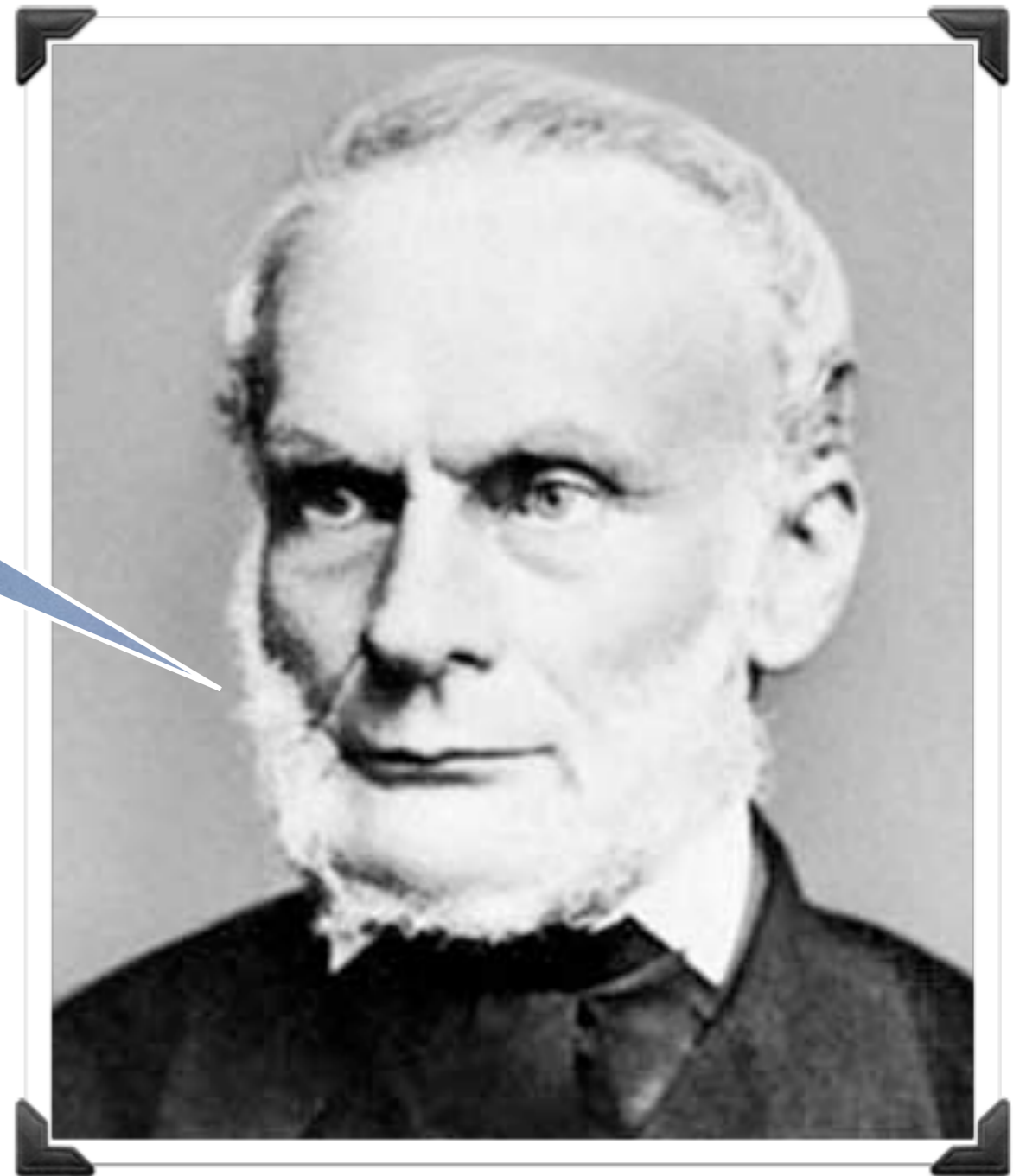


# Principles of Astrophysics and Cosmology

Welcome Back  
to PHYS 3368



**Rudolf Clausius**

**January 2, 1822 – August, 24 1888**

# Announcements

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- Reading Assignments: Chapter 3.3 - 3.12.
- Problem Set 4 is due Wednesday, February 18th.
- Problem Set 3 is due Monday, February 16th.
- Next lab is Monday, February 23rd. Be sure to report to FOSC 032 that day.
- If your homework was marked correct, but you have more “red ink” than “pencil, black or blue”, you will be in trouble (meaning, you will lose points) for the next time I grade papers. Read the homework policy!

