Welcome back to PHY 3305

Today's Lecture:

Transmission & Reflection

Max Born

1882 - 1970

He is the one who figured out $P(x) = \psi^*\psi$ and got the 1954 Nobel prize for it.

What is a bound state?

Cases where a particle's motion is restricted by a force. The motion is restricted to a finite region.

- States that are NOT free of forces. They are states that act under the influence of forces.
- These forces have only a spacial component. They can be described by adding a space-dependent potential, $U(x)$, to the SWE.

What is a unbound state?

Cases where a particle's motion is unrestricted by a force. The motion is NOT restricted to a finite region.

Where is the plane wave going?

Recall the wave function for a plane wave.

$$\Psi(x,t) = Ae^{i(kx-\omega t)}$$

We can use the momentum operator to tell us what direction it is traveling.

$$\hat{p}\Psi(x,t) = (-i\hbar\frac{\partial}{\partial x})\Psi(x,t) = Ak\hbar e^{i(kx-\omega t)} = k\hbar\Psi(x,t)$$

The momentum returned from this operator is

$$p = \hbar k$$ positive!

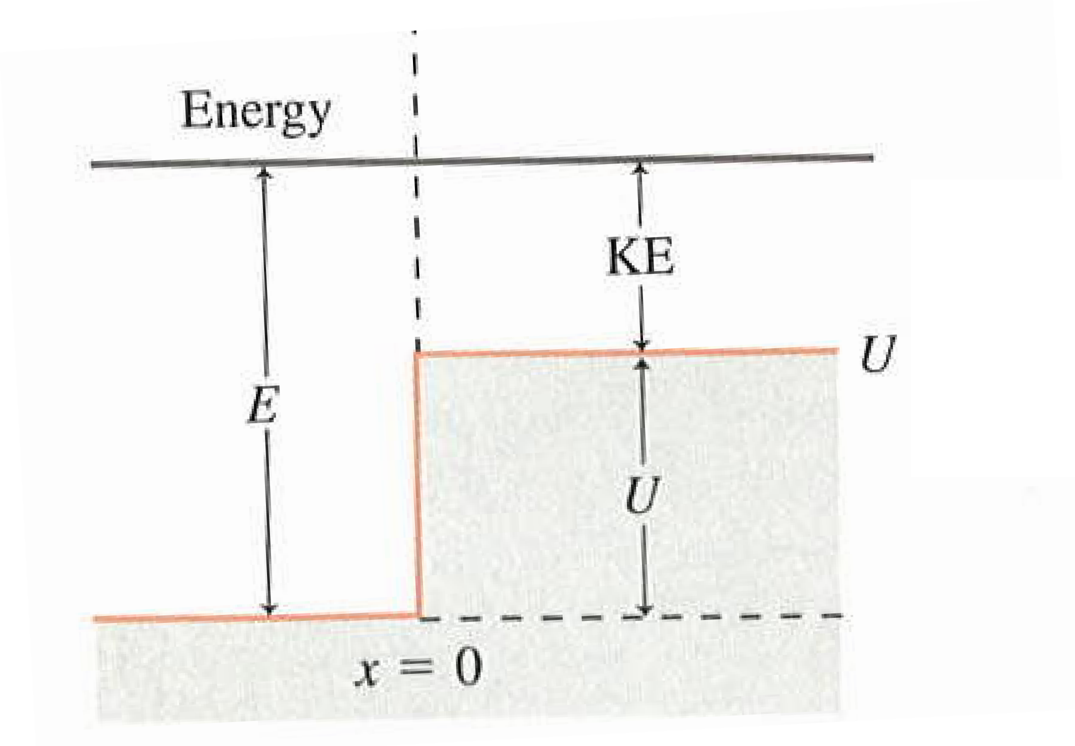
What is the wave function for a plane wave in the negative direction?

$$\Psi(x,t) = Ae^{-i(kx-\omega t)}$$

Potential Step

What is a step?

We will define a <u>step</u> as a region where the potential energy suddenly increases over its value in other parts of space.

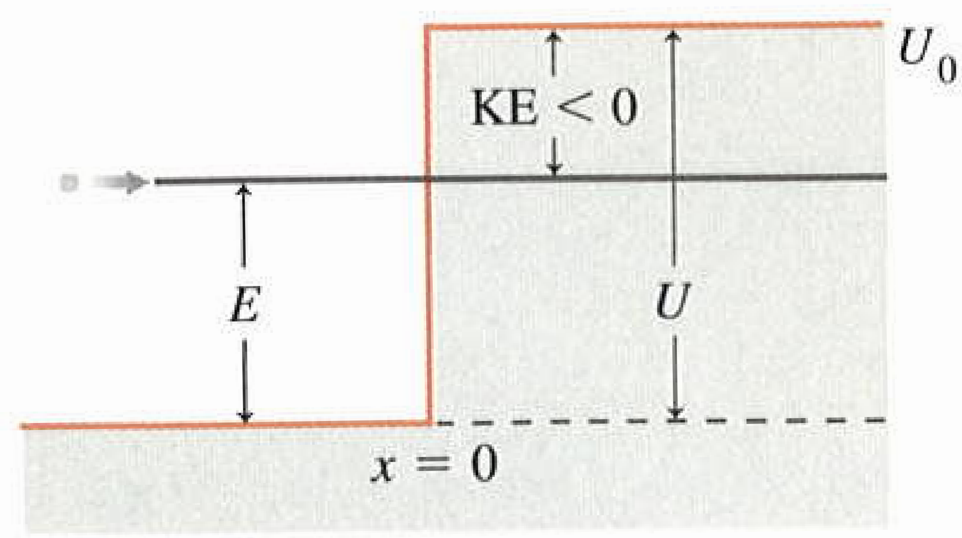


$$U(x) = \begin{cases} 0 & x < 0 \\ U_0 & x > 0 \end{cases}$$

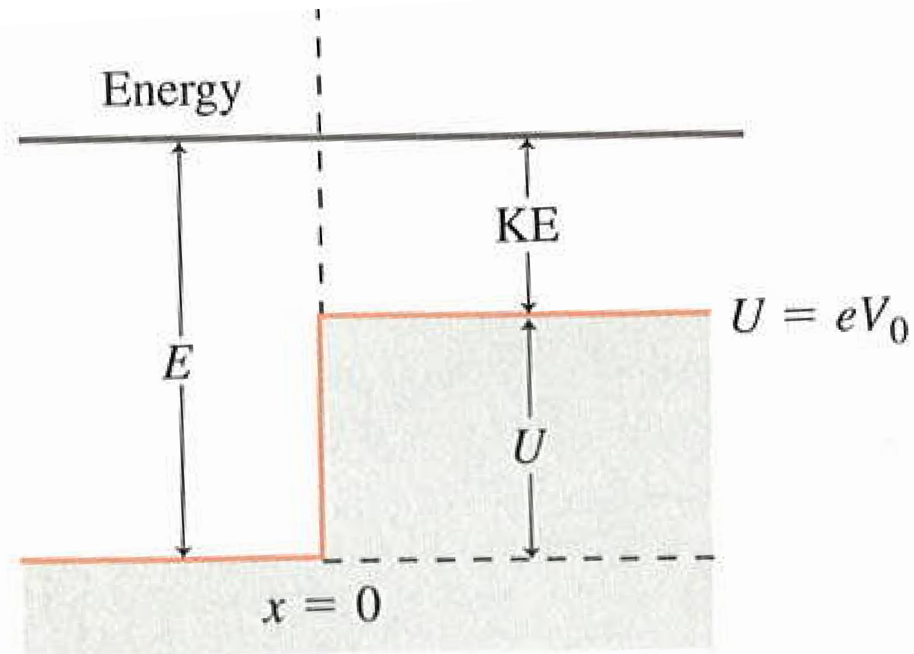
Step Potential

Consider a particle entering from the left. What happens classically?

- Classically, the particle would reflect back to the left at x=0.

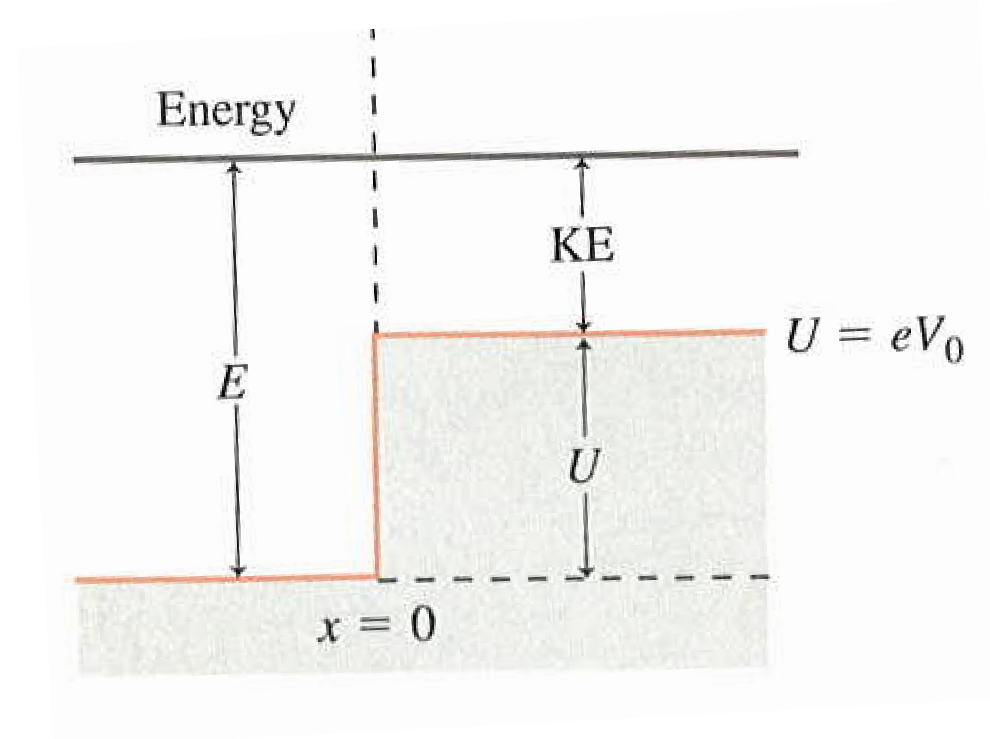


- Classically, the particle would continue on with $KE = E - U = eV_0$.



