Principles of Astrophysics and Cosmology

Welcome Back to PHYS 3368

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From Last Time:

We can more generally write the number densities of H, He and metals.

$$n_H = \frac{X\rho}{m_H}, \;\; n_{\rm He} = \frac{Y\rho}{4m_H}, \;\; n_A = \frac{Z_A\rho}{Am_H}$$

How many particles results from the complete ionization of hydrogen?

2 = 1 proton + 1 electron

Helium?

3 = 2 electrons + 1 nucleus

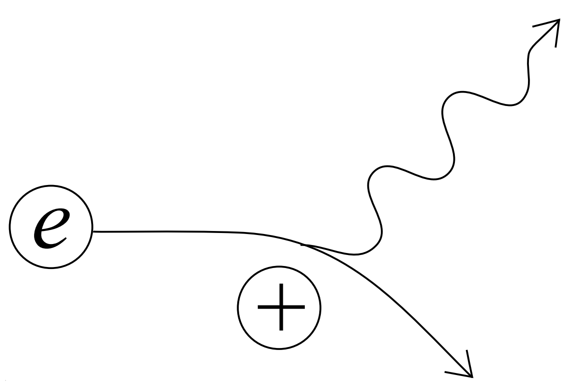
Thus, for an ionized gas:

$$n = 2n_H + 3n_{\rm He} + \sum \frac{A}{2} n_A$$

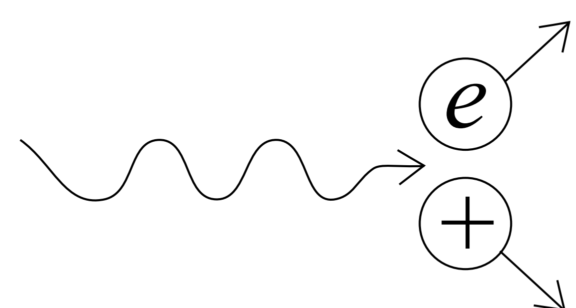
Opacity

How can a photon scatter (lose energy or be absorbed)?

- Thomson Scattering - photon scatters off free electron
- Interaction with an atom
  - Bound-Bound Absorption - electron stays bound
  - Bound-Free Absorption - electron is ejected = photoionization
- Free-Free Absorption - photon absorbed by free electron or ion, and as a result radiates (**Bremsstrahlung**)



Free electron accelerating in a Coulomb potential of an ion emits Bremsstrahlung radiation .



A photon is absorbed by a free electron. This is the inverse process of Bremsstrahlung.

Thomson scattering has a constant cross-section, but we need to know the number density of free electrons.

This calculation is similar to that we did for hydrogen:

$$n_e = n_{\mathrm{H}} + 2n_{\mathrm{He}} + \sum \frac{A}{2} n_A$$

Recall,

$$n_H = \frac{X\rho}{m_H}, \quad n_{\mathrm{He}} = \frac{Y\rho}{4m_H}, \quad n_A = \frac{Z_A \rho}{A m_H}$$

Substitute

$$n_e = \frac{\rho}{m_H}\left(X + \frac{2}{4}Y + \frac{1}{2}Z\right) = \frac{\rho}{2m_H}(1+X)$$

Recalling the relationship between mean free path and opacity: $\ell = \frac{1}{\rho\kappa} = \frac{1}{n\sigma}$

$$\kappa_{\mathrm{es}} = \frac{n_e \sigma_T}{\rho} = \frac{\sigma_T}{2m_H}(1+X) = (1+X)\, 0.2\ \mathrm{cm}^2\ \mathrm{g}^{-1}$$

Bound-free and free-free absorption are approximately described by **Kramer's Law**

$$\bar{\kappa}_{\mathrm{bf,ff}} \sim \frac{\rho}{T^{3.5}}$$

This law is only an approximation that holds over a limited range in temperature and density, but it is often used in modeling stellar structure.

