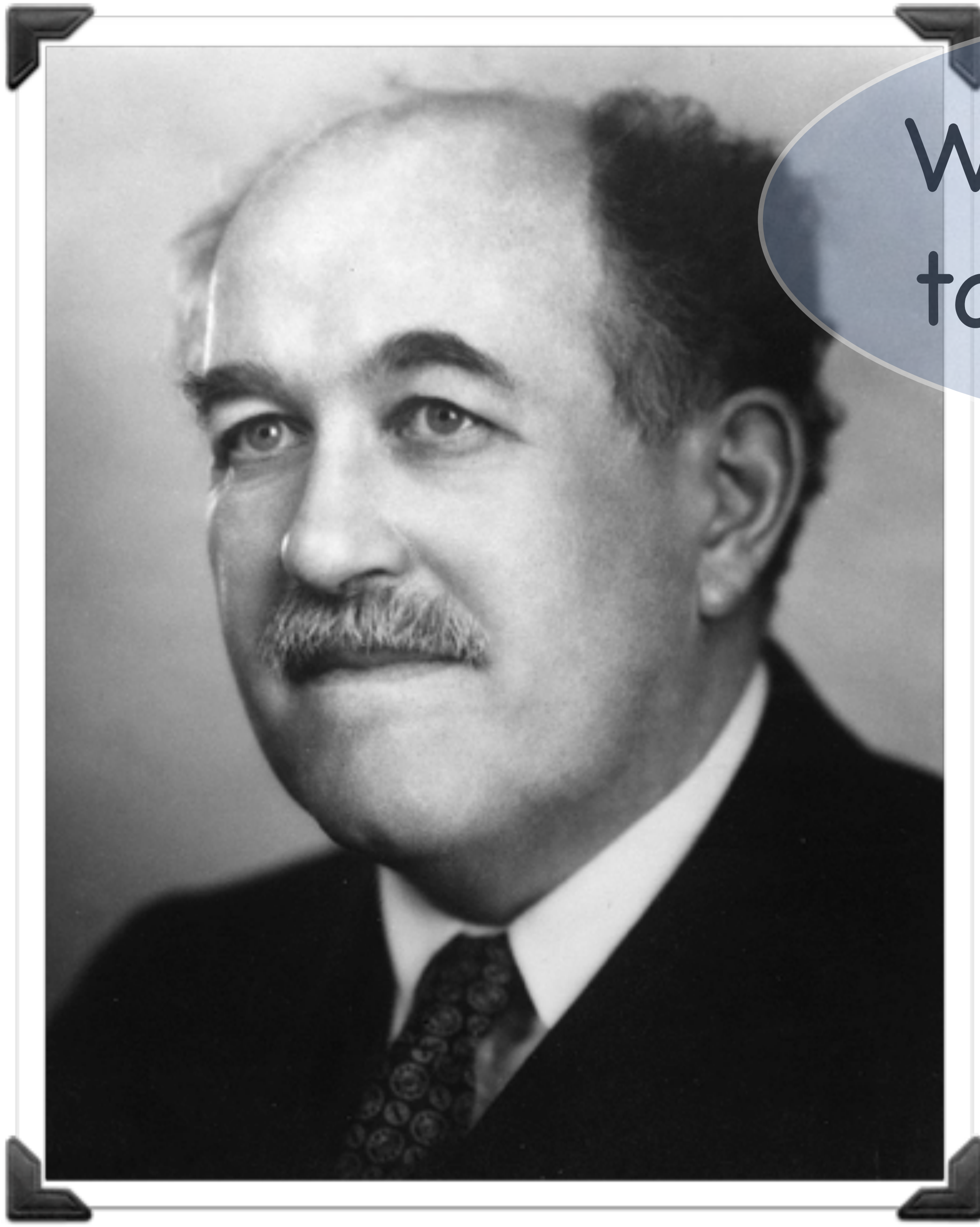
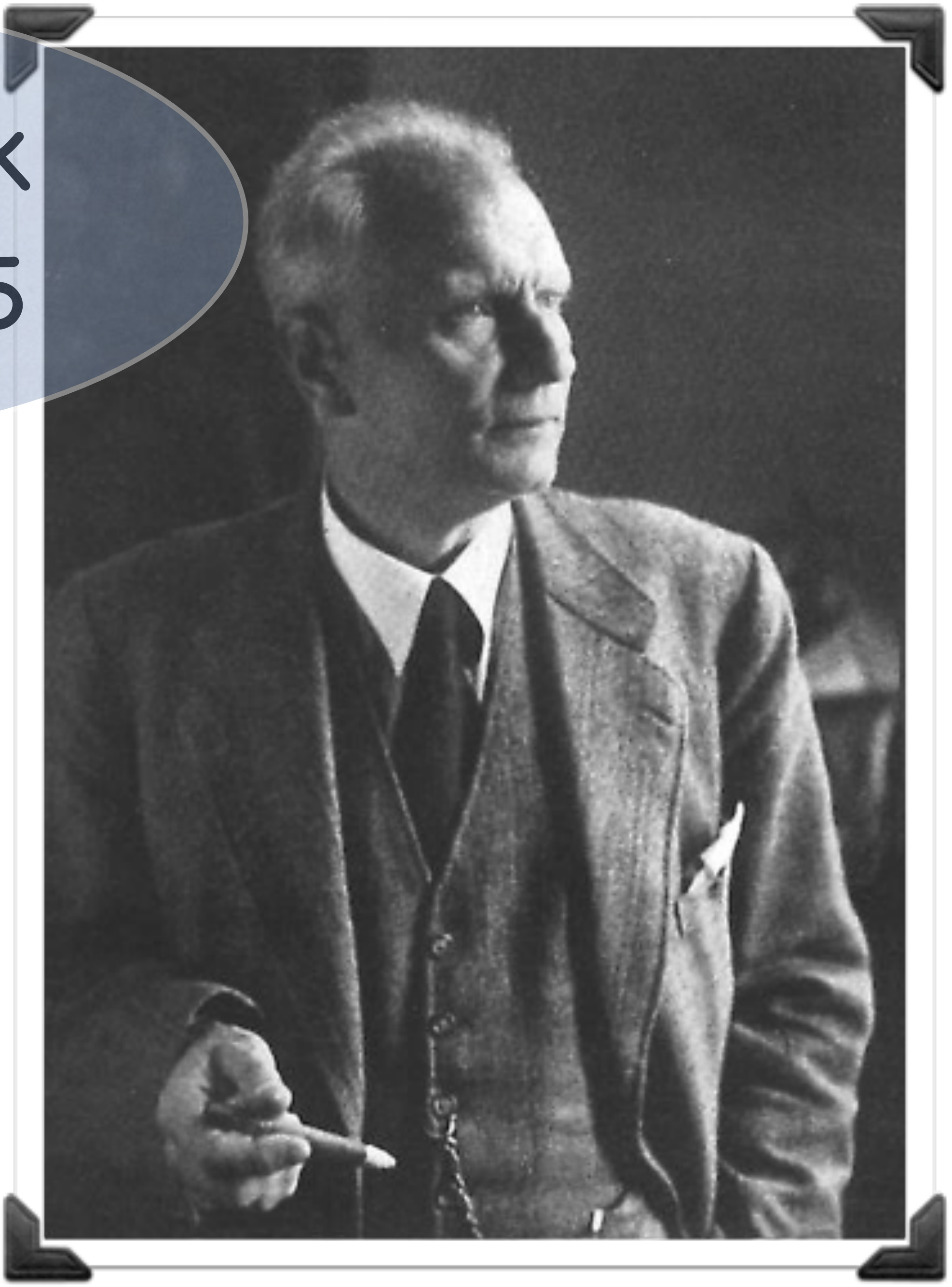
Welcome back to PHYS 3305