But, what does Quantum Mechanics say?

Case E > U_0:



What form does the SWE take?

$$\frac{d^2\psi(x)}{dx^2} = -\frac{2m(E-U_0)}{\hbar^2}\psi(x)$$

Solutions are dependent on which region you are in.

$$\psi(x)|_{x<0} = \underbrace{Ae^{ikx}}_{\text{incident}} + \underbrace{Be^{-ikx}}_{\text{reflected}} \qquad \text{with} \qquad k = \frac{2mE}{\hbar}$$

$$\psi(x)|_{x>0} = \underbrace{Ce^{ik'x}}_{\text{transmitted}} \qquad \text{with} \qquad k' = \frac{2m(E-U_0)}{\hbar}$$

Reflection and Transmission Probabilities:

$$\psi(x)|_{x<0} = Ae^{ikx} + Be^{-ikx}$$
$$\psi(x)|_{x>0} = Ce^{ik'x}$$

What are the probability densities of incident, reflection and transmission?

$$|\psi|^2_{inc} = A^*A \qquad |\psi|^2_{refl} = B^*B \qquad |\psi|^2_{trans} = C^*C$$

We can solve for the coefficients A, B and C by using boundary conditions.

$$\psi_{x<0}(0) = \psi_{x>0}(0) \longrightarrow Ae^{+ik0} + Be^{-ik0} = Ce^{+ik'0}$$

$$A + B = C$$

$$\left.\frac{d\psi_{x<0}}{dx}\right|_{x=0} = \left.\frac{d\psi_{x>0}}{dx}\right|_{x=0} \longrightarrow ikAe^{+ik0} - ikBe^{-ik0} = ik'Ce^{+ik'0}$$

$$k(A - B) = k'C$$

Examine: $k(A-B) = k^{'}C$

In the case of a bound particle, imposing the physical requirements lead to quantization.

$$\psi(x) = A\sin(kx) \quad \text{(particle in infinite well)}$$

Boundary condition: $A\sin(kL) = 0$

$$kL = n\pi \;\longrightarrow\; E = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

In the case we have here, there is no restriction on k. Hence, there is no restriction on E! No quantization.

The boundary conditions can give us the reflection and transmission probabilities.

Transmission Probability:

$$T = \frac{\frac{\text{number transmitted}}{\text{time}}}{\frac{\text{number incident}}{\text{time}}} = \frac{|\psi|^2_{\text{trans}} k_{x>0}}{|\psi|^2_{\text{inc}} k_{x<0}} = \frac{C^*C}{A^*A}\frac{k'}{k}$$

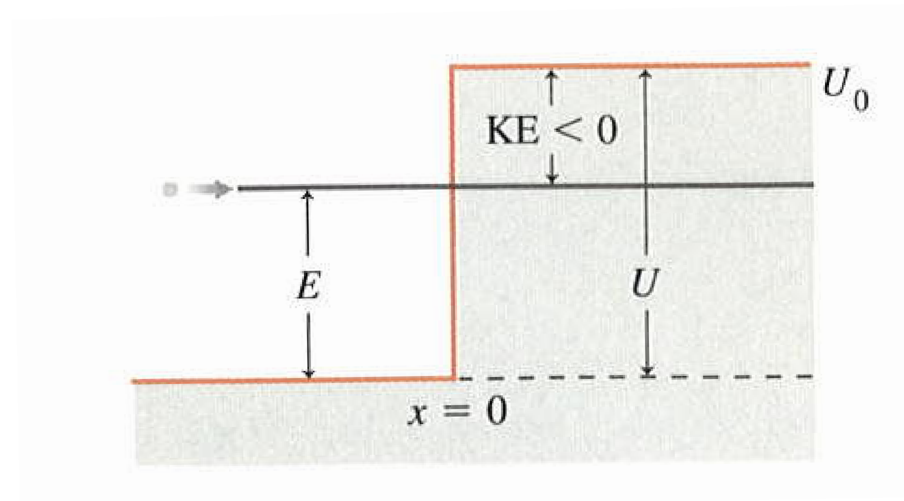
Reflection Probability:

$$R = \frac{\frac{\text{number reflected}}{\text{time}}}{\frac{\text{number incident}}{\text{time}}} = \frac{|\psi|^2_{\text{refl}} k_{x<0}}{|\psi|^2_{\text{inc}} k_{x<0}} = \frac{B^*B}{A^*A}$$

With a little bit of algebra (exercise for the student), one can get

$$T = 4\frac{\sqrt{E(E-U_0)}}{\left(\sqrt{E}+\sqrt{E-U_0}\right)^2} \qquad R = \frac{\left(\sqrt{E}-\sqrt{E-U_0}\right)^2}{\left(\sqrt{E}+\sqrt{E-U_0}\right)^2} \qquad (6\text{-}7)$$

Case E < U_0:



Again, SWE takes the form

$$\frac{d^2\psi(x)}{dx^2} = \frac{2m(E-U_0)}{\hbar^2}\psi(x)$$

And we have

$$\psi(x)|_{x<0} = \underbrace{Ae^{ikx}}_{\text{incident}} + \underbrace{Be^{-ikx}}_{\text{reflected}} \qquad \text{with} \qquad k = \frac{2mE}{\hbar}$$

$$\psi(x)|_{x<0} = Ae^{ikx} + Be^{-ikx}$$

What about the transmitted portion? We know that for a finite barrier some of the wave function can penetrate into the classically forbidden region.

$$\psi_{x>0}(x) = Ce^{-\alpha x}$$ where $$\alpha \equiv \sqrt{\frac{2m(U_0 - E)}{\hbar^2}}$$

If we solve for the boundary conditions, we have

$$\psi_{x<0}(0) = \psi_{x>0}(0) \longrightarrow Ae^{+ik0} + Be^{-ik0} = Ce^{-\alpha 0}$$

$$A + B = C$$

$$\left.\frac{d\psi_{x<0}}{dx}\right|_{x=0} = \left.\frac{d\psi_{x>0}}{dx}\right|_{x=0} \longrightarrow ikAe^{+ik0} - ikBe^{-ik0} = -\alpha Ce^{-\alpha 0}$$

$$ik(A - B) = \alpha C$$

If you work through the algebra (pg 201 in your text), you will find

$$|B| = |A|$$

This gives a reflection probability of

$$R = \frac{B^*B}{A^*A} = 1$$

It follows that the transmission probability must be zero.

However, we know that the wave penetrates the step, so there is a probability of finding particles on the "wrong" side of the step. We must simply accept that as long as no attempt is made to find a particle, we will have an undisturbed wave that is completely reflected and penetrates the classically forbidden region.

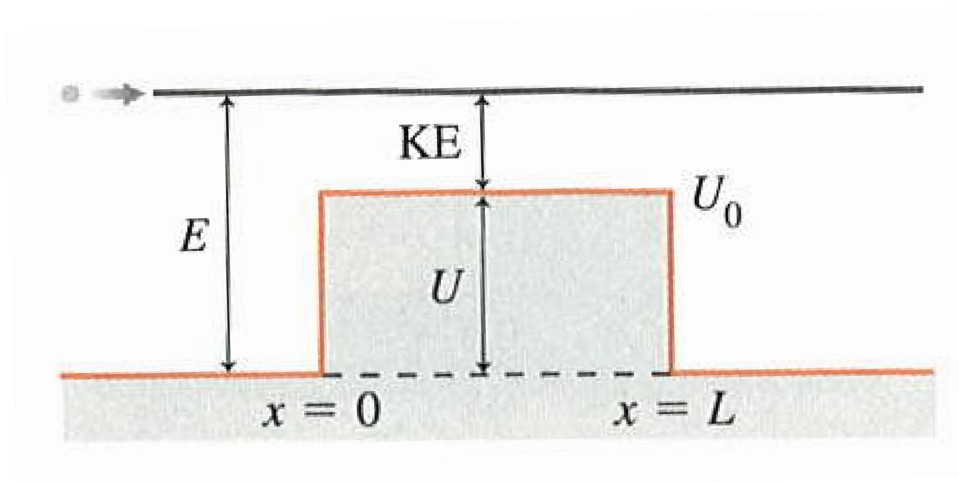
$$\delta = \frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(U_0 - E)}}$$