# Goals for this Class

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1. Calculate the free-fall time of the sun.
2. Calculate how long it takes a photon to travel from the center of the sun and emerge at its surface.

# Free-Fall Timescale of Sun

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## Free-fall timescale:

The time it would take a star (or cloud) to collapse to a point if there was no outward pressure to counteract gravity.

We can calculate the free-fall timescale of the sun.

Consider a mass element  $dm$  at rest in the sun at a radius  $r_0$ . What is its potential energy?

$$dU = -\frac{GM(r_0)dm}{r_0}$$

Next, apply the conservation of energy:

$$E_i = E_f$$

$$U_i + K_i = U_f + K_f$$

$$-\frac{GM(r_0)dm}{r_0} + \frac{1}{2}dm\left(\frac{dr}{dt}\right)^2 = -\frac{GM(r_0)dm}{r}$$

Remember, we are looking for free-fall time. Simplify:

$$\left(\frac{dr}{dt}\right)^2 = 2\left[\frac{GM(r_0)}{r_0} - \frac{GM(r_0)}{r}\right]$$

$$\left(\frac{dr}{dt}\right) = \left[-2GM(r_0)\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]^{\frac{1}{2}}$$

Continuing to simplify:

$$\left(\frac{dr}{dt}\right) = [-2GM(r_0)\left(\frac{1}{r} - \frac{1}{r_0}\right)]^{\frac{1}{2}}$$

$$dt = [-2GM(r_0)\left(\frac{1}{r} - \frac{1}{r_0}\right)]^{-\frac{1}{2}} dr$$

To find  $t_{ff}$  we integrate:

$$\tau_{ff} = \int_0^{\tau_{ff}} dt = - \int_{r_0}^0 \left[ 2GM(r_0) \left( \frac{1}{r} - \frac{1}{r_0} \right) \right]^{-1/2} dr$$

As 5 points extra credit (due at the beginning of next class), show that from the above equation, you get the free-fall time of

$$\tau_{ff} = \left( \frac{3\pi}{32G\bar{\rho}} \right)^{1/2}$$

Let's examine the solution. Does it contain the correct units (use cgs)?

$$\tau_{\text{ff}} = \left( \frac{3\pi}{32G\bar{\rho}} \right)^{1/2}$$

$$G = [\text{erg}][\text{cm}][\text{g}^{-2}]$$

$$\rho = [\text{g}][\text{cm}^{-3}]$$

$$\text{erg} = [\text{g}][\text{cm}^2][\text{s}^{-2}]$$

$$\text{units} = \sqrt{[\cancel{\text{g}} \cancel{\text{cm}^2} \text{s}^{-2}][\cancel{\text{cm}}][\cancel{\text{g}}^{-2}][\cancel{\text{g}}][\cancel{\text{cm}}^{-3}]}$$

$$\text{units} = [\text{s}]$$



For the parameters of the Sun, calculate the free-fall timescale.

$$\tau_{\text{ff}\odot} = \left( \frac{3\pi}{32 \times 6.7 \times 10^{-8} \text{ cgs} \times 1.4 \text{ g cm}^{-3}} \right)^{1/2} = 1800 \text{ s}$$

Observation: Without pressure support, the sun would collapse very quickly!

The sun does not collapse because it is in **hydrostatic equilibrium**. This means that the sun is in a state of balance by which the internal pressure exactly balances the gravitational pressure.



# Hydrostatic Equilibrium

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Back to Introductory Mechanics!

Consider a small cylinder-shaped star mass element of area  $A$  and height  $dr$ .