Calculate how far you could see through the Earth's atmosphere if it had the opacity of the solar photosphere. The opacity of the sun's photosphere is 0.3 $cm^2\,g^{-1}$ and the density of the Earth's atmosphere is 1.2×10^{-3} $g\,cm^{-3}$.

$$\ell = \frac{1}{\rho\kappa} = \frac{1}{(1.2 \times 10^{-3})(0.3)}$$

$$\ell = 2.7 \times 10^{3} cm$$

Scaling Relations on the Main Sequence

Equations of Stellar Structure

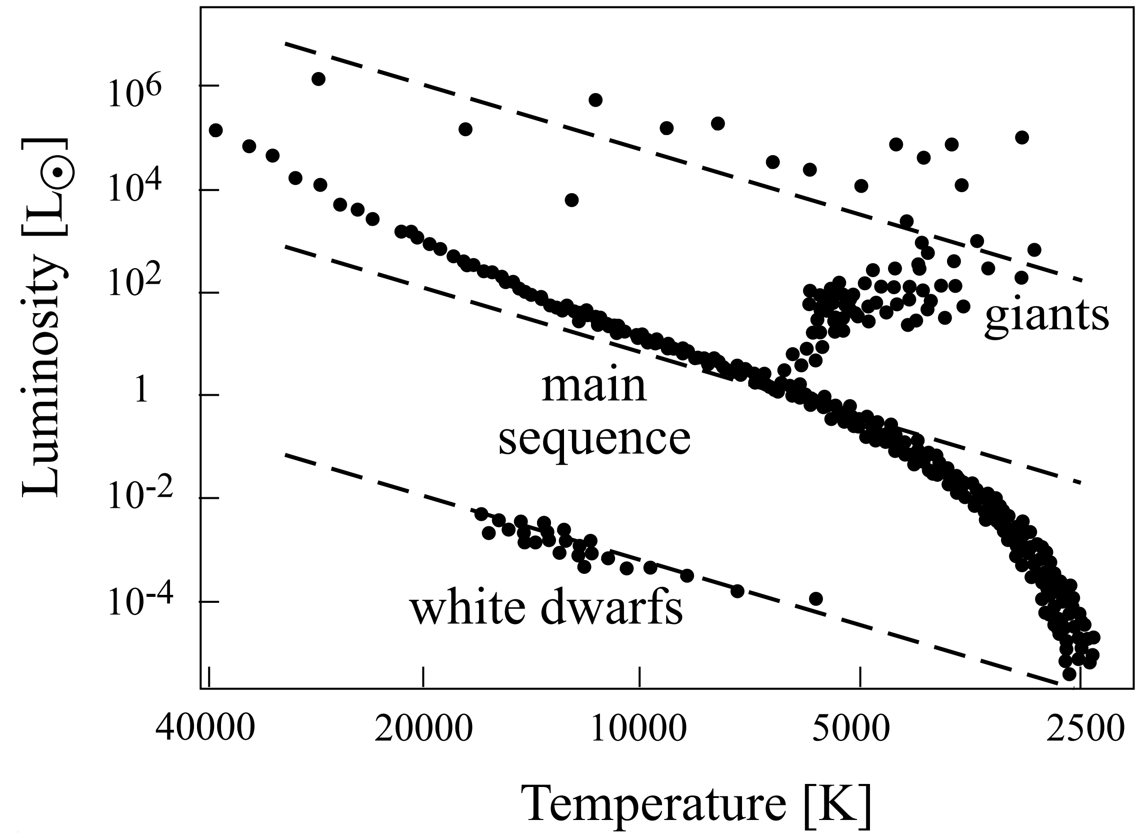
$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2},$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r),$$

$$\frac{dT(r)}{dr} = -\frac{3L(r)\kappa(r)\rho(r)}{4\pi r^2 4acT(r)^3},$$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r).$$

HR Diagram



Can we understand the observed forms of the mass-luminosity and temperature-luminosity relations?