penetration depth

Potential Barrier

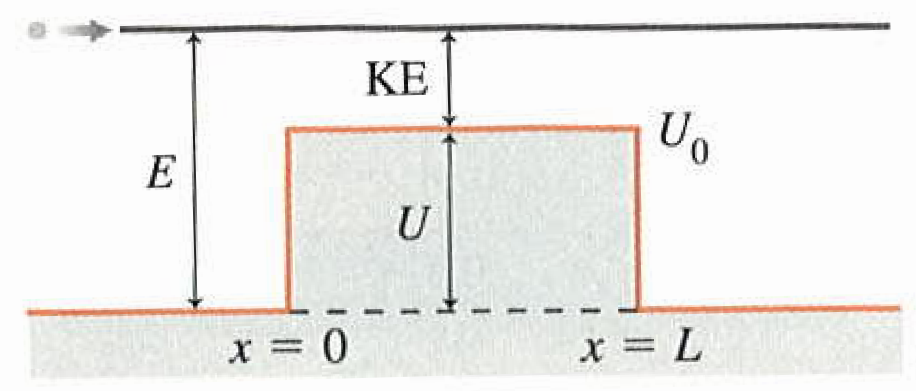
What is a barrier?

We will define a barrier as a region where the potential energy suddenly increases over its value in other parts of space, but only temporarily.



$$U(x) = \begin{cases} 0 & x < 0, x > L \\ U_0 & 0 < x < L \end{cases}$$

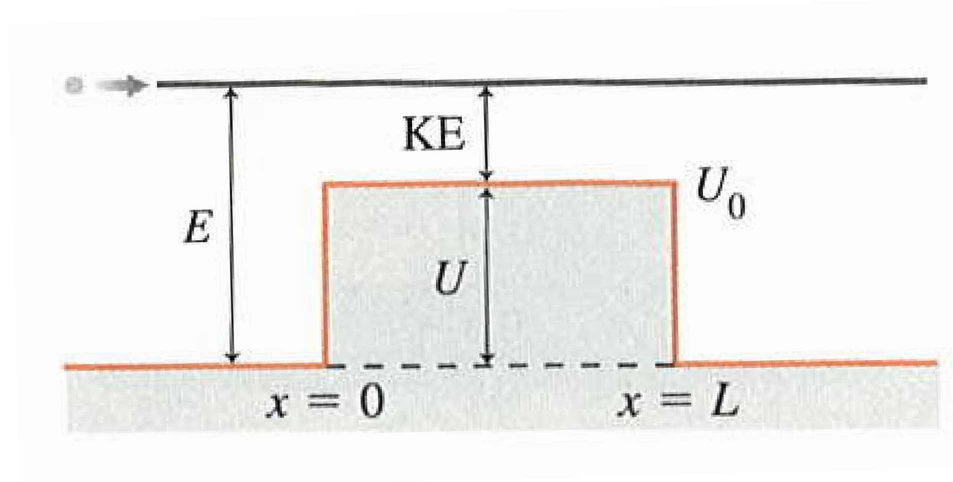
Region left of barrier (x < 0):



$$\psi(x)|_{x<0} = \underbrace{Ae^{+ikx}}_{\text{incident}} + \underbrace{Be^{-ikx}}_{\text{reflected}}$$

The **reflection coefficient** is the ratio of the intensities of theses two waves.

$$R = \frac{|B|^2}{|A|^2}$$



Region right of barrier (x > L):

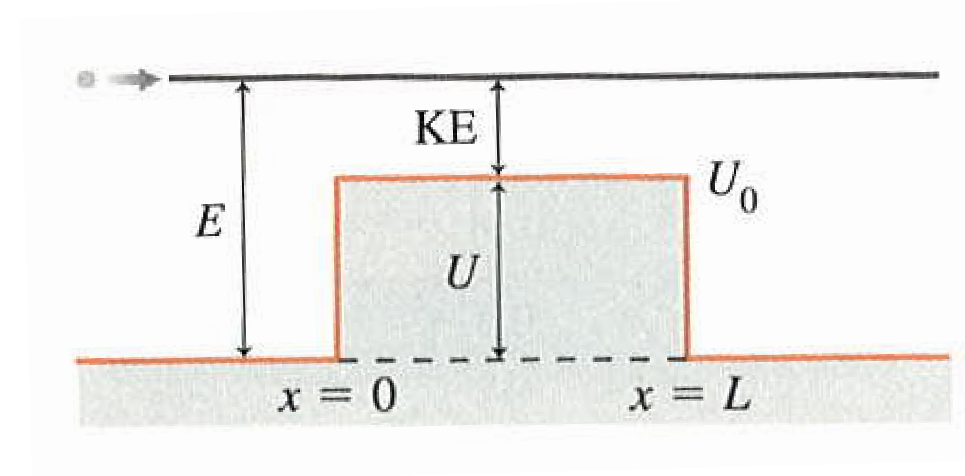
$$\psi(x)|_{x>L} = \underbrace{Fe^{+ikx}}_{\text{transmitted}}$$

The **transmission coefficient** is the ratio of the intensities of theses two waves.

$$T = \frac{|F|^2}{|A|^2}$$

Note that the transmission + reflection coefficients must add to 1. The particle must be found somewhere.

Region 0 < x < L):



$$\psi(x)|_{0<x<L} = Ce^{+ik'x} + De^{-ik'x}$$

To solve for the constants A, B, C, D and F, we apply the smoothness conditions at each of the boundaries.

$$\psi_{x<0} = \psi_{0<x<L}$$

$$Ae^{+ik0} + Be^{-ik0} = Ce^{+ik'0} + De^{-ik'0}$$

$$\boxed{A + B = C + D}$$

$$\psi_{0<x<L} = \psi_{x>L}$$

$$\boxed{Ce^{+ik'x} + De^{-ik'x} = Fe^{ikx}}$$

$$\left.\frac{d\psi_{x<0}}{dx}\right|_{x=0} = \left.\frac{d\psi_{0<x<L}}{dx}\right|_{x=0}$$

$$ikAe^{+ik0} - ikBe^{-ik0} = ik'Ce^{+ik'0} - ik'De^{-ik'0}$$

$$\boxed{k(A - B) = k'(C - D)}$$

$$\left.\frac{d\psi_{0<x<L}}{dx}\right|_{x=L} = \left.\frac{d\psi_{x>L}}{dx}\right|_{x=L}$$

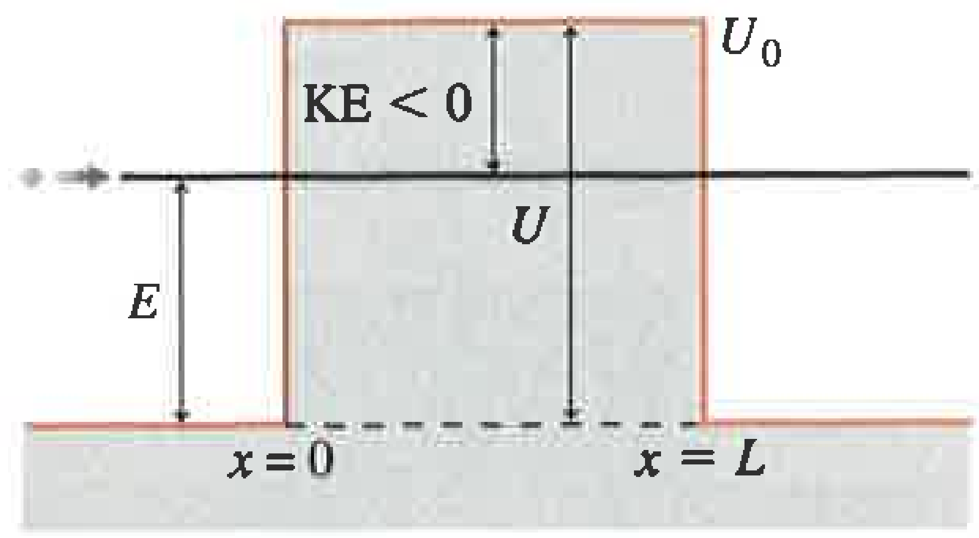
$$\boxed{ik'(Ce^{+ik'x} - De^{-ik'x}) = ikFe^{ikx}}$$

Details of these calculation are on page 203 of your textbook. Here I will just state the results.

$$R = \frac{\sin^2\left[\sqrt{2m(E - U_0)}\,L/\hbar\right]}{\sin^2\left[\sqrt{2m(E - U_0)}\,L/\hbar\right] + 4(E/U_0)\left[(E/U_0) - 1\right]}$$

$$T = \frac{4(E/U_0)\left[(E/U_0) - 1\right]}{\sin^2\left[\sqrt{2m(E - U_0)}\,L/\hbar\right] + 4(E/U_0)\left[(E/U_0) - 1\right]} \qquad (6\text{-}13)$$