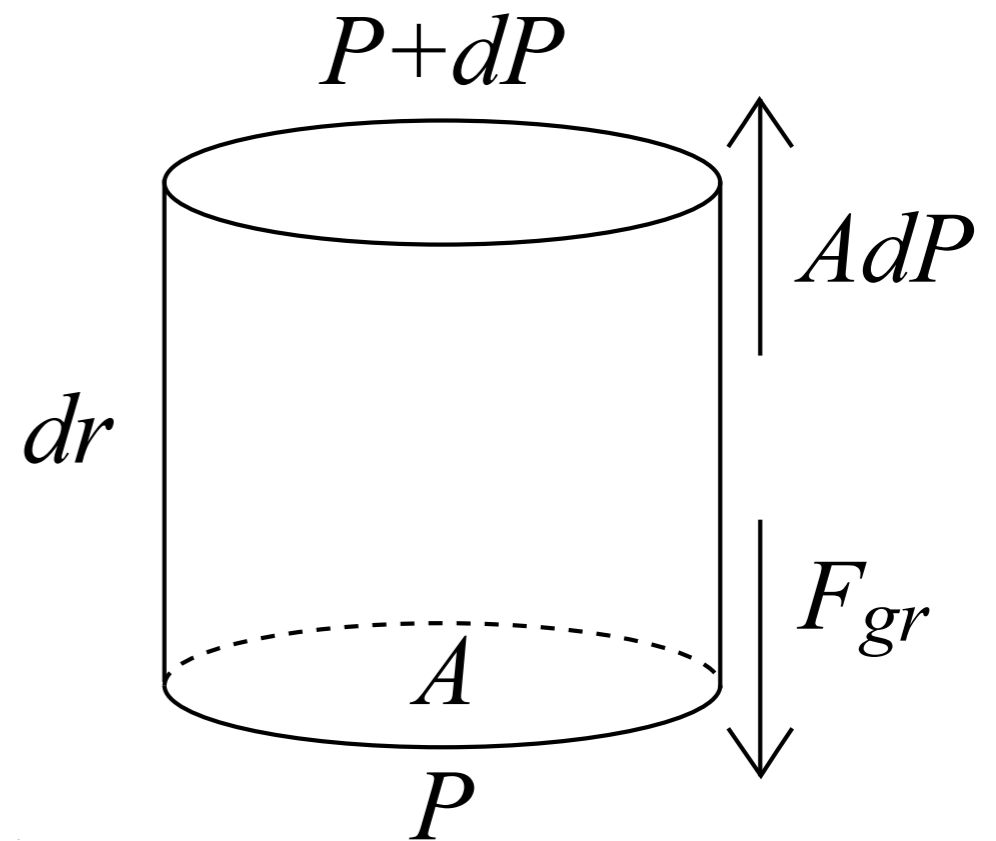
The pressure difference between the top and bottom of the cylinder is  $dP$ . It leads to a net force (due to pressure)

$$F_{pressure} = AdP$$

Equilibrium will exist if there is no net force.

$$\Sigma F = 0$$

$$-\frac{GM(r)dm}{r^2} - AdP = 0$$



The mass element  $dm$  is given by the definition of density

$$dm = \rho(r) A dr$$

Combining with our expression for equilibrium, we find

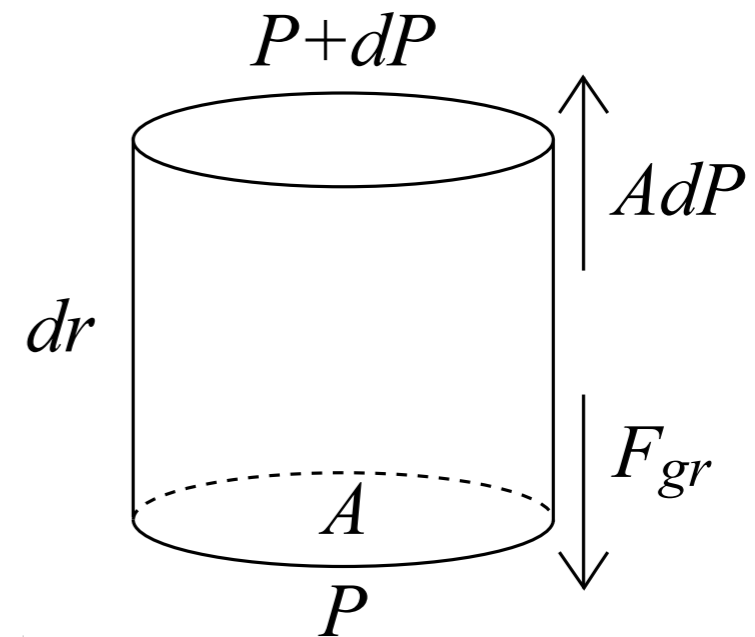
$$AdP = -\frac{GM(r)\rho(r)}{r^2}$$

Simplifying gives us the **Equation of Hydrostatic Equilibrium**, the first equation of stellar structure that we will study.

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

Stop and Think: This pressure gradient is negative. Does that make sense?

$$-\frac{GM(r)dm}{r^2} - AdP = 0$$



$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

Now let's add some thermodynamics (for fun):

First, let's introduce a cleaver 1. Multiply both sides of our equation by  $4\pi r^3 dr$ .

$$4\pi r^3 \frac{dP}{dr} dr = \frac{GM(r)\rho(r)}{r^2} 4\pi r^3 dr$$

Simplify and integrate from the star's interior to its radius:

$$\int_0^{r_*} 4\pi r^3 \frac{dP}{dr} dr = - \int_0^{r_*} \frac{GM(r)\rho(r)4\pi r^2 dr}{r}$$

This side we will have to integrate by parts to solve.

