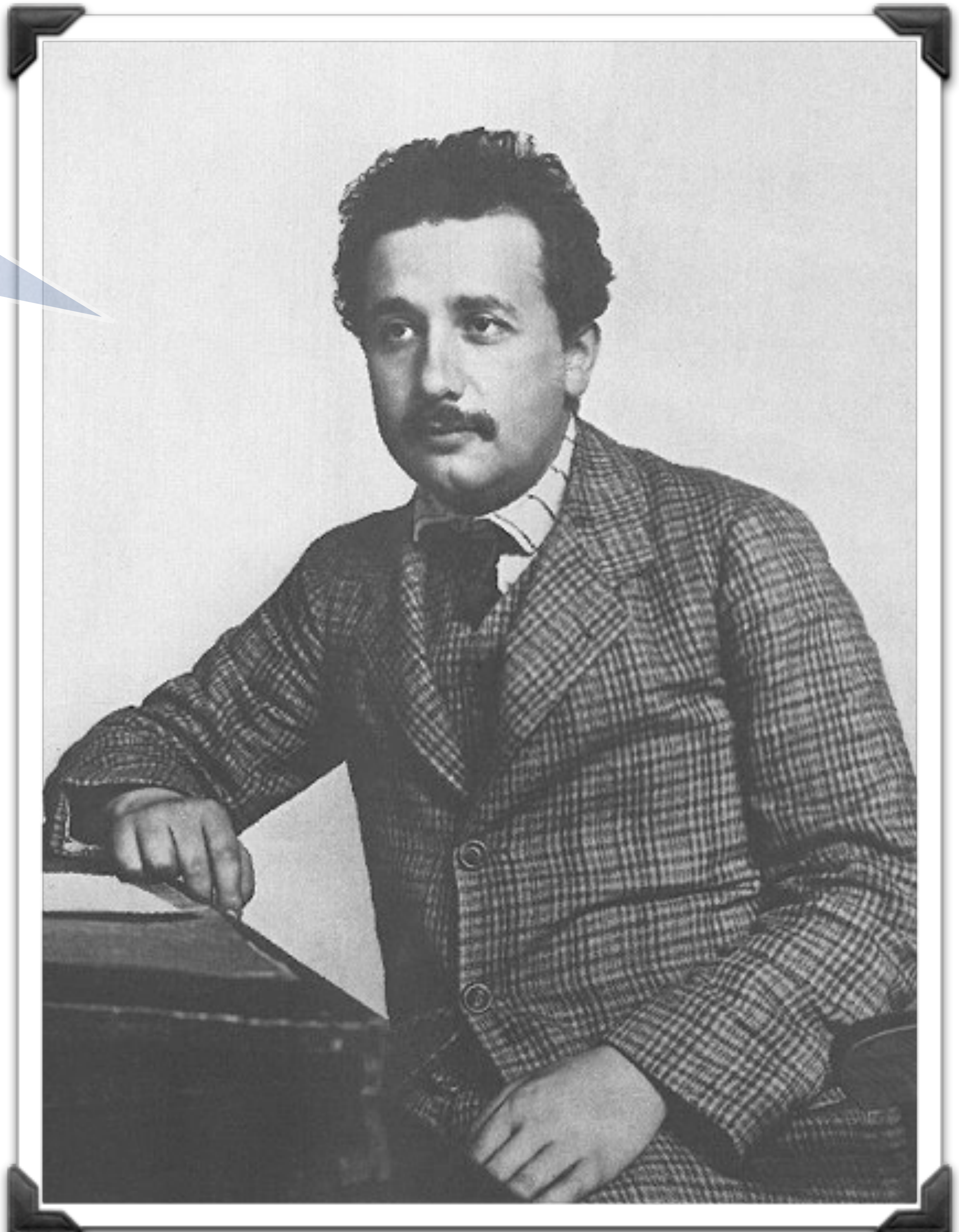
Welcome back to PHY 3305



Today's Lecture:

Momentum and Energy Conservation

Albert Einstein
1879-1955

Review: Transforming Velocity

Remember:

$$u = \frac{dx}{dt} \qquad x = \gamma_\nu(x' + vt') \qquad t = \gamma_\nu(\frac{v}{c^2}x' + t')$$

From this we can derive the following relations. (Exercise for the student).

$$u = \frac{u' + v}{1 + \frac{vu'}{c^2}} \qquad \text{and} \qquad u' = \frac{u - v}{1 - \frac{vu}{c^2}}$$

MOMENTUM

In classical physics,

$$\sum \vec{p}_i = \sum \vec{p}_f$$

Is momentum conserved in classical physics when you apply the Galilean Transformation?

YES!

MOMENTUM

Consider the collision of two objects of mass m_1 and m_2.