Tunneling: E < U



Region left of barrier (x < 0):

$$\psi(x)|_{x<0} = \underbrace{Ae^{+ikx}}_{\text{incident}} + \underbrace{Be^{-ikx}}_{\text{reflected}}$$

Region right of barrier (x > L):

$$\psi(x)|_{x>L} = \underbrace{Fe^{+ikx}}_{\text{transmitted}}$$

Region (0 < x < L):

$$\psi(x)|_{0<x<L} = Ce^{+\alpha x} + De^{-\alpha x}$$

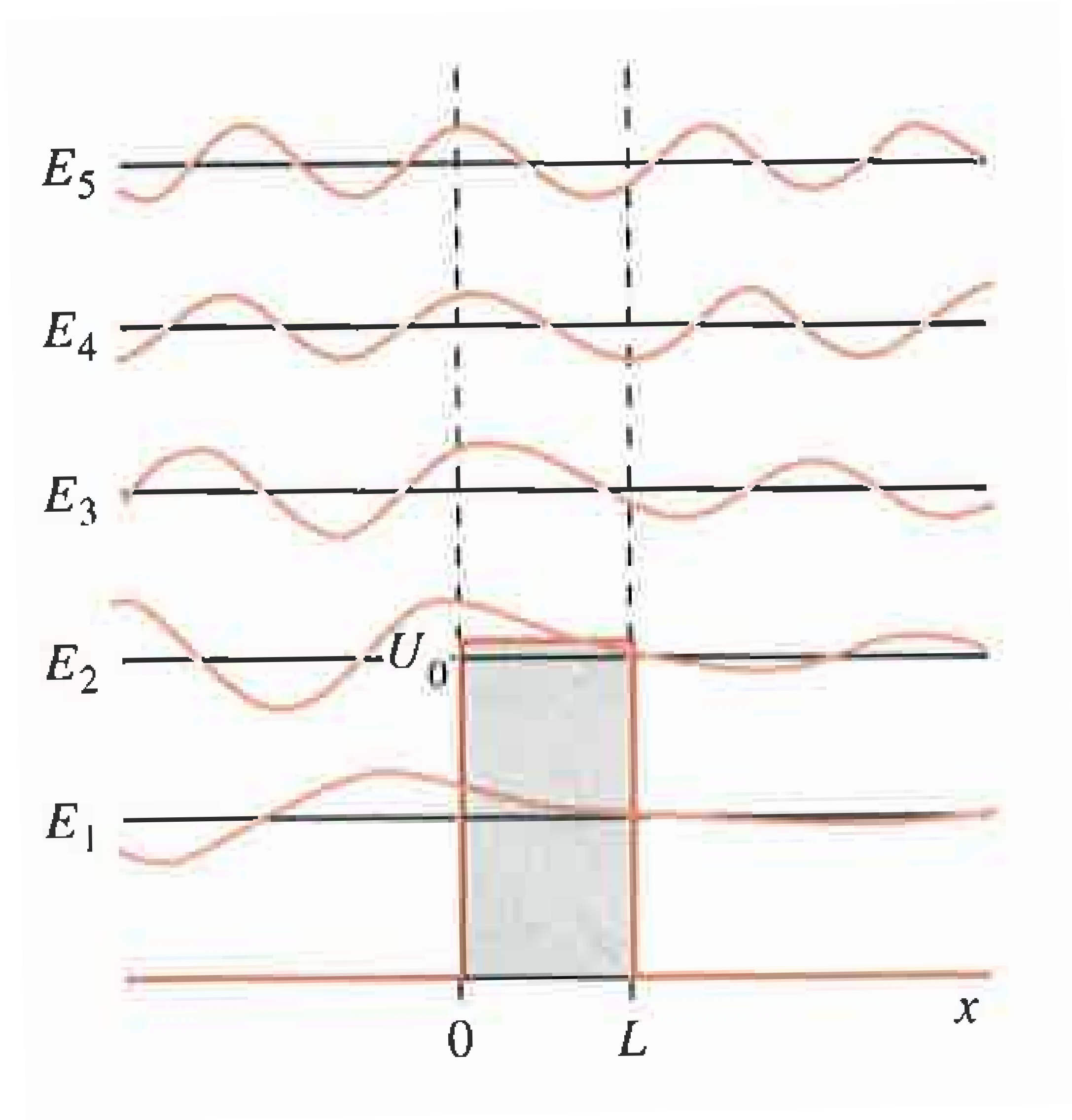
Again you would use smoothness conditions to solve. Here I just state results.

$$R = \frac{\sinh^2\left[\sqrt{2m(U_0 - E)}\,L/\hbar\right]}{\sinh^2\left[\sqrt{2m(U_0 - E)}\,L/\hbar\right] + 4(E/U_0)(1 - E/U_0)}$$

$$T = \frac{4(E/U_0)(1 - E/U_0)}{\sinh^2\left[\sqrt{2m(U_0 - E)}\,L/\hbar\right] + 4(E/U_0)(1 - E/U_0)} \qquad (6\text{-}16)$$

Tunneling Observations

- Wavelengths decrease as kinetic energy increases.
- At E_1 little transmission
- At E_2 wave decays less rapidly, larger "transmitted tail"
- At E_3 see evidence of reflection. Wave is smaller to the right of barrier than left. Wave is longer over barrier because of smaller speed.



Special Case: Tunneling through Wide Barriers

The discussion of tunneling in Harris is lengthy and detailed. You are **STRONGLY** encouraged to read through it!

What does it mean to be a "wide barrier"?

The length, L, of the barrier must be significantly larger than the penetration depth of the wave function in the barrier.

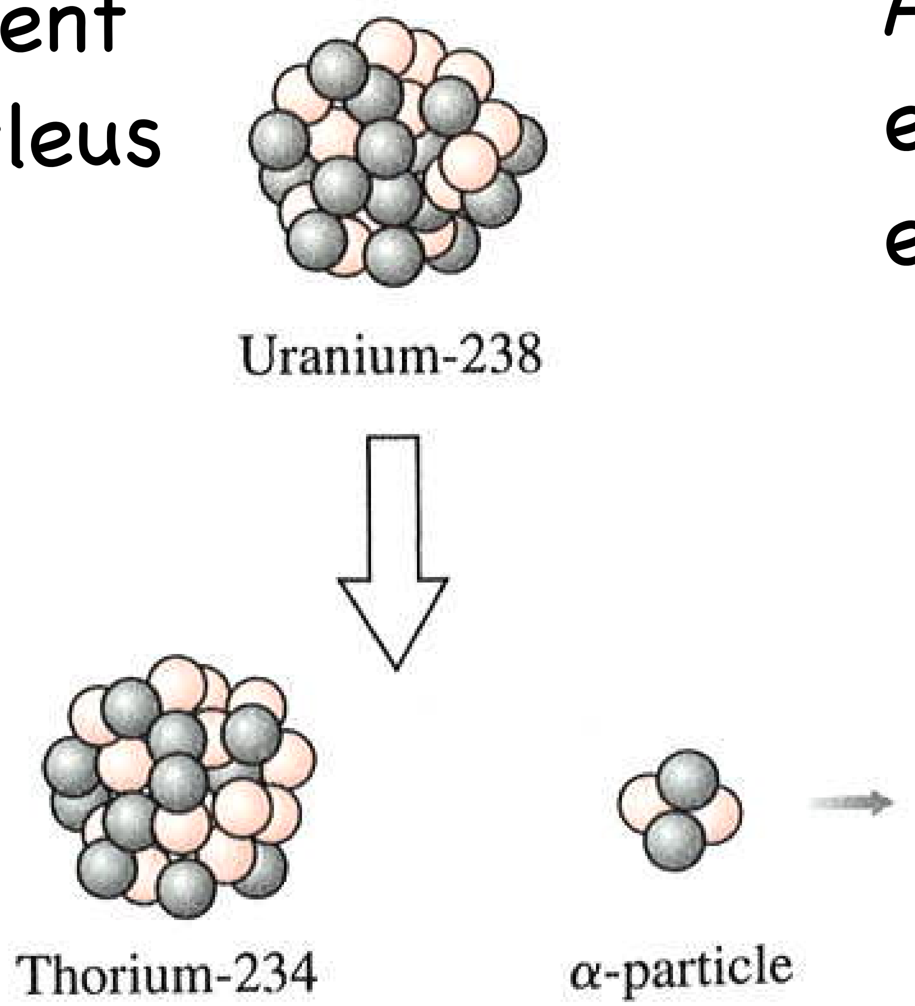
$$1 << \frac{L}{\delta} = \alpha L = \frac{\sqrt{2m(U_0 - E)}}{\hbar} L$$

In this case, the transmission probability becomes

$$T = 16 \frac{E}{U_0} (1 - \frac{E}{U_0}) e^{-2L\sqrt{2m(U_0 - E)}/\hbar}$$

Alpha Decay

parent
nucleus



daughter
nucleus

2 protons
2 neutrons

An unstable ^{238}U nuclei eliminates excess energy by spontaneously emitting an alpha particle.

^{238}U:

92 p + 146 n = 238 nucleons

^{234}Th:

90 p + 144 n = 234 nucleons

What is the minimum energy we would expect the alpha particle to have classically?

