Assume that a muon is created in the atmosphere 3 km above Earth’s surface, traveling downward at 0.98c. It survives 2.2 µs in its own reference frame before decaying.

Define the S reference frame as the surface of Earth with the x-direction along the path of the muon. The S’ reference frame is that of the muon.

This means that the muon is moving at +v as seen from Earth and Earth is moving at -v as seen from the muon.
Classical Physics:

1. How far will the muon travel before it decays?

\[ d = vt = 0.98(3.00 \times 10^8 \text{m/s})(2.2 \times 10^{-6} \text{s}) \]

\[ d = 647 \text{m} \]

The muon would decay before reaching Earth’s surface.

2. How much longer would it have to live in order to reach Earth?

\[ t = \frac{d}{v} = \frac{3000 \text{m} - 647 \text{m}}{0.98(3.00 \times 10^8) \text{m/s}} \]

\[ t = 8.0 \times 10^{-6} \text{s} = 8.0 \mu \text{s} \]
Special Relativity:

1. According to an observer on Earth, how long will the muon survive before decaying?

We can approach the problem in 2 ways - Lorentz transformation equations or time dilation formula (which is derived from Lorentz transformations).

\[ t = \gamma_{\nu} \left( \frac{v}{c^2} x' + t' \right) \]

\[ \Delta t = \gamma_{\nu} \left( \frac{v}{c^2} \Delta x' + \Delta t' \right) \]

In the muon's frame, it is standing still. \( \Delta x' = 0 \).

\[ \Delta t = \gamma_{\nu} \Delta t' \]

Note: Proper time is assigned to the \( S' \) reference frame where events occur in the same location. This agrees with our formula here. \( \Delta t = \gamma_{\nu} \Delta t_0 \)
Now just calculate. Start with the Lorentz factor.

\[ \gamma_\nu = \frac{1}{\sqrt{1 - \left(\frac{0.98c}{c}\right)^2}} = 5.03 \]

Then, apply to our case.

\[ \Delta t = \gamma_\nu \Delta t' = 5.03 \times (2.2 \times 10^{-6} \text{ s}) \]

\[ \Delta t = 11 \mu\text{s} \]

2. Will the muon reach the surface?

\[ d = vt = 0.98(3.00 \times 10^8 \frac{m}{s}) \times (11 \times 10^{-6} \text{ s}) = 3.2 \text{ km} \]

Yes, the muon will reach the surface.
3. Now, let’s ask ourselves how long it takes to reach Earth from the muon’s perspective.

Use the Lorentz transformation equations to find $\Delta t'$.

$$ \Delta x = \gamma v (\Delta x' + v \Delta t') $$

In the muon’s frame, it sees itself standing still. Thus, $\Delta x' = 0$.

$$ \Delta t' = \frac{\Delta x}{v \gamma} = \frac{3000 \text{m}}{0.98 \times (3.00 \times 10^8 \text{m/s}) \times 5.03} $$

$$ \Delta t' = 2.0 \times 10^{-6} \text{s} = 2.0 \mu\text{s} $$

Thus, from the perspective of the muon, it reaches Earth, penetrates Earth, and decays 0.2 $\mu$s later.
3. Suppose that we observe $10^4$ muons at an altitude of 3000 m in some time interval. How many muons would we observe at sea level from a classical perspective?

Muons decay according to the statistical law of radioactivity.

$$N(t) = N_0 e^{(-t/\tau)}$$

$N_0 =$ original number of muons  
$N(t) =$ number of muons at time $t$  
$\tau =$ mean lifetime of the muon (a proper time interval)
The time it takes muon at an altitude of 3000 m which are traveling 0.98c to reach sea level is

\[ t = \frac{L}{v} = \frac{3000}{0.98 \times 3 \times 10^8} \approx 10 \mu s \]

The mean lifetime for muons is 2.2 µs. Thus, it take approximately 4.5 lifetimes to reach sea level and the number of particles to reach sea level is

\[ N = N_0 e^{-t/\tau} = 10^4 e^{-4.5} = 111 \]

\[ N = 111 \]
4. Suppose that we observe $10^4$ muons at an altitude of 3000 m in some time interval. How many muons would we observe at sea level taking into account special relativity?

Due to length contraction, the distance is contracted to a length of

$$L = \frac{L_0}{\gamma_v} = \frac{3000}{5.03} = 600 \text{ m}$$

Thus, the actual time it takes muon to reach sea level is

$$t = \frac{L}{v} = \frac{600 \text{ m}}{0.98 \times 10^8 \text{ m/s}} \approx 2 \mu s$$

Plugging into our formula for the number at sea level yields

$$N = N_0 e^{-t/\tau} = 10^4 e^{-2.0/2.2} \sim 3 \times 10^4$$

$$N = 3 \times 10^4$$ closer to real measurements