Otto Stern
1888 - 1969



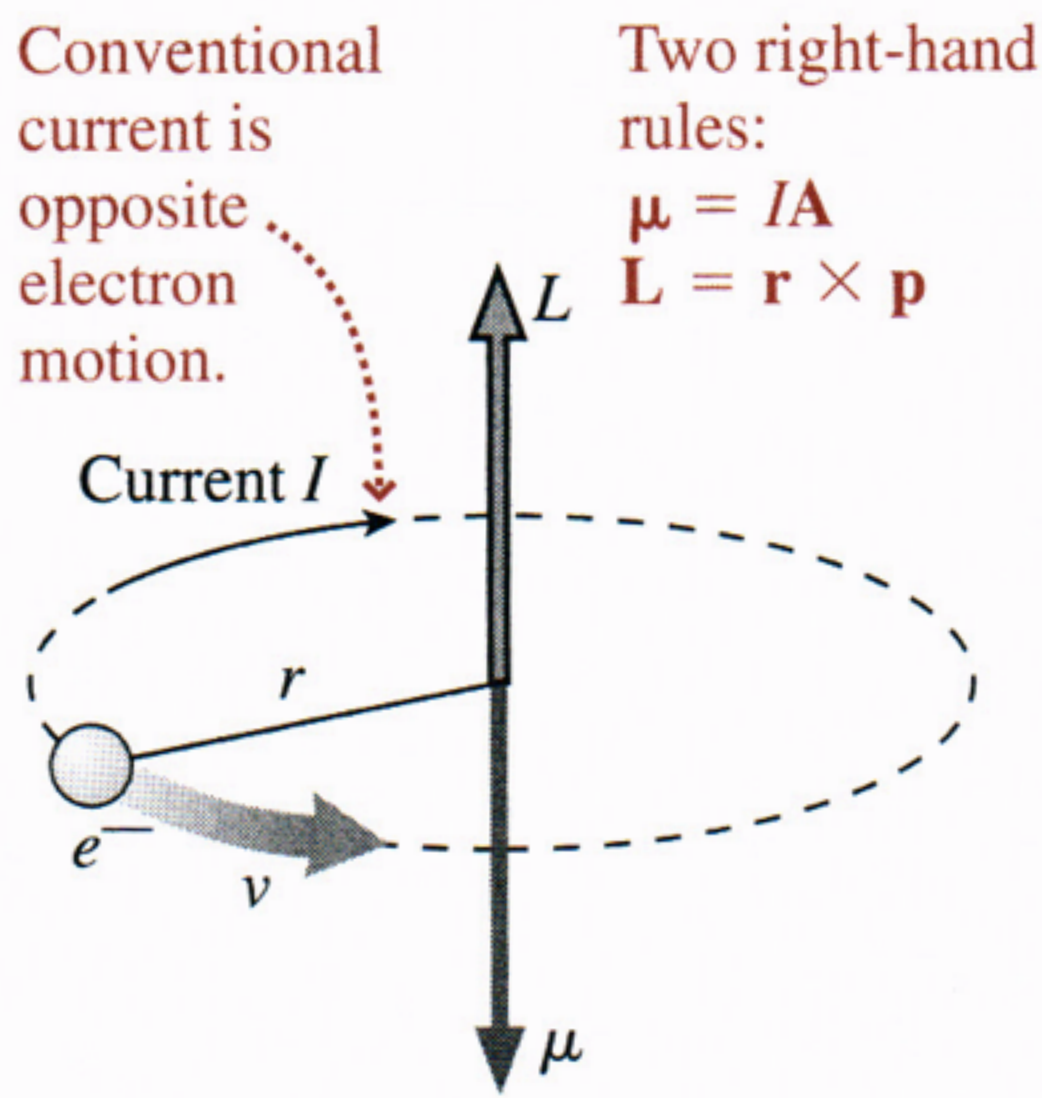
Walther Gerlach
1889 - 1979

Today's Lecture:

Angular Momentum Quantization

Stern-Gerlach Experiment

Review: Orbital Dipole Moments



The **magnetic moment** for an electron orbiting CCW is given by

$$\mu = IA$$

A = area enclosed by the loop
I = current produced
I = electric charge / period

$$\mu = \frac{e}{T}\pi r^2 = \frac{e}{2\pi r/v}\pi r^2 = \frac{e}{2}vr = \frac{e}{2m_e}(m_e vr) = \frac{e}{2m_e}L$$

$$\vec{\mu_L} = -\frac{e}{2m_e}\vec{L}$$

A **magnetic dipole** in a magnetic field experiences a torque.

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

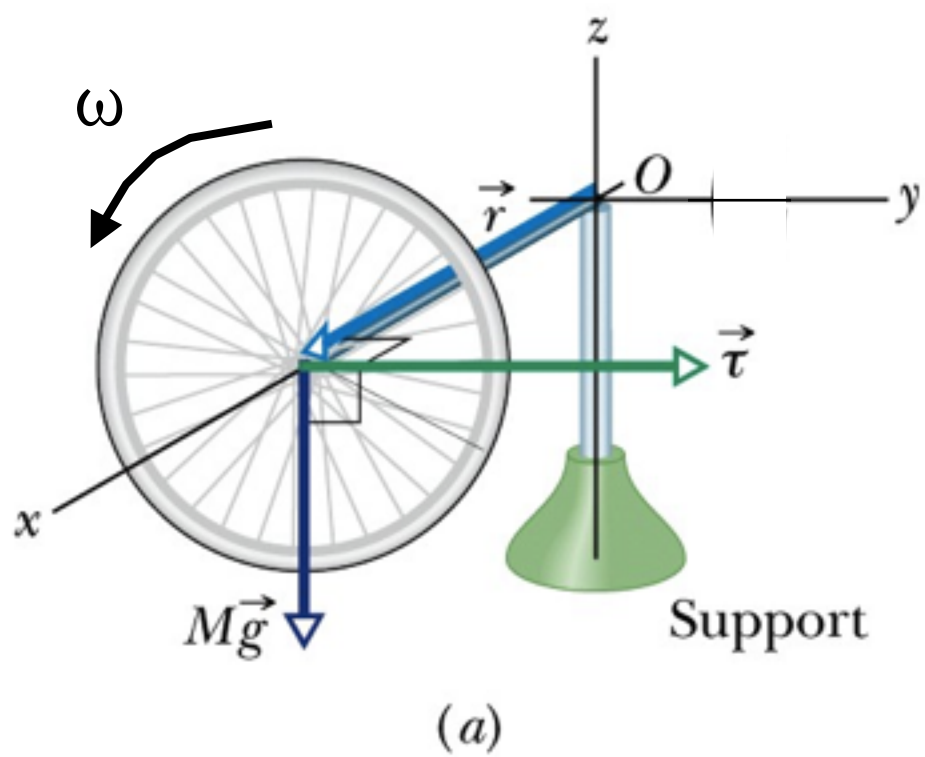
This torque causes the angular momentum to precess about the B-field line.

This works much like the precession of a gyroscope.

Gyroscope

- Device for measuring or maintaining orientation.
- Uses:
  - Wii remotes (wii plus)
  - navigation equipment
  - bicycles
- Mechanical gyroscope is essentially a spinning wheel whose axle is free to rotate.

Demo: Gyroscope



Initially L is directed to the left (positive x-direction.

The weight of the wheel produces a torque about the origin.

$$\vec{\tau} = \vec{r} \times \vec{F} \; = \vec{r} \times m\vec{g}$$

$|\vec{\tau}| = mgr$ (positive y-direction)

Demo: Gyroscope

- Angular momentum and torque are related to each other.

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

- The change in L is in direction only.