This is the gravitational self-potential energy of the star. It is equal to  $E_{gr}$ .

Recall integration by parts:

$$\int u dv = uv - \int v du$$

$$\int_0^{r_*} 4\pi r^3 \frac{dP}{dr} dr$$

Let

$$u = 4\pi r^3$$

$$dv = \frac{dP(r)}{dr} dr$$

$$du = 3(4\pi r^2) dr$$

$$v = \int \frac{dP(r)}{dr} dr = P(r)$$

Putting it together:

$$\begin{aligned} \int_0^{r_*} 4\pi r^3 \frac{dP(r)}{dr} dr &= [4\pi r^3 P(r)]_0^{r_*} - \int_0^{r_*} P(r) (3)(4)\pi r^2 dr \\ &= -3 \int_0^{r_*} P(r) 4\pi r^2 dr = -3 \frac{\bar{P}}{V} \end{aligned}$$

This is the volume-averaged pressure divided by the volume of the star

Putting it all together:

$$\int_0^{r_*} 4\pi r^3 \frac{dP}{dr} dr = - \int_0^{r_*} \frac{GM(r)\rho(r)4\pi r^2 dr}{r}$$
$$-3 \frac{\bar{P}}{V} = E_{gr}$$

Which gives the one form of the viral theorem for a gravitationally bound system.

$$\bar{P} = -\frac{1}{3} \frac{E_{gr}}{V}$$

This tells us that the pressure inside a star is one third its gravitational energy density.

Now let's add some classical thermodynamics. If a star is composed of a classical, non relativistic, mono-atomic ideal gas of  $N$  particles, what is the gas equation of state of the star?

$$PV = NkT \quad \dots(1)$$

What is its thermal energy?

$$E_{\text{th}} = \frac{3}{2}NkT \quad \dots(2)$$

Substituting  $NkT$  from (2) into (1) we find

$$P = \frac{2}{3} \frac{E_{\text{th}}}{V}$$

The local pressure is equal to  $2/3$  the local thermal energy density.

Multiply by  $4\pi r^2$  and integrating over the volume of the star, we find

$$\bar{P}V = \frac{2}{3}E_{\text{th}}^{\text{tot}}$$

Recall the our first form of the viral theorem:

$$\bar{P} = -\frac{1}{3}\frac{E_{\text{gr}}}{V}$$

Substituting gives another form of the viral theorem:

$$E_{\text{th}}^{\text{tot}} = -\frac{E_{\text{gr}}}{2}$$

The second form of the viral theorem says when a star contracts and loses energy, its self gravity becomes more negative.

# Pressure Inside a Star

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Let's examine the following equation again.

$$E_{\text{gr}} = - \int_0^{r_*} \frac{GM(r)\rho(r)4\pi r^2 dr}{r}$$

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This is the gravitational self-potential energy of the star. It is equal to  $E_{\text{gr}}$ .

Let's assume that the density profile is constant, then

$$E_{\text{gr}} = - \int_0^{r_*} \frac{GM(r)\rho(r)4\pi r^2 dr}{r} = - \int_0^{r_*} \frac{G \frac{4\pi}{3} r^3 \rho^2 4\pi r^2 dr}{r}$$

recall:

$$\rho_{\text{const}} = \frac{M}{V} = \frac{M_*}{\frac{4}{3}\pi r_*^3}$$

A little math and we get ....

$$E_{\text{gr}} = -\frac{3}{5} \frac{GM_*^2}{r_*}$$



# Mean Pressure of the Sun

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Take a characteristic  $E_{gr} \sim -GM^2/r$  and calculate the mean pressure in the sun.

Using the first form of the virial theorem, we get

$$\bar{P} = \frac{1}{3} \frac{E_{gr}}{V} \sim \frac{1}{3} \frac{GM_{\odot}^2}{\frac{4}{3}\pi r_{\odot}^3 r_{\odot}} = \frac{GM_{\odot}^2}{4\pi r_{\odot}^4}$$

$$\bar{P} = \frac{(6.7 \times 10^{-8} \text{ erg cm g}^{-2})(2.0 \times 10^{33} \text{ g})^2}{4\pi(7.0 \times 10^{10} \text{ cm})^4} = 8.9 \times 10^{14} \text{ erg cm}^{-3}$$

$$P \sim 8.9 \times 10^{14} \text{ erg cm}^{-3}$$

Note: Your textbook uses units of dyne  $\text{cm}^{-2}$ . 1 dyne = 1 erg  $\text{cm}^{-1}$ .