To be ejected from the nucleus, the alpha particle would need energy of 35 MeV to overcome the strong force.

$$U_{\text{elec}} = \frac{q_1 q_2}{4\pi\varepsilon_0 r} = \frac{(2 \times 1.6 \times 10^{-19}\,\text{C})(90 \times 1.6 \times 10^{-19}\,\text{C})}{4\pi(8.85 \times 10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)(7.4 \times 10^{-15}\,\text{m})}$$
$$= 5.6 \times 10^{-12}\,\text{J} = 35\,\text{MeV}$$

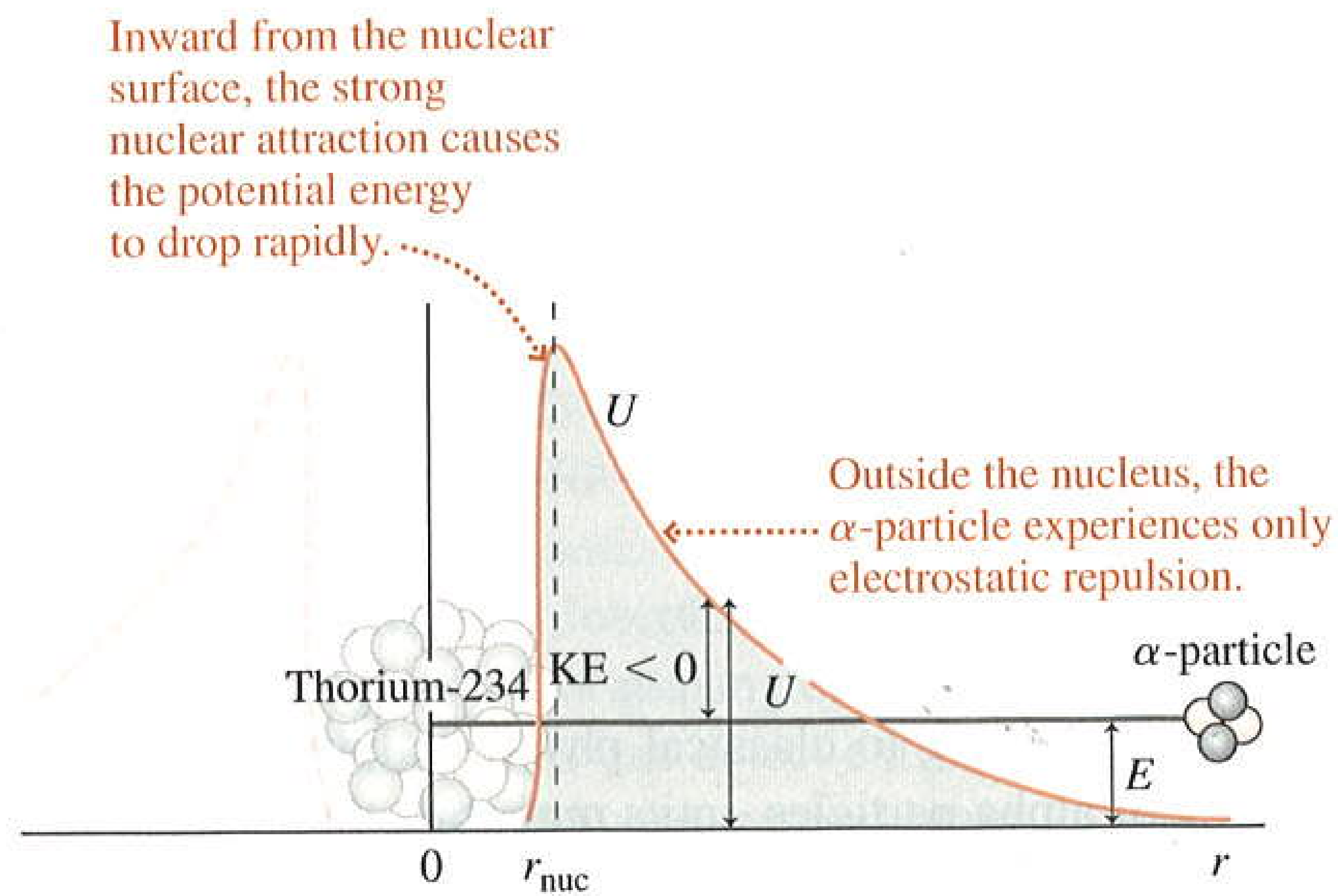
This means that a alpha particle should never have less than 35 MeV of energy.

Experimentally, the alpha particle from this decay has been found to be 4.3 MeV. How is this possible?

You don't have to have 35 MeV to escape the nucleus!

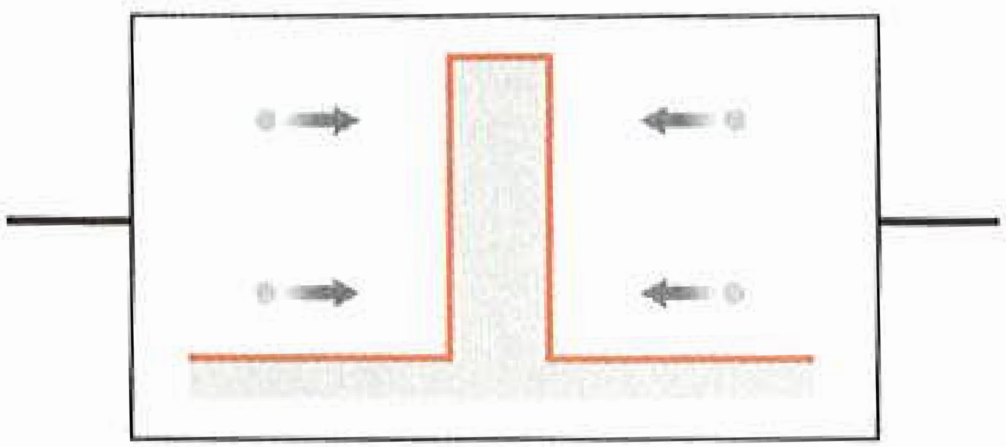
Quantum Mechanics:

You just need to have energy and run into the potential barrier a sufficient number of times to tunnel out of the nucleus.



Calculations of decay rates of the nucleus using tunneling agree perfectly with the experimentally the observed value.

The Tunnel Diode



$\Delta V = 0, I = 0$

Electrons on one end of the diode are separated by an electrostatic potential barrier.

Describe what happens when no voltage is applied.

Tunneling occurs equally in both directions. No, net flow.

Describe what happens when voltage (a potential difference) is applied.

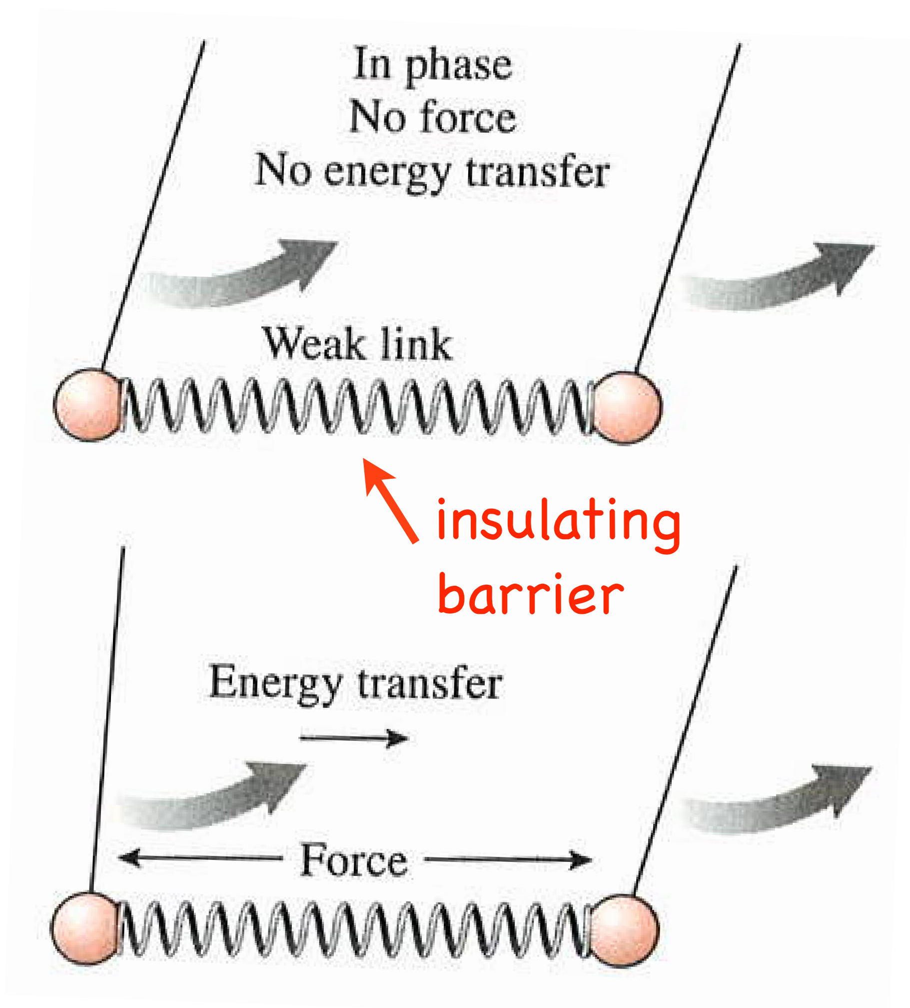
Right and left side tunneling are asymmetric. A net current flows.