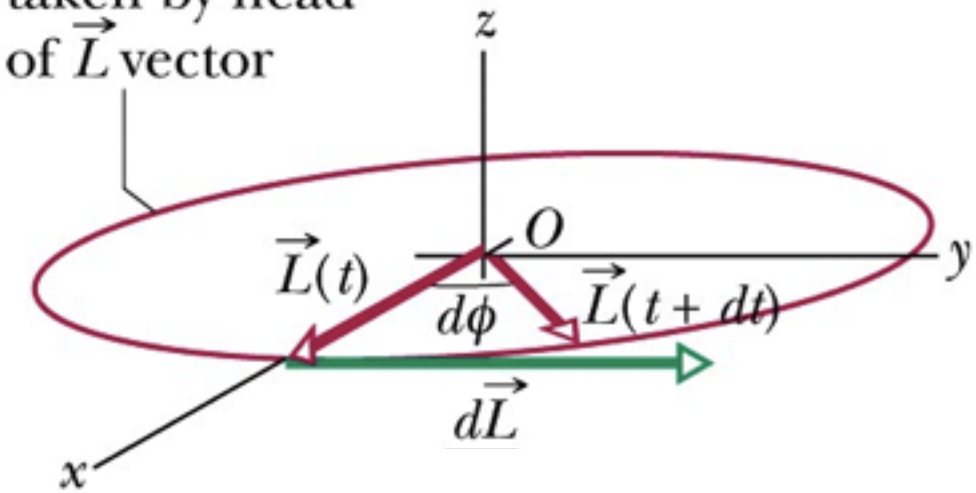
$$|L_i| = |L_f| = I\omega$$

- This change in direction is the same as the direction of the torque.

- The only way for L to change direction w/o a change in magnitude is for L to rotate about the z-axis.

Demo: Gyroscope

We can calculate the speed of the precession. Recall, for small angles arc length gives



$$dL = L d\phi$$

$$d\phi = \frac{dL}{L} = \frac{\sum \tau dt}{L}$$

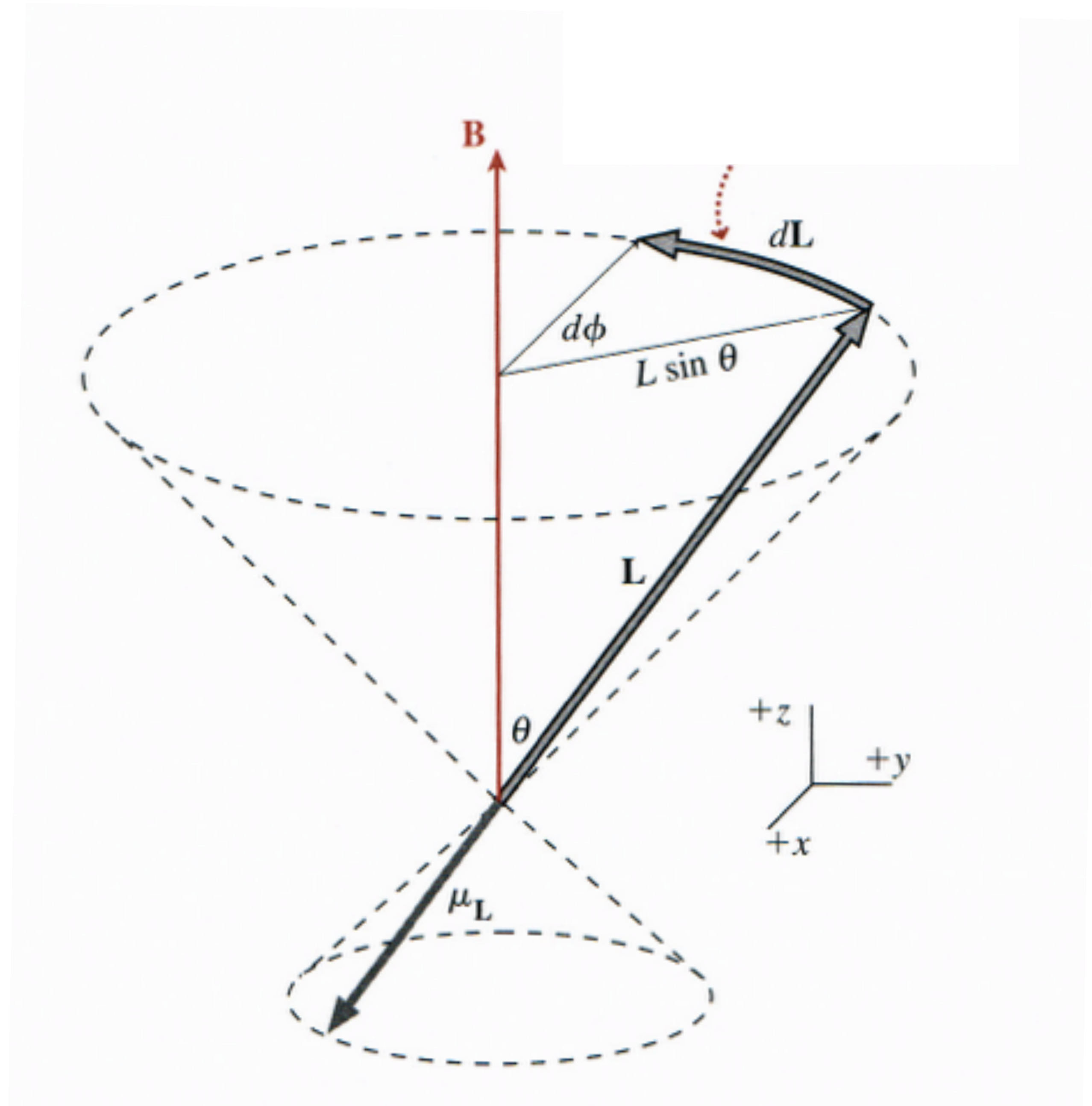
$$d\phi = \frac{mgr(dt)}{L}$$

$$\frac{d\phi}{dt} = \frac{mgr}{L} = \frac{mgr}{I\omega} \longrightarrow \Omega = \frac{mgr}{I\omega}$$

A dipole in a magnetic field also feels a torque: $|\vec{\tau}| = |\vec{\mu_L} \times \vec{B}|$

What direction is the torque?

Counterclockwise



We also know that torque is related to angular frequency by the rotational second law:

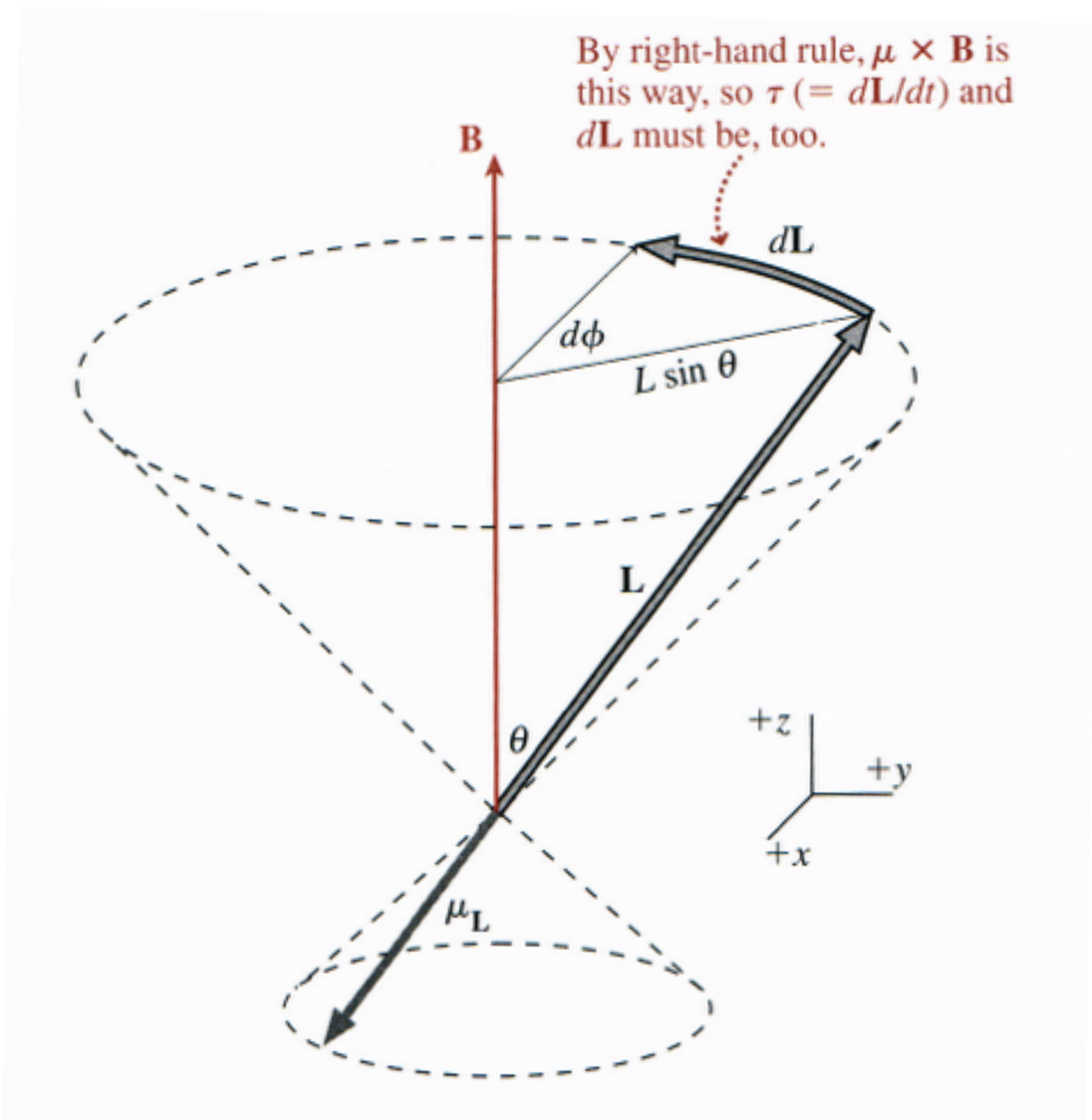
$$|\vec{\tau}| = \left|\frac{d\vec{L}}{dt}\right| = \left| -\frac{e}{2m_e}\vec{L} \times \vec{B}\right|$$

$$\frac{dL}{dt} = \frac{e}{2m_e}LB\sin\theta$$

$$\frac{L\sin\theta d\phi}{dt} = \frac{e}{2m_e}LB\sin\theta$$

$$\frac{d\phi}{dt} = \frac{eB}{2m_e}$$

In the case of the **magnetic dipole** the rate of precession is known as the **Larmor frequency** and is given by



$$\frac{d\phi}{dt} = \frac{eB}{2m_e}$$

Recall: The magnetic moment is in the opposite direction of L.

$$\vec{\mu_L} = -\frac{e}{2m_e}\vec{L}$$

The B-field is in the z-direction.

RH rule says that the change in L must be CCW.

Why do we care?

- Whichever direction B points (z-axis, by convention) becomes a special axis.
- Only the component of μ along the B-axis is fixed.
- μ and L are directly related.
- So, it follows that we can not observe the quantization of L along any other axis.

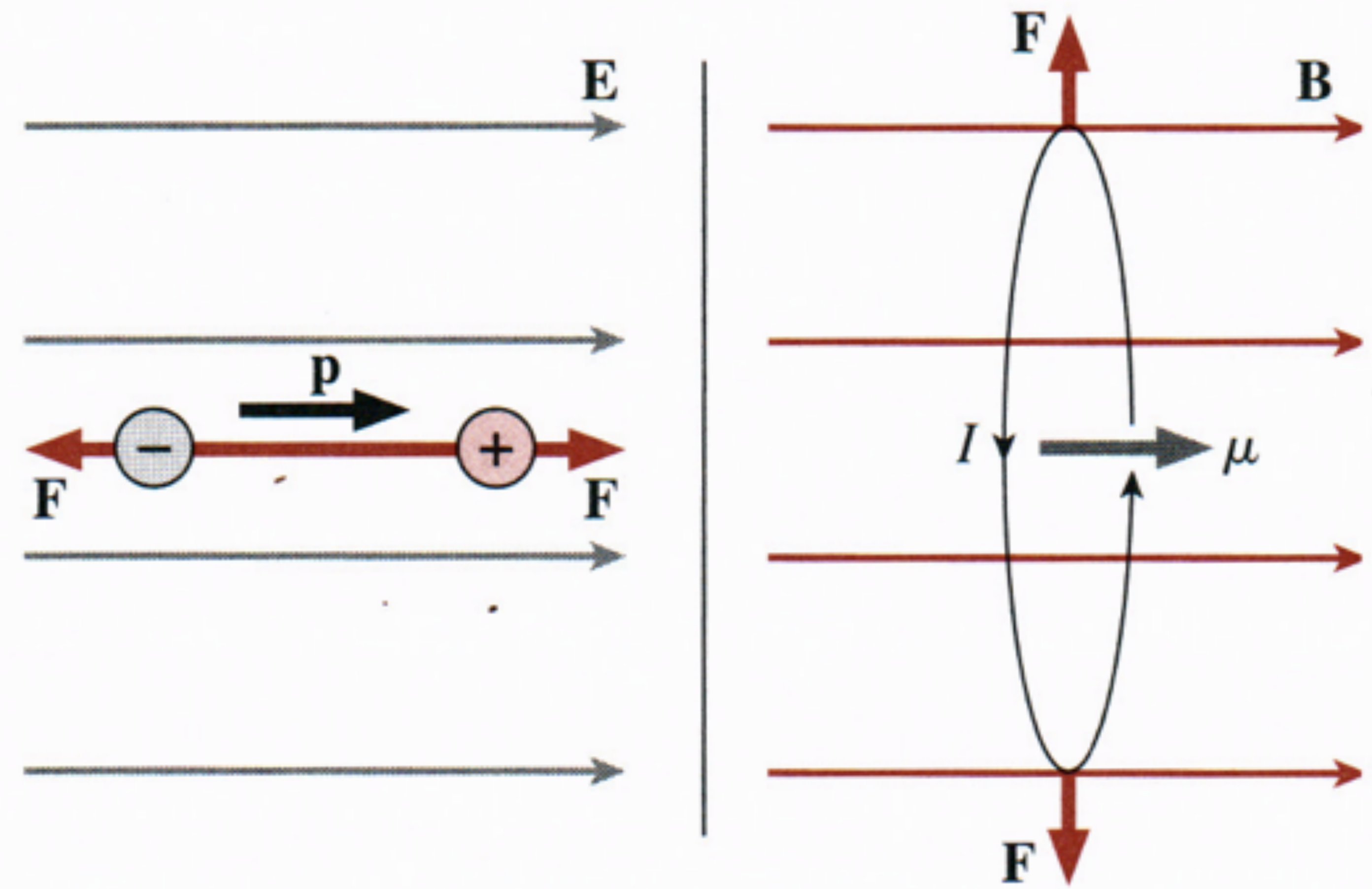
Uncovered intriguing truth from solving the hydrogen atom Schrödinger equation - only one component of angular momentum is quantized. This is in accordance with our experimental inability to observe quantization of more than one component.

All means of probing possible angular momentum quantization would employ some sort of inherently directional influence, so there is none that does not itself establish a preferred direction -- that is z-axis.

Magnetic Dipole in a Magnetic Field

What is the net force on our magnetic dipole in a uniform field?