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Scaling Relations:

P(r), M(r), ϱ(r) and T(r) are roughly power laws

$$P \sim \frac{M\rho}{r}$$

$$M \sim r^3 \rho$$

$$T \sim \frac{M}{r}$$

$$L \sim \frac{T^4 r}{\kappa\rho}$$

$$L \sim M^3$$

For moderately massive stars, kinetic gas pressure dominates and opacity is dominated by electron scattering.

$P \sim \rho T$ and $\kappa = \text{const}$

$$P \sim \frac{M\rho}{r} \qquad M \sim r^3\rho \qquad T \sim \frac{M}{r} \qquad L \sim \frac{T^4 r}{\kappa\rho}$$

Consider the relation below for the case of a star contracting under its own gravity and heating up.

- Once the temperature of the core is high enough for nuclear reactions to occur, the star will stop contacting and equilibrium will be set up.
- The nuclear power density depends mainly on T.
- For any initial mass, the radius will stop shrinking when a particular core T is reached.

Thus, the internal T is comparable in all main-sequence stars. So,

$$r \sim M$$

Given this, how does the density of the star scale with mass?

$$\rho \sim M^{-2}$$

$P \sim \frac{M\rho}{r}$

$\rho \sim M^{-2}$

$r \sim M$

$L \sim \frac{T^4 r}{\kappa \rho}$

Low Mass Stars:

These stars have high density which means there is a dominant role for bound-free and free-free opacity.

$$\kappa \sim \frac{\rho}{T^{3.5}}$$

Keeping in mind that $T \sim const$, how are L and M scaled?

$$L \sim \frac{T^4 r}{\kappa \rho} \sim \frac{r}{\rho \rho} \sim \frac{r}{\rho^2} \sim M^5$$

$$L \sim M^5$$

$$P \sim \frac{M\rho}{r}$$ $\rho \sim M^{-2}$ $r \sim M$ $$L \sim \frac{T^4 r}{\kappa\rho}$$

High Mass Stars:

In these stars, the low gas density makes radiation pressure dominant. The opacity will primarily be due to electron scattering.

$P \sim T^4$ κ =const

What can we say about the scaling relation between L and M?

$$L \sim \frac{T^4 r}{\kappa\rho} \sim \frac{Pr}{\kappa\rho} \sim \frac{M\rho}{r}\frac{r}{\rho} \sim M$$

$$L \sim M$$

$P \sim \frac{M\rho}{r}$ $\rho \sim M^{-2}$ $r \sim M$ $L \sim \frac{T^4 r}{\kappa\rho}$

Functional Dependence of Main Sequence:

For low mass: $L \sim M^5$

For moderate mass: $L \sim M^3$

So, it seems reasonable to take an intermediate slope, $L \sim M^4$ as representative.

$$\sigma T_E^4 = \frac{L}{4\pi r_*^2} \sim \frac{M^4}{M^2} \sim M^2 \sim L^{1/2}$$

$$L \sim T_E^8$$

Nuclear Energy Production

The last equation of stellar structure is energy production. In the homework, you considered how long the sun could shine at its current luminosity via the release of gravitational energy alone.