In frame S:

$$m_1 u_{1i} + m_2 u_{2i} = m_1 u_{1f} + m_2 u_{2f}$$

In frame S':

$$m_1 u'_{1i} + m_2 u'_{2i} = m_1 u'_{1f} + m_2 u'_{2f}$$

Use Galilean velocity transformation to rewrite u' in terms of u. (u' = u - v)

$$m_1(u_{1i} - \cancel{v}) + m_2(u_{2i} - \cancel{v}) = m_1(u_{1f} - \cancel{v}) + m_2(u_{2f} - \cancel{v})$$

$$m_1 u_{1i} + m_2 u_{2i} = m_1 u_{1f} + m_2 u_{2f}$$

Momentum conservation is invariant under Galilean Transformation

What if we use the relativistic Lorentz Transformations instead?

In frame S':

$$m_1 u'_{1i} + m_2 u'_{2i} = m_1 u'_{1f} + m_2 u'_{2f}$$

Use our velocity transformation

$$u' = \frac{u - v}{1 - \frac{vu}{c^2}}$$

$$m_1\left(\frac{u_{1i} - v}{1 - \frac{u_{1i}v}{c^2}}\right) + m_2\left(\frac{u_{2i} - v}{1 - \frac{u_{2i}v}{c^2}}\right) \stackrel{?}{=} m_1\left(\frac{u_{1f} - v}{1 - \frac{u_{1f}v}{c^2}}\right) + m_2\left(\frac{u_{2f} - v}{1 - \frac{u_{2f}v}{c^2}}\right)$$

Momentum is not conserved!

If p = mu is wrong, what is the correct form?

- We need to to reduce to p = mu at low speeds. (experimental evidence tells us so)

Do we know of a quantity which behaves this way, which we could multiply by mu?

$$\gamma_u = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

So, we can postulate that

$$\boxed{p = \gamma_u m u}$$

Examine our relationship for relativistic momentum in the S′ reference frame.

$$u' = \frac{u - v}{1 - \frac{vu}{c^2}}$$

$$\gamma_{\mu'} = \frac{1}{\sqrt{1 - \frac{u'}{v}}} = (1 - \frac{uv}{c^2})\gamma_v\gamma_\mu$$

$$P'_{tot} = \sum_i \gamma_{\mu'} m_i \mu'_i$$

$$= \frac{1}{1 - \frac{uv}{c^2}} \sum_i \gamma_{\mu'_i} m_i u_i - \frac{v}{1 - \frac{uv}{c^2}} \sum_i \gamma_{\mu'_i} m_i$$

$$= \frac{1}{1 - \frac{uv}{c^2}} \sum_i (1 - \frac{uv}{c^2})\gamma_\mu\gamma_v m_i u_i - \frac{v}{1 - \frac{uv}{c^2}} \sum_i (1 - \frac{uv}{c^2})\gamma_\mu\gamma_v m_i$$

$$= \gamma_v P_{tot} - v\gamma_v \boxed{\sum_i \gamma_{\mu_i} m_i}$$

So, it seems that that this quantity must also be conserved.

Any ideas on what it might be?

ENERGY

If we multiply by c^2 we get something with units of energy.

$$E = \gamma m c^2$$

A consequence is that mass measures energy. If an object is stationary, its total energy is

$$E_{internal} = mc^2$$

The kinetic energy is then

KE = energy moving - energy at rest

$$KE = \gamma_\mu mc^2 - mc^2 = (\gamma_\mu - 1)mc^2$$

ENERGY

Einstein sought an expression for the total energy of a body in motion. Starting with conservation of energy, he made some arguments about what must happen as a consequence of relativity.

$$E = constant + \frac{1}{2}mu^2$$

What he found was

$$E^2 = (mc^2)^2 + (pc)^2$$

EXAMINE ENERGY

Postulates tell us the laws of physics are invariant for observers in relative motion.

$$E_{1b} + E_{2b} = E_{1a} + E_{2a}$$

$$p_{1b} + p_{2b} = p_{1a} + p_{2a}$$

Classically:

$$E = E(0) + \frac{1}{2}mu^2$$

$$p = mu$$

Where energy comes to us from definition of work.

$$W = \int F dx = \int (\frac{dp}{dt}) dx = \int m(\frac{du}{dt})(udt) = \int mudu = C + \frac{1}{2}mu^2$$

Generalize energy and momentum in order to deduce the correct relativistic form:

$$p = \mathcal{M}(u)$$

$$E = \mathcal{E}(u)$$

$\mathcal{M}$ & $\mathcal{E}$ are unknown functions of the motion of the relative frames. We do know that E & p should approach their classical values as u approaches 0.

$$\lim_{u \to 0} \mathcal{M}(u) = mu$$

$$\lim_{u \to 0} \frac{\partial \mathcal{E}(u)}{\partial u^2} = \frac{1}{2} m u^2$$

Momentum:

We already saw that this is straight forward.

$$p = \mathcal{M}(u) = \gamma_u m u$$

Energy:

More tricky due to the nonlinear dependence on velocity of the body in the frame (u^2).