Answer: zero

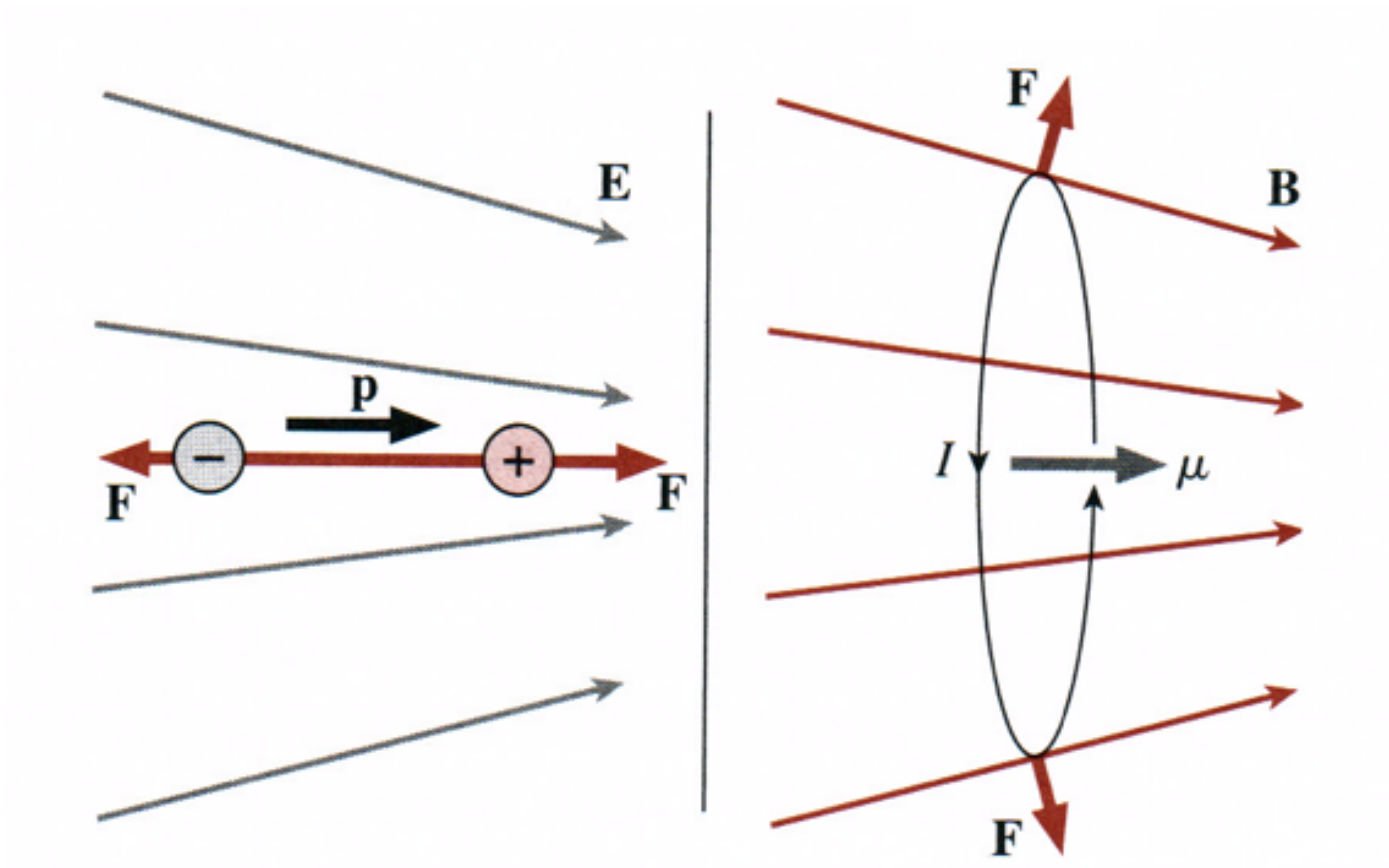


Dipoles in uniform fields $F_{net} = 0$

p = electric dipole moment

Magnetic Dipole in a Magnetic Field

Does the net force change if we immerse our magnetic dipole in a non-uniform magnetic field which is changing as a function of z?



Dipoles in nonuniform fields $F_{net} \neq 0$

Answer: Yes, the dipole will experience now experience a net force.

The potential energy of a magnetic dipole μ in an external magnetic field is given by

$$U = -\boldsymbol{\mu} \cdot \mathbf{B}$$

note: U is a scalar quantity

How are force and potential related?

$$\mathbf{F} = -\nabla(-\boldsymbol{\mu} \cdot \mathbf{B})$$

In our case, we said the magnetic field was changing in the z-direction -- thus we can simplify to

$$\mathbf{F} = \mu_z \frac{\partial B_z}{\partial z}\hat{\mathbf{z}} = \left(-\frac{e}{2m_e}L_z\right)\frac{\partial B_z}{\partial z}\hat{\mathbf{z}}$$

Substitute in $L_z = m_\ell \hbar$

$$\mathbf{F} = \left(-\frac{e}{2m_e} L_z \right) \frac{\partial B_z}{\partial z} \hat{\mathbf{z}}$$

$$\mathbf{F} = -\frac{e}{2m_e} m_\ell \hbar \frac{\partial B_z}{\partial z} \hat{\mathbf{z}} \qquad m_\ell = -\ell, \ldots, +\ell$$

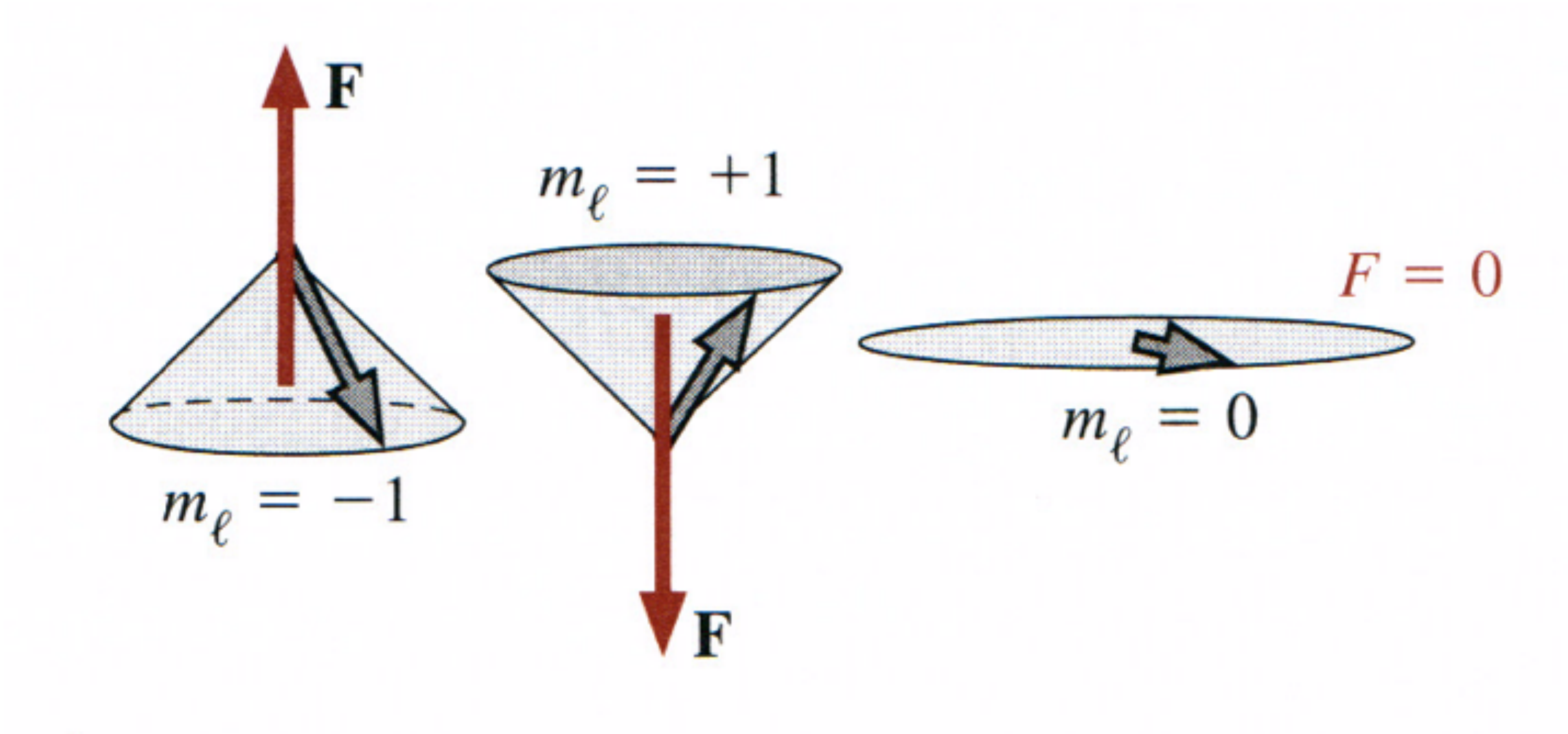
The **magnetic quantum number, m_l,** is an important factor for governing the effect of the magnetic field.