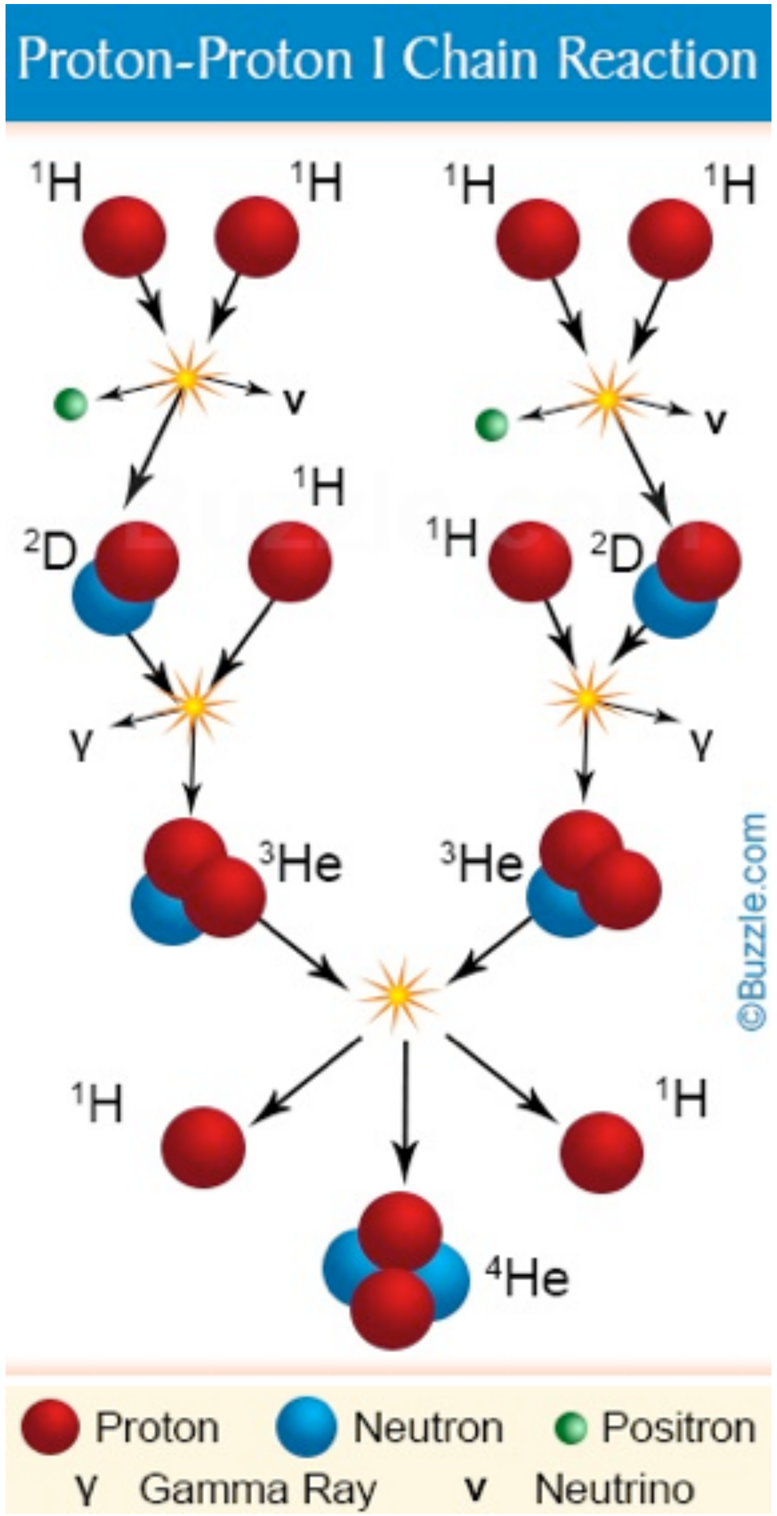
Kelvin-Helmholtz timescale:

$$\tau_{kh} = 9.6 \times 10^6 \; years$$

Geological records indicate that the Earth has existed for over 4 billion years and that the sun has been shining with similar luminosity over that time.

Where is the energy coming from?

p-p chain



This reaction produces 99% of the energy in the sun.