The Tunnel Diode

- The distinct feature of the tunnel diode is not that voltage flows, but how the current varies with voltage.
- It does not steadily increase as voltage is increased.
- Applying voltage almost instantaneously changes the transmission rates -- a desirable feature at high frequencies.
- Applications in a variety of modern electronics.
  - trigger circuits in oscilloscopes
  - high speed counter circuits
  - pulse generating circuits
  - space applications (resistant to nuclear radiation)

SuperConductivity

Superconductivity is described by the long-distance pairing of electrons in a solid.

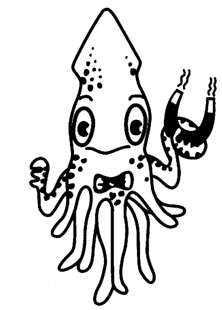


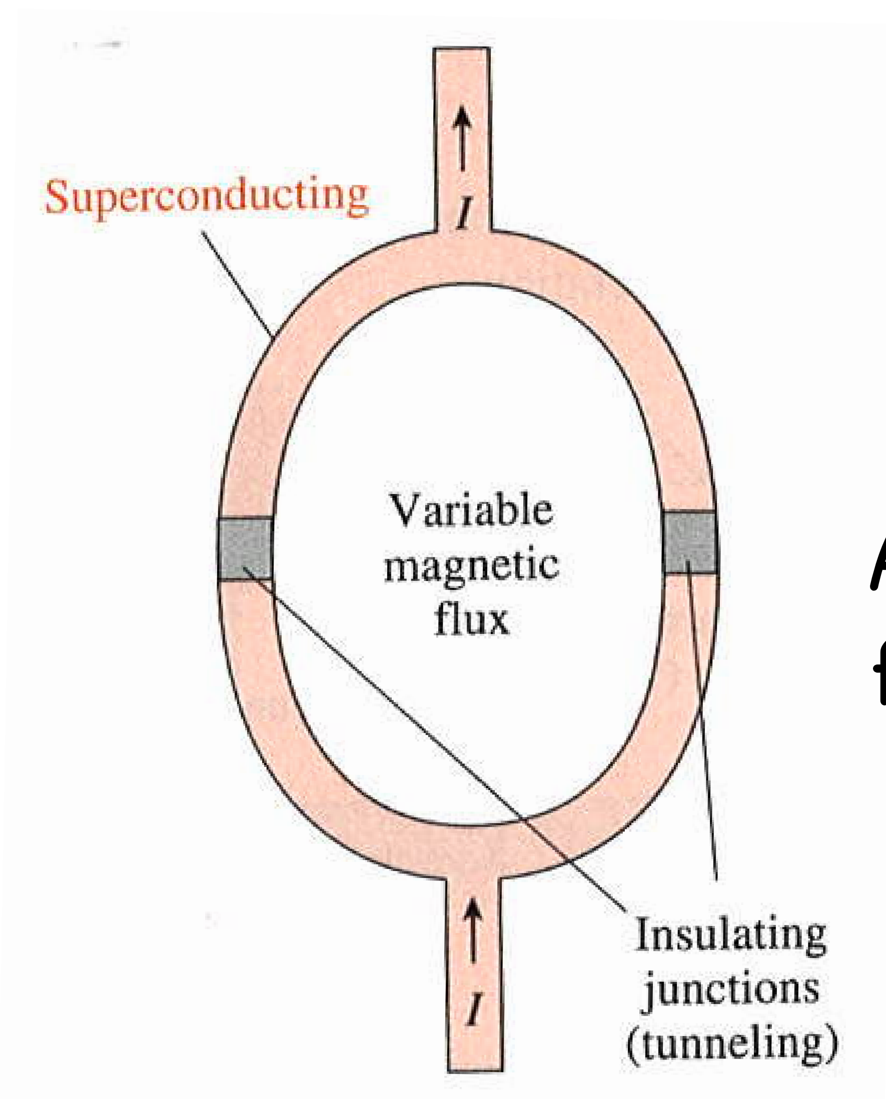
Josephson Junction

- composed of two semiconductors separated by an insulating barrier.
- Pairs of electrons tunnel through the barrier.
- Electrons are coupled to each other, similar to a weak spring connected to two pendulums.

SQUIDS

Superconducting QUantum Interface DeviceS





A SQUID combines two Josephson Junctions and makes the relationship between the electron pairs very sensitive to things like magnetic fields.

A small change in the magnetic flux, produces a change in current.

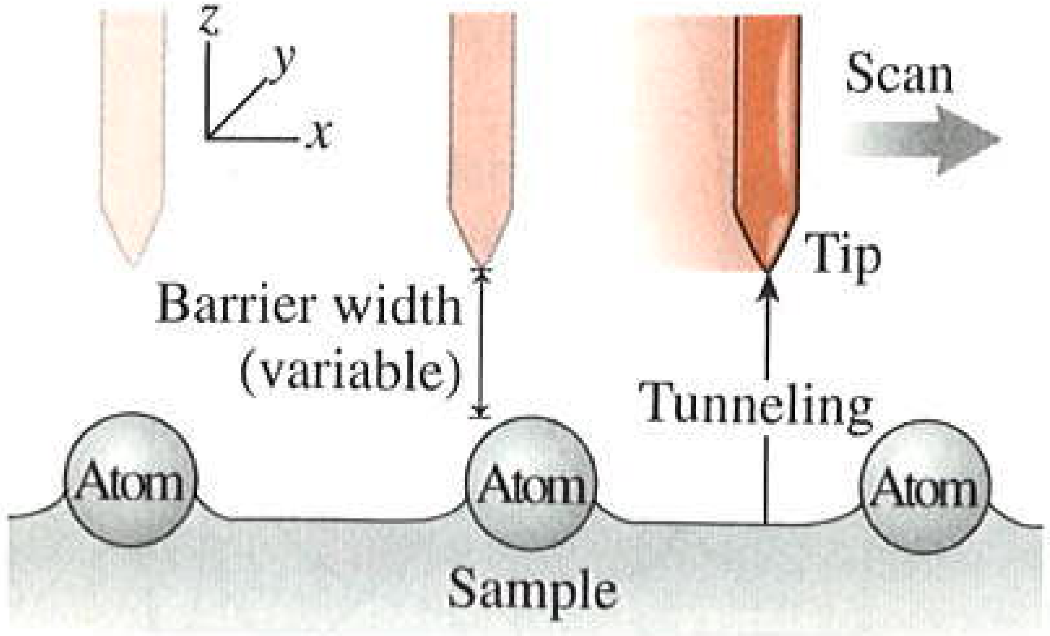
A SQUID can detect fields as small as 5×10^{-18} T (fridge magnets are 0.01 T). They can detect magnetic fields from the human heart and brain.

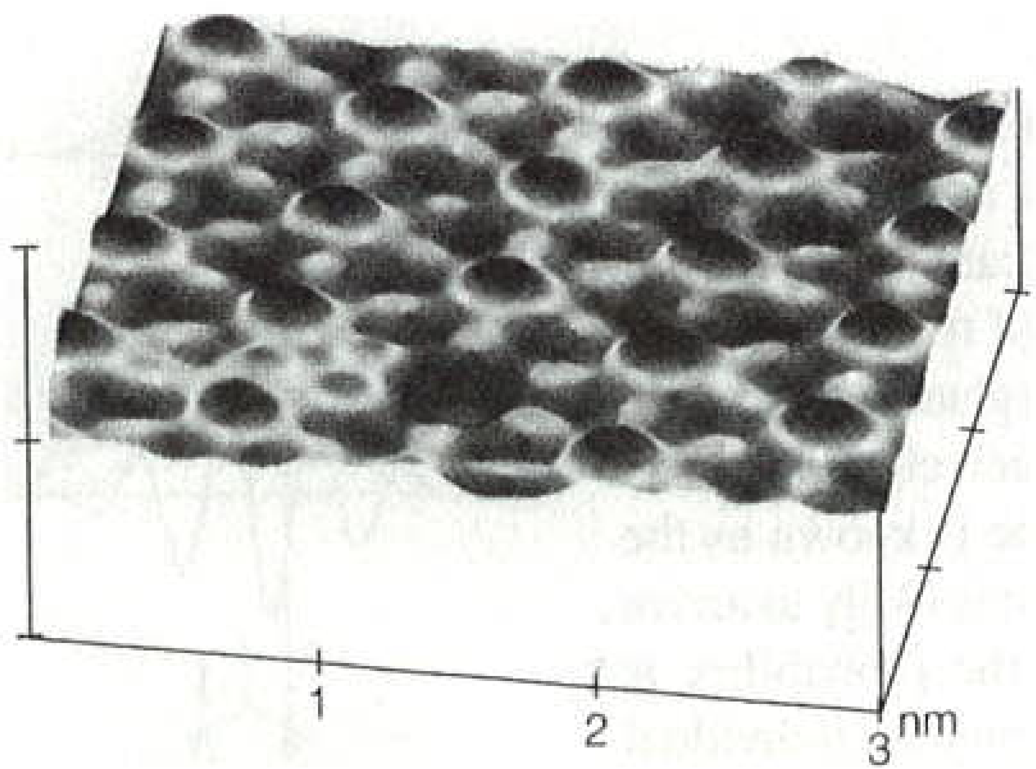
Scanning Tunneling Microscope

Slender tip is positioned near the sample.