Question: Consider hydrogen in the ground state. What force do we expect and why?

Answer: F = 0. For the ground state, ℓ = 0, thus, m_ℓ would equal zero.

Question: What would we expect for hydrogen in the ℓ=1 state?

Answer: Either an upward or downward force depending on the sign of m_ℓ (or zero if m_ℓ=0).

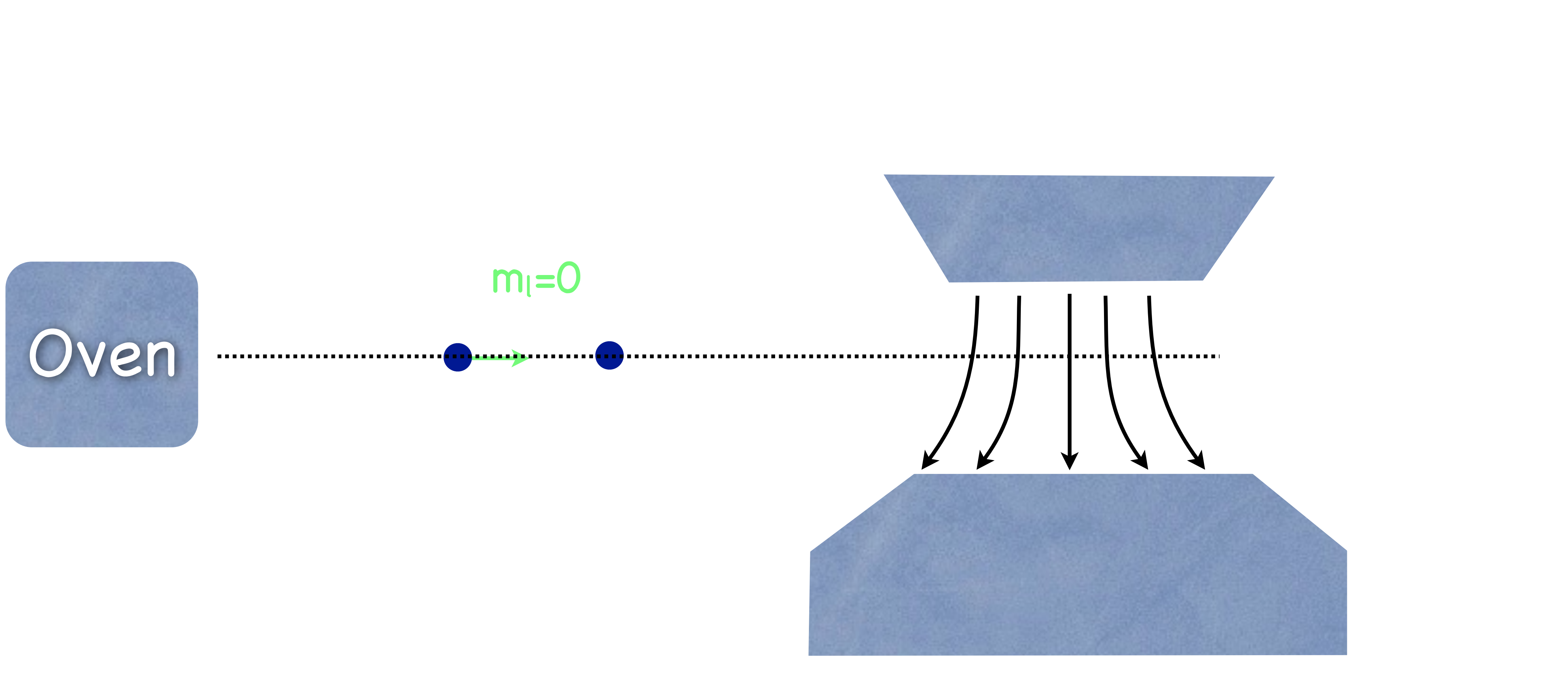


Stern-Gerlach Experiment

- Shoot a beam of atoms though a region of inhomogeneous magnetic field.
- Atoms get deflected according to the value of m_l.
- How many lines would we expect to see on a screen placed after the atoms exit the magnetic field?

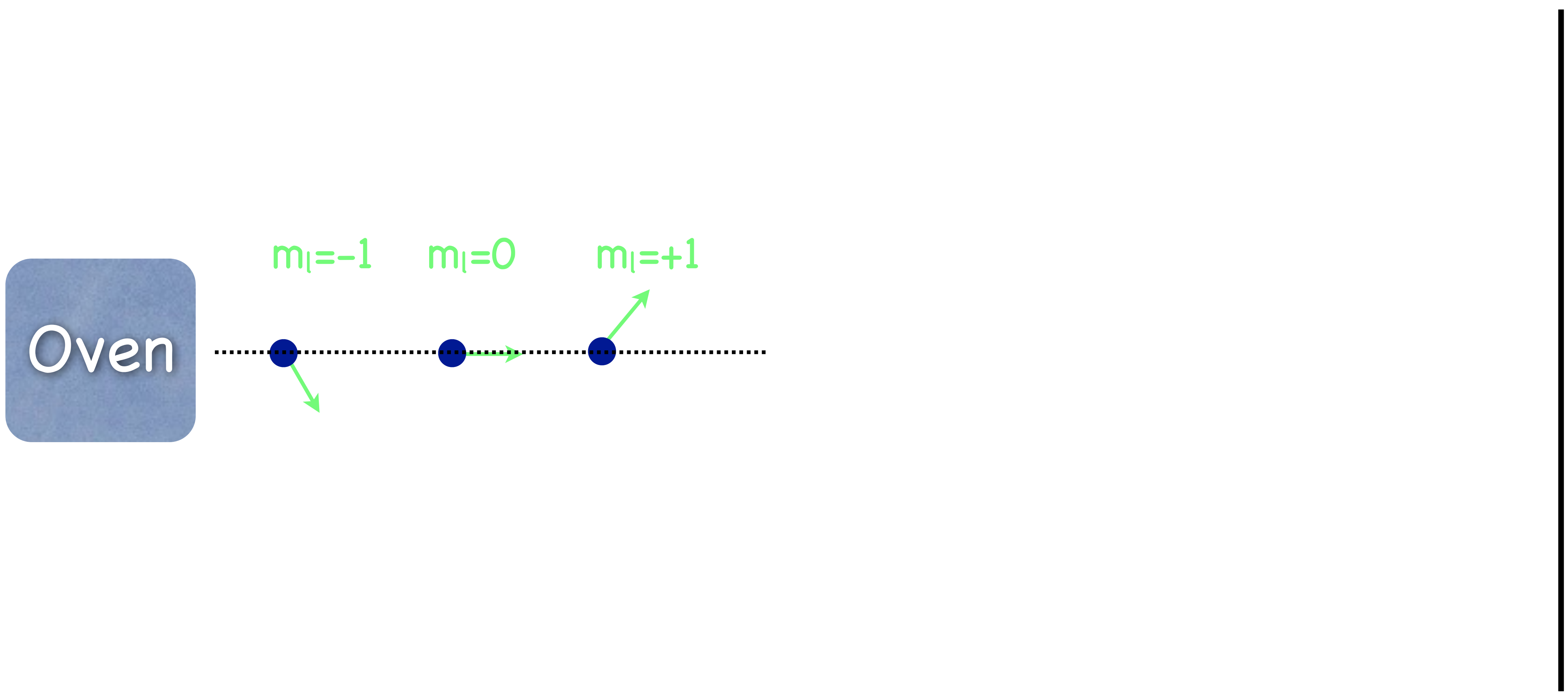
If we shoot a beam of H-atoms all in their ground state, how many spots would we expect to see on the screen?