Step 1:

$$p + p \rightarrow d + e^+ + \nu_e$$

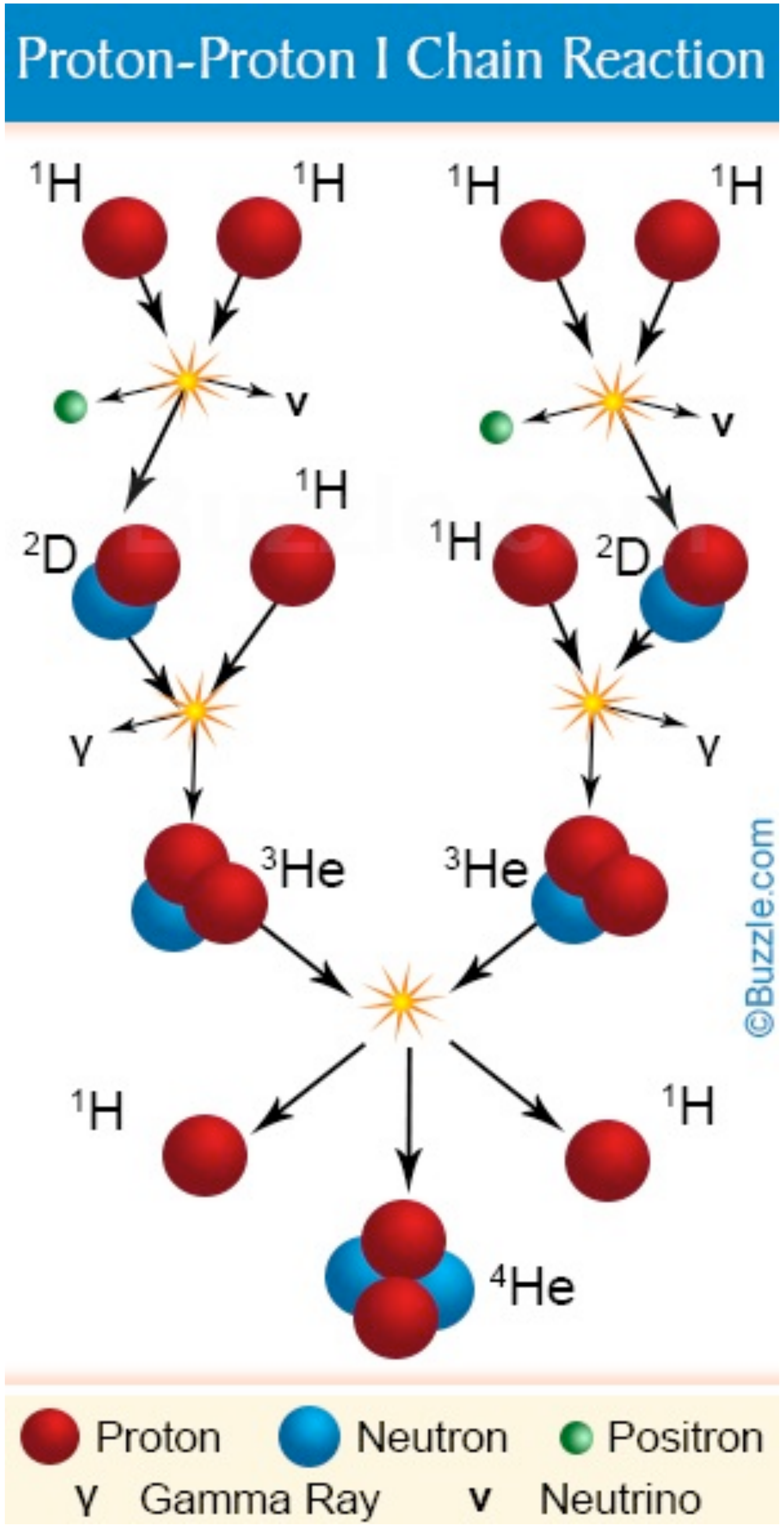
- This is a weak interaction with a timescale of ~10^{10} years in sun's core.

- Energy released = 0.425 MeV.

- The positron quickly annihilates with an ambient electron.

$$e^+ + e^- \longrightarrow \gamma + \gamma$$

Each γ has ~0.511 MeV energy.



Step 2:

$$p + d \rightarrow {}^3\text{He} + \gamma$$

- This is an electromagnetic interaction with a timescale of ~1 s in sun's core.