# Typical Temperature

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**Viral Temperature** is the typical temperature inside a star. To find the viral temperature we start with our second form of the virial theorem.

$$\frac{3}{2} N k T_{\text{vir}} = E_{\text{th}}^{\text{tot}} = -\frac{E_{\text{gr}}}{2} \sim \frac{1}{2} \frac{G M_{\odot}^2}{r_{\odot}}$$

If the particles have a mean mass  $\bar{m}$  we can write:

$$\frac{3}{2} N k T_{\text{vir}} \sim \frac{1}{2} \frac{G M_{\odot} N \bar{m}}{r_{\odot}}$$

The sun is comprised of mostly ionized hydrogen gas, consisting of an equal number of protons and electrons.

$$\bar{m} = \frac{m_e + m_p}{2} = \frac{m_H}{2}$$

Note:

$m_e \ll m_p$ , thus  $m_H \sim 1/2 m_p$ .

Putting this together, we have

$$\frac{3}{2} N k T_{\text{vir}} \sim \frac{1}{2} \frac{G M_{\odot} N \bar{m}}{r_{\odot}}$$

$$k T_{\text{vir}} \sim \frac{G M_{\odot} m_H}{6 r_{\odot}} = \frac{6.7 \times 10^{-8} \text{ cgs} \times 2 \times 10^{33} \text{ g} \times 1.7 \times 10^{-24} \text{ g}}{6 \times 7 \times 10^{10} \text{ cm}}$$
$$= 5.4 \times 10^{-10} \text{ erg}$$

Divide by  $k = 1.4 \times 10^{-16} \text{ erg K}^{-1}$

$$T_{\text{vir}} \sim 4 \times 10^{-6} \text{ K}$$

Nuclear reactions take place at temperatures of this order of magnitude. So nuclear reactions can take place and thus replenish the thermal energy that a star radiates away. This, temporarily halts the gravitational collapse.

# Mass Continuity

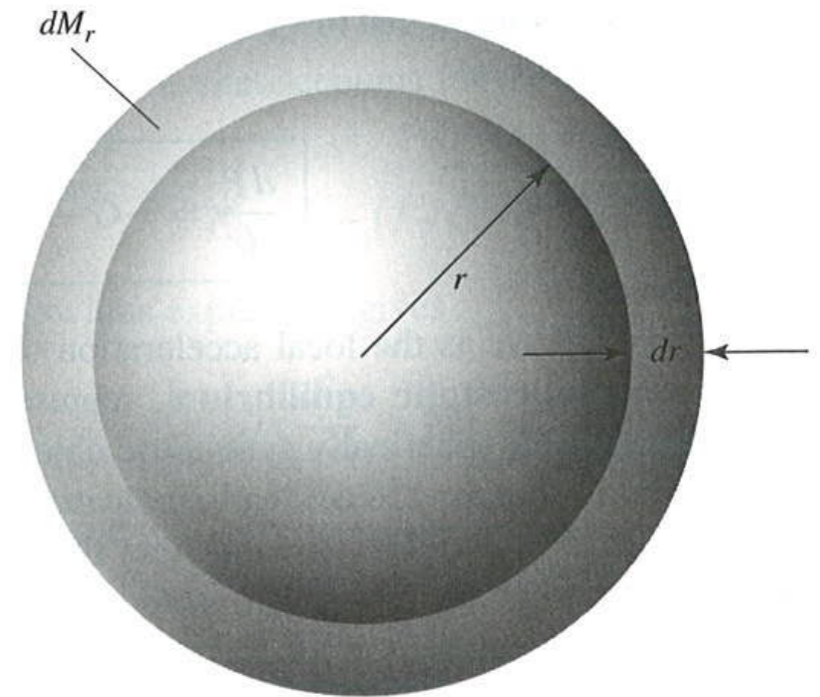
For a spherically symmetric star, consider a shell of mass  $dM_r$  and thickness  $dr$  located a distance  $r$  from the center. If the shell is thin ( $dr \ll r$ ) the volume of the shell can be approximated as  $dV = 4\pi r^2 dr$ .

If the local density is given by  $\rho(r)$ , then the shell's mass is given by

$$dM(r) = \rho(r) 4\pi r^2 dr$$

Re-arranging terms yields the **Equation of Mass Continuity**.

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$



# Next Up

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- Radiative Transport
- Conservation of energy (and our 4th equation of stellar structure).
- The equation of state
- And more awesome stuff....

**Stay Tuned!**