- In their ground state n=1, l=0 and $m_l=0$: we expect one spot and no deflection.



What do we expect if we shoot a beam of H-atoms in the n=2 state though the magnetic field:

- For n=2, l=0, 1 and m_l= -1, 0, 1: we expect three spots.



Stern Gerlach:

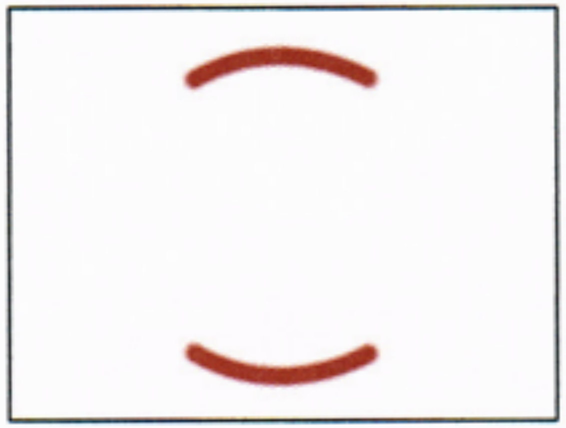
- Take hydrogen atoms in the l = 0 state and pass them through the magnetic field.
- Expect to see 1 line on the screen corresponding to the only possible m_l state (m_l = 0) (no magnetic moment).

What was seen -- 2 lines!

- Take hydrogen atoms in the l = 1 state and pass them through the magnetic field.
- Expect to see 3 lines on the screen corresponding to the 3 possible m_l states.

What was seen -- Only 2 lines!

$\ell = 0$ expectation



$\ell = 0$ observation

Stern Gerlach:

- Where did the $m_l = 0$ line go in the $l = 1$ experiment?
- And where did the lines come from in the $l = 0$ experiment?

This suggests a magnetic moment we had not considered.

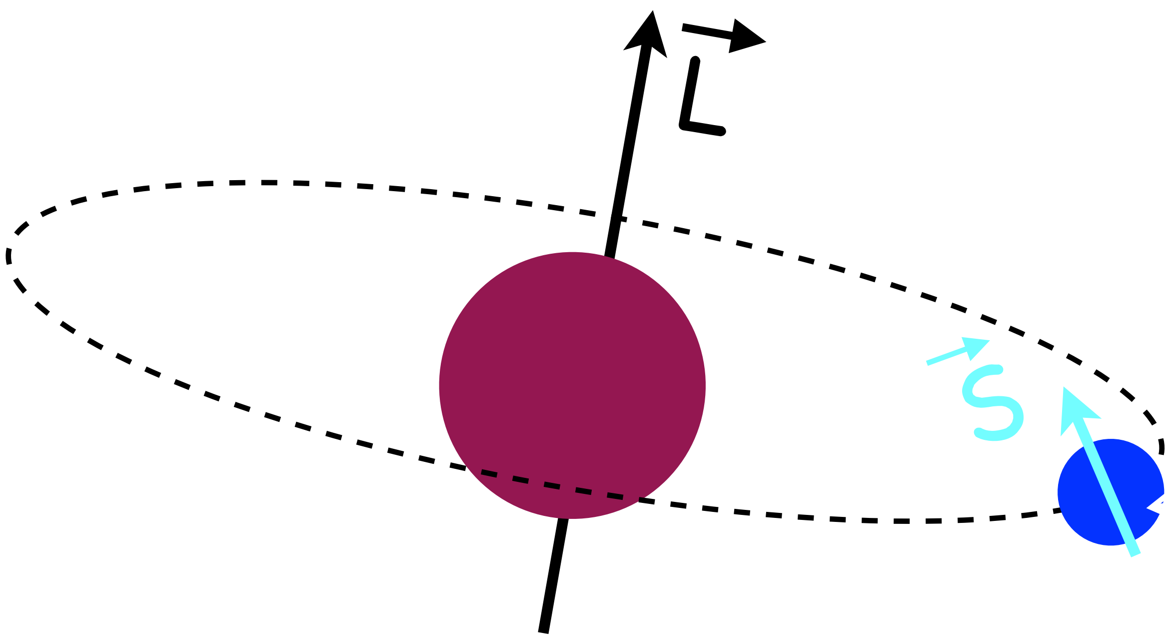
intrinsic magnetic dipole moment = this is inherent to the electron

if it has an intrinsic magnetic dipole moment, it must have an intrinsic angular momentum which is call **SPIN**.

Spin

- **Intrinsic angular momentum** is call **SPIN (S)**.
- An intrinsic property is one that is fundamental to a particle's nature and cannot be taken away (think mass and charge).
- What does that mean?
  - Originally, it was thought the electron was spinning. However, if that were the case, the intrinsic energy of the electron would be far greater.
  - Each particle with spin has an intrinsic magnetic moment, something that makes it respond under the influence of a magnetic field.

One way to visualize this in a classical sense is to think of the electron orbiting the nucleus in the same way Earth orbits the sun. The electron would then have both orbital angular momentum AND an intrinsic magnetic moment.



Similar to orbital angular momentum, the z-component of **intrinsic angular momentum** is quantized.

$$S_z = m_s \hbar \qquad m_s = -s, -s+1, \ldots, s-1, s$$

Spin is a new Quantum Number bringing the total number to 4. The **spin quantum number** is given by m_s.

$$(n, \ell, m_\ell, m_s)$$

Like total orbital angular momentum, the **magnitude of the particles intrinsic spin** is given by

$$S = \sqrt{s(s+1)}\hbar$$

The spin of many particles has been measured.