Use Binomial Expansion for γ_v.

$$\gamma_u = 1 + \frac{1}{2}\frac{u^2}{c^2} + \mathcal{O}(u^4)$$

$$\gamma_u = 1 + \frac{1}{2}\frac{u^2}{c^2} + \mathcal{O}(u^4)$$

$$E = \mathcal{E}(u)$$

Remember - we want something that as u approaches 0 gives

$$constant + \frac{1}{2}m\frac{u^2}{c^2}$$

Try γ_u.

$$\mathcal{E} = \gamma_u \mathcal{E}(0)$$

In the low velocity limit

$$\lim_{u \to 0} \mathcal{E}(u) \approx \mathcal{E}(0) + \frac{1}{2}\mathcal{E}(0)\frac{u^2}{c^2}$$

Which leads us to conclude

$$\mathcal{E}(0) = mc^2$$

E = mc^2 for an object at rest.

This is remarkable. It says that the total energy of a body at rest can be described by its mass. When energy is given off by an object, its mass decreases correspondingly by

$$\Delta m = \frac{\Delta E}{c^2}$$

LAST DETAIL

$$\mathcal{E} = \gamma_u \mathcal{E}(0)$$
$$\mathcal{E}(0) = mc^2$$

What if the object is moving? Is there a general equation we can use for an object in any inertial reference frame?

$$\mathcal{E}^2 = \gamma_u^2 \mathcal{E}(0)^2 = \gamma_u^2 (mc^2)^2 = \frac{1}{1-\frac{u^2}{c^2}}(mc^2)^2$$

Use the Binomial Expansion of $1/(1-x^2)$ to expand ($x = u/c$).

$$\mathcal{E}(u)^2 = (mc^2)^2(1 + \frac{u^2}{c^2} + \frac{u^4}{c^4} + ...)$$

$$\mathcal{E}(u)^2 = m^2c^4 + m^2u^2c^2 + m^2u^4 + m^2u^6(\frac{1}{c^2}) + ...$$

$$\mathcal{E}(u)^2 = (mc^2)^2 + m^2u^2c^2(1 + \frac{u^2}{c^2} + \frac{u^4}{c^4} + ...)$$

$$\mathcal{E}(u)^2 = (mc^2)^2 + (\gamma_u mu)^2c^2 = \boxed{(mc^2)^2 + p^2c^2}$$

As a final step, redefine

$$\mathcal{E}(u) \equiv E$$

and then

$$\boxed{E^2 = m^2c^4 + p^2c^2}$$

This equation describes the total energy of any inertial reference frame.

CONSEQUENCES

1) If the object is stationary, this simplifies to $E = mc^2$.

   - This implies that if you heat an object, (i.e. it gains internal energy through an increase in T, but NOT KE through motion), the mass of the object increases.

2) What if the mass of an object is zero? Is that allowed?

   - When mass is zero, $E = pc = \gamma_u mu$. The only way the energy of such an object is non-zero is if $u = c$ and γ_u = infinity. Light is a massless particle. (Further discussions of momentum and energy of photons later in the course.)

SUMMARY

The total energy of any reference frame:

$$\boxed{E = \gamma mc^2 \longrightarrow E^2 = m^2c^4 + p^2c^2}$$

Energy of a stationary object (internal energy):

$$\boxed{E = mc^2}$$

Kinetic energy = total energy - stationary object energy:

$$\boxed{KE = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2}$$

If a particle has no mass, (i.e. photons) then

$$\boxed{E = pc}$$

MASS-ENERGY EQUIVALENCE

Stationary Mass Energy:

$$E = \gamma mc^2 \longrightarrow E^2 = m^2c^4 + p^2c^2$$

$$E = mc^2$$

Mass and energy are not the same thing. However, these equations tell us that they are equivalent in the sense that mass can be transformed to energy and energy can be transformed into mass so long as total energy is conserved.

UNITS

Take the case of a proton:

$$m_p = 1.6726217 \times 10^{-27} \text{ kg}$$

$$\boxed{E = mc^2}$$

Atomic mass unit:

$$u = 1.6605389 \times 10^{-27} \text{ kg} \quad = 931.4941 \text{ MeV}/c^2$$

Notice that 1 proton = 1.007276 u

$$m_p = 1.67 \times 10^{-27} \text{ kg} = 1.007 \text{ u} = 938.3 \text{ MeV}/c^2$$

If I multiply by c^2, this is the proton's stationary energy.

ELECTRON VOLT

In particle and atomic physics the usual unit of energy is the **electronvolt (eV)**.

1eV is the energy gained by an electron accelerated through a potential difference of 1 volt
(KE = W = qV).

$$\boxed{1\ \text{eV} = (1.602 \times 10^{-19}\ C)(1.000\ V) = 1.602 \times 10^{-19}\ J}$$

and the usual prefixes apply

$1\ \text{MeV} = 10^{6}\ \text{eV}$ $\quad$ $1\ \text{GeV} = 10^{9}\ \text{eV}$

THE END
(FOR TODAY)