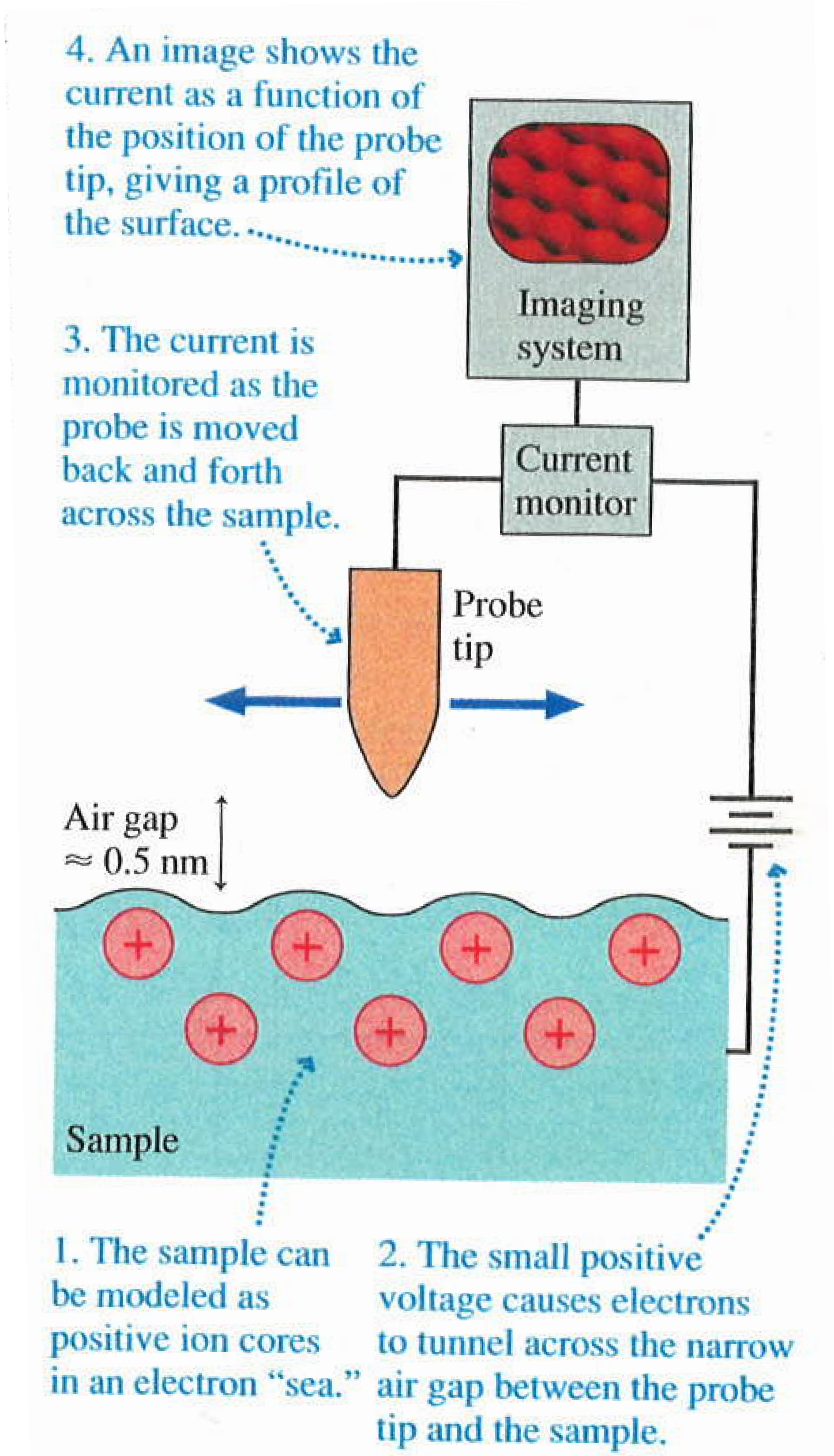
The space between the tip and the sample is a potential barrier.

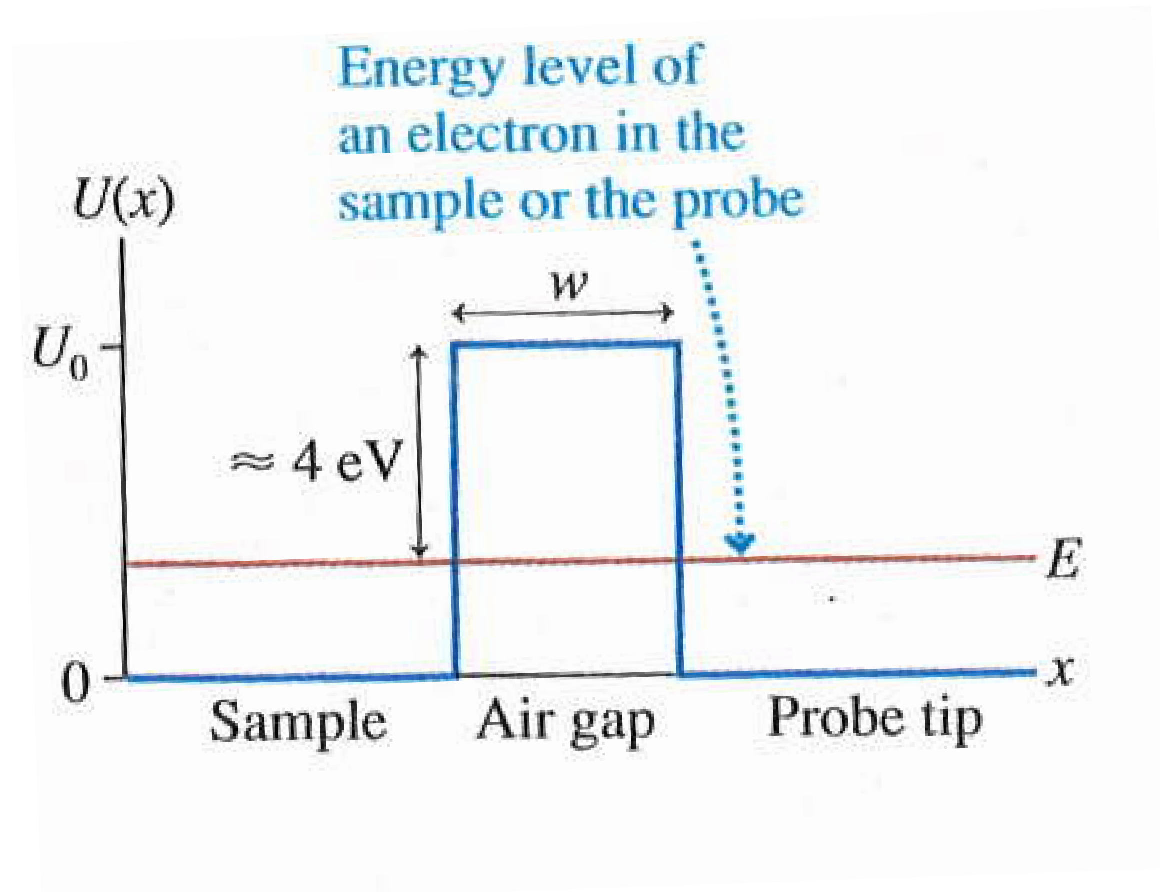
Variations in the tip-sample separation translate into measurable tunneling current.





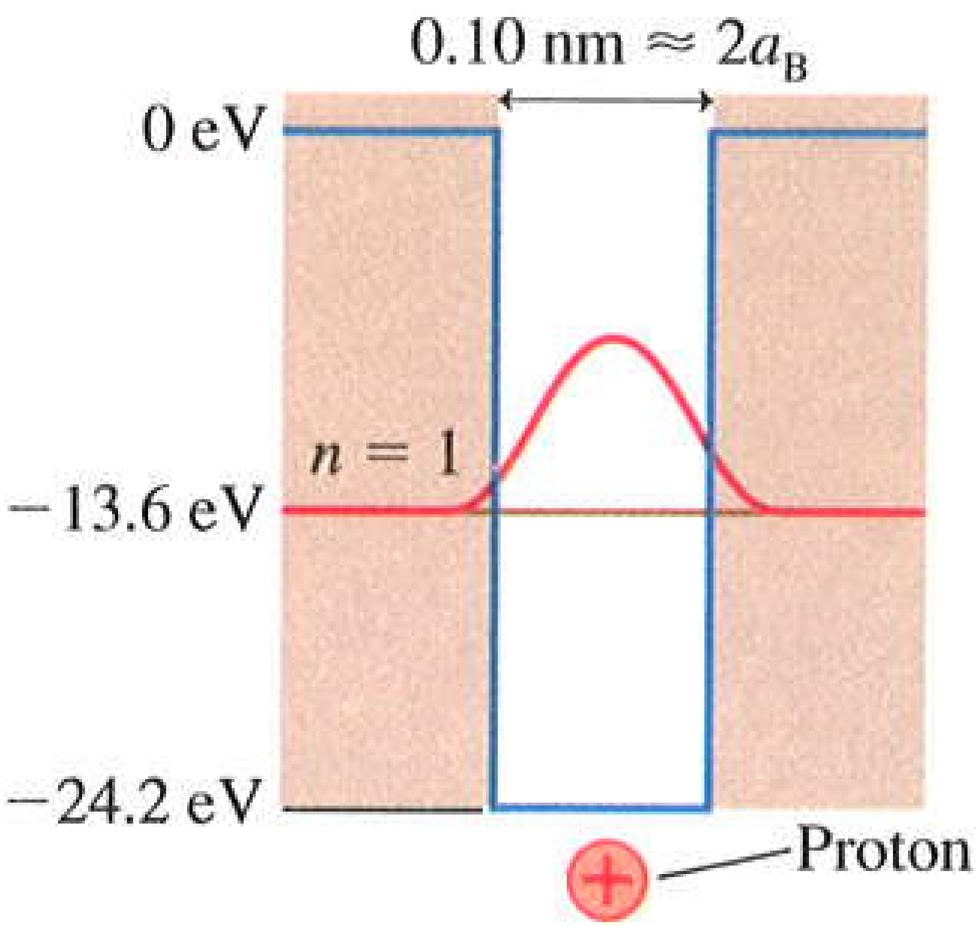
Scanning Tunneling Microscope



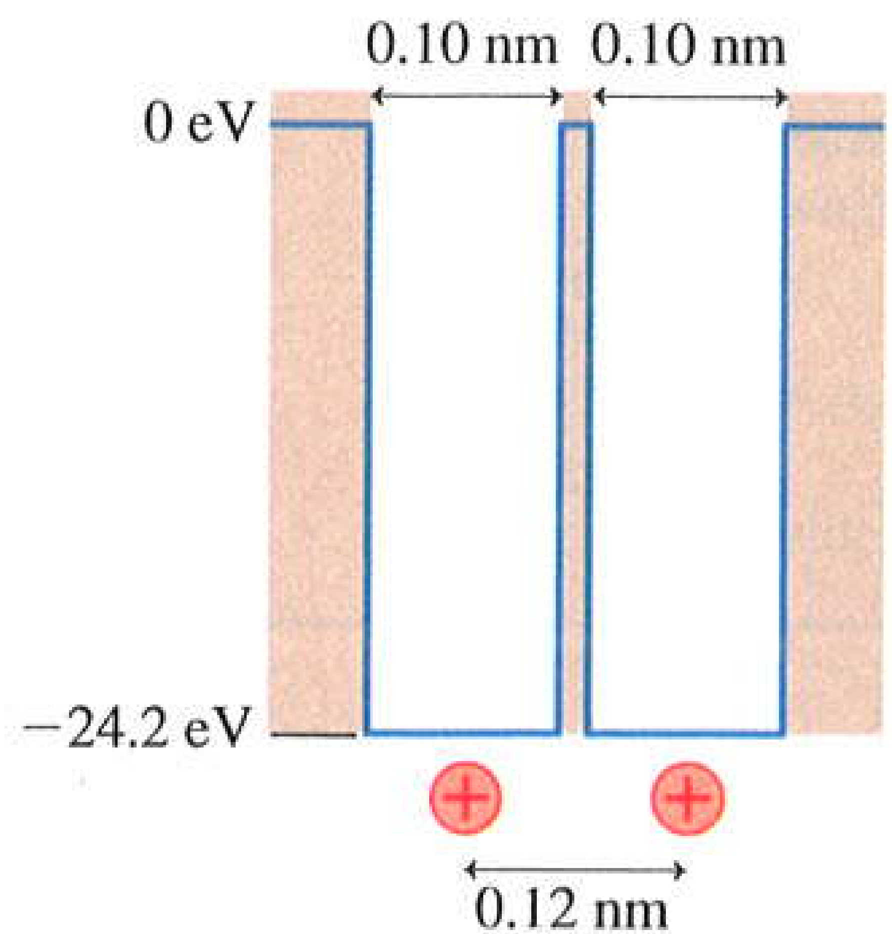


Variations in the tip-sample separation (barrier width) translate into measurable tunneling current.

Bonds in Molecules



Simple 1D model of hydrogen atom.



H_2^+ molecule modeled as an electron with two protons separated by 0.12 nm

Bonds in Molecules

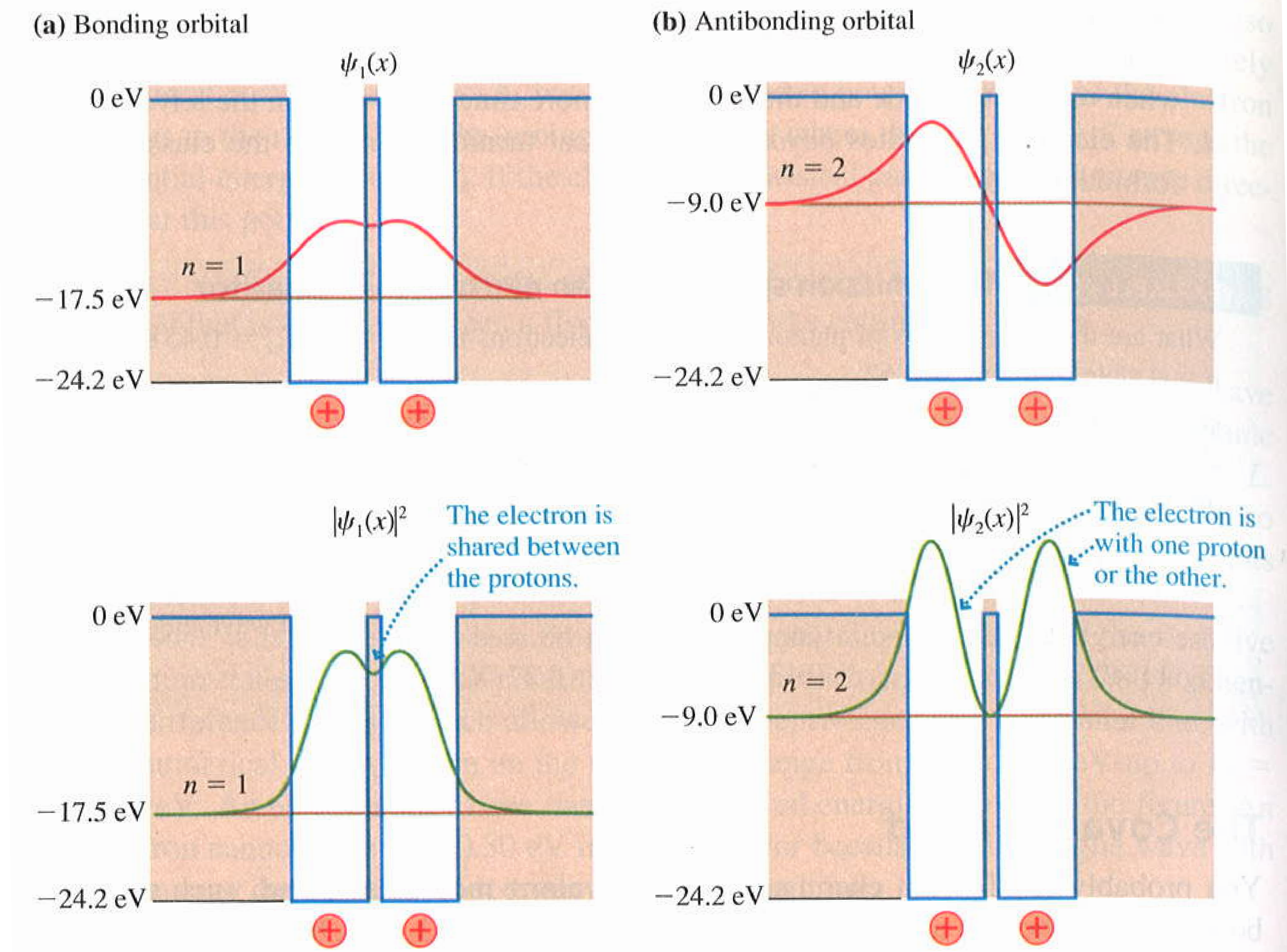


FIGURE 40.28 The wave functions and probability densities of the electron in H_2^+.

potential between two protons

$$U = \frac{q_1 q_2}{4\pi\epsilon_0 r}$$

$$E_{\text{mol}} = E_{\text{p-p}} + E_{\text{elec}} = \begin{cases} 12.0\text{ eV} - 17.5\text{ eV} = -5.5\text{ eV} & n = 1 \\ 12.0\text{ eV} - \ \ 9.0\text{ eV} = +3.0\text{ eV} & n = 2 \end{cases}$$

SWE Summary

- Bound States (chapter 5)
  - Particle in a box infinite walls, Particle in a box finite walls, harmonic oscillator
  - Energy was quantized.
- Unbound States (chapter 6)
  - potential steps, barriers and tunneling
  - Energy is NOT quantized.