TABLE 8.1 Spins of some selected particles

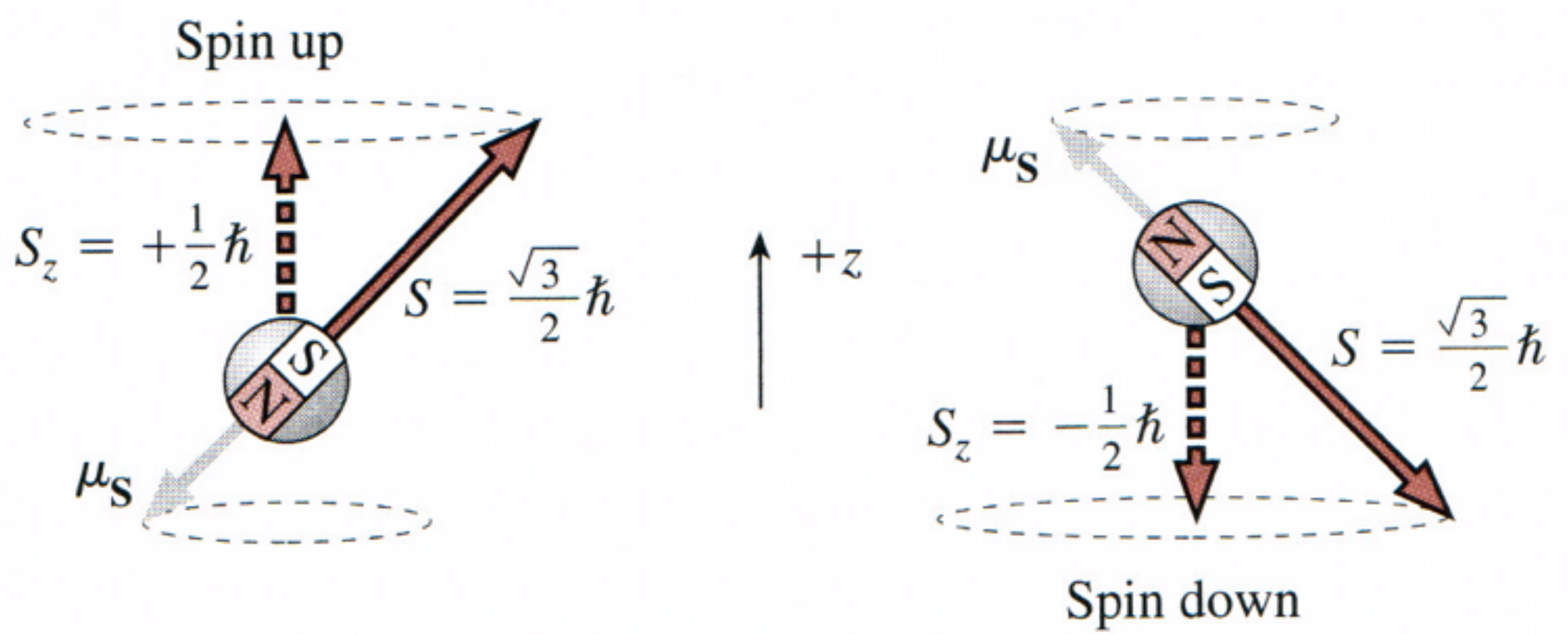
Fermions (Half-integral spin)		Bosons (Integral spin)	
Particle	s	**Particle**	s
Electron, e^-	$\frac{1}{2}$	Pion, π^0	0
Proton, p	$\frac{1}{2}$	Alpha particle, α (helium nucleus)	0
Neutron, n	$\frac{1}{2}$	Photon, γ	1
Neutrino, ν	$\frac{1}{2}$	Deuteron, d (bound n-p)	1
Omega, Ω^-	$\frac{3}{2}$	Graviton	2

Note that spin, m_s, may be integer or 1/2. However, it increments in integer numbers. Thus, for an electron, m_s can be -1/2 or +1/2 but NOT 0. For the W^- particle it can be -1, 0, or +1.

Calculate the two possible spin states (S and S_z) for an electron.

$$S = \sqrt{s(s+1)}\hbar \quad = \sqrt{\frac{1}{2}(\frac{1}{2}+1)}\hbar \quad = \frac{\sqrt{3}}{2}\hbar$$

$$S_z = m_s\hbar \quad = \frac{1}{2}\hbar \quad \text{or} \quad -\frac{1}{2}\hbar$$



Revisit Wave Functions

Now that we have a 4th quantum number, we need to adjust our wave functions.

$$\psi_{n,\ell,m_\ell,m_s} = \psi_{n,\ell,m_\ell} m_s$$

spacial state

spin state

For an electron (spin 1/2), m_s can take on only two values:

$$\psi_{n,\,\ell,\,m_\ell,\,+\frac{1}{2}} = \psi_{n,\,\ell,\,m_\ell}(r,\theta,\phi)\uparrow \qquad \text{Spin up}$$

$$\psi_{n,\,\ell,\,m_\ell,\,-\frac{1}{2}} = \psi_{n,\,\ell,\,m_\ell}(r,\theta,\phi)\downarrow \qquad \text{Spin down}$$

Example: Stern-Gerlach

In a Stern-Gerlach type of experiment, the magnetic field varies with distance in the z direction according to $dB_z/dz = 1.4$ T/mm. Silver atoms travel a distance $x = 3.5$ cm through the magnet. The most probable speed of the atoms emerging from the oven is $v = 750$ m/s. Find the separation of the two beams as they leave the magnet. The mass of a silver atom is 1.8×10^{-25} kg, and its magnetic moment is about 1 Bohr magneton.

Hint: Use a combination of what we just learned about Forces and B-fields and what we know of kinematics to solve this problem.

Step 1: Find the force felt by the atom.

The magnetic field is acting only in the z-direction. So, we can use the expression we just derived to relate the magnetic moment and the magnetic field to the force.

$$\mathbf{F} = \mu_z \frac{\partial B_z}{\partial z} \hat{\mathbf{z}}$$

Step 2: Relate force to the acceleration of the atom.

The acceleration of a silver atom of mass m as it passes through the magnet is

$$a = \frac{F_z}{m} = \frac{\mu_z (dB_z/dz)}{m}$$

Step 3: The vertical deflection of the beam can be found from kinematics.

$$a = \frac{F_z}{m} = \frac{\mu_z(dB_z/dz)}{m}$$

$$\Delta z = \frac{1}{2}at^2$$

The time, t, to transverse the magnet is given by

$$t = \frac{x}{v} \longrightarrow \Delta z = \frac{1}{2}a(\frac{x}{v})^2$$

The spread in the beam is two times the distance Δz.

$$d = \frac{\mu_z(dB_z/dz)x^2}{mv^2}$$

$$= \frac{(9.27 \times 10^{-24}\ \text{J/T})(1.4 \times 10^3\ \text{T/m})(3.5 \times 10^{-2}\ \text{m})^2}{(1.8 \times 10^{-25}\ \text{kg})(750\ \text{m/s})^2}$$

$$d = 0.16mm$$

A particle's intrinsic magnetic dipole moment is related to its intrinsic angular momentum.

$$\vec{\mu_s} = g\frac{q}{2m}\vec{S}$$

q = charge
m = mass
g = gyromagnetic ratio

NOTE: μ_S is related to the intrinsic angular momentum, S
μ_L is related to the orbital angular momentum, L

Values of g depend upon the particle. For an electron the value is close to 2. For a proton it is 5.6.

The predominant effect in the Stern-Gerlach experiment is due to the hydrogen atom's electron. Why?

Although the gryromagnetic ratio is greater for the proton than for the electron, the mass of the proton is much, much larger than the mass of the electron. Thus, it's magnetic moment is quite small compared to that of the electron.

Summary:

Intrinsic Angular Momentum is given the name SPIN, $(\vec{S})$.

$$S = \sqrt{s(s+1)}\hbar$$ (length of spin vector)

Remember: an intrinsic property is one that is fundamental to the particles nature.

The magnitude of a particle's spin vector depends on a dimensionless value s.

TABLE 8.1 Spins of some selected particles

Fermions (Half-integral spin)		**Bosons** (Integral spin)	
Particle	s	**Particle**	s
Electron, e^-	$\frac{1}{2}$	Pion, π^0	0
Proton, p	$\frac{1}{2}$	Alpha particle, α (helium nucleus)	0
Neutron, n	$\frac{1}{2}$	Photon, γ	1
Neutrino, ν	$\frac{1}{2}$	Deuteron, d (bound n-p)	1
Omega, Ω^-	$\frac{3}{2}$	Graviton	2

s is not a quantum number and can not take on different values!

Summary:

Similar to orbital angular momentum, the z-component of **intrinsic angular momentum** is quantized.

$$S_z = m_s \hbar$$

$$m_s = -s, (-s+1), ...(s-1), s$$

Spin is a new Quantum Number bringing the total number to 4. The **spin quantum number** is given by **m_s**.

$$(n, \ell, m_\ell, m_s)$$

The end
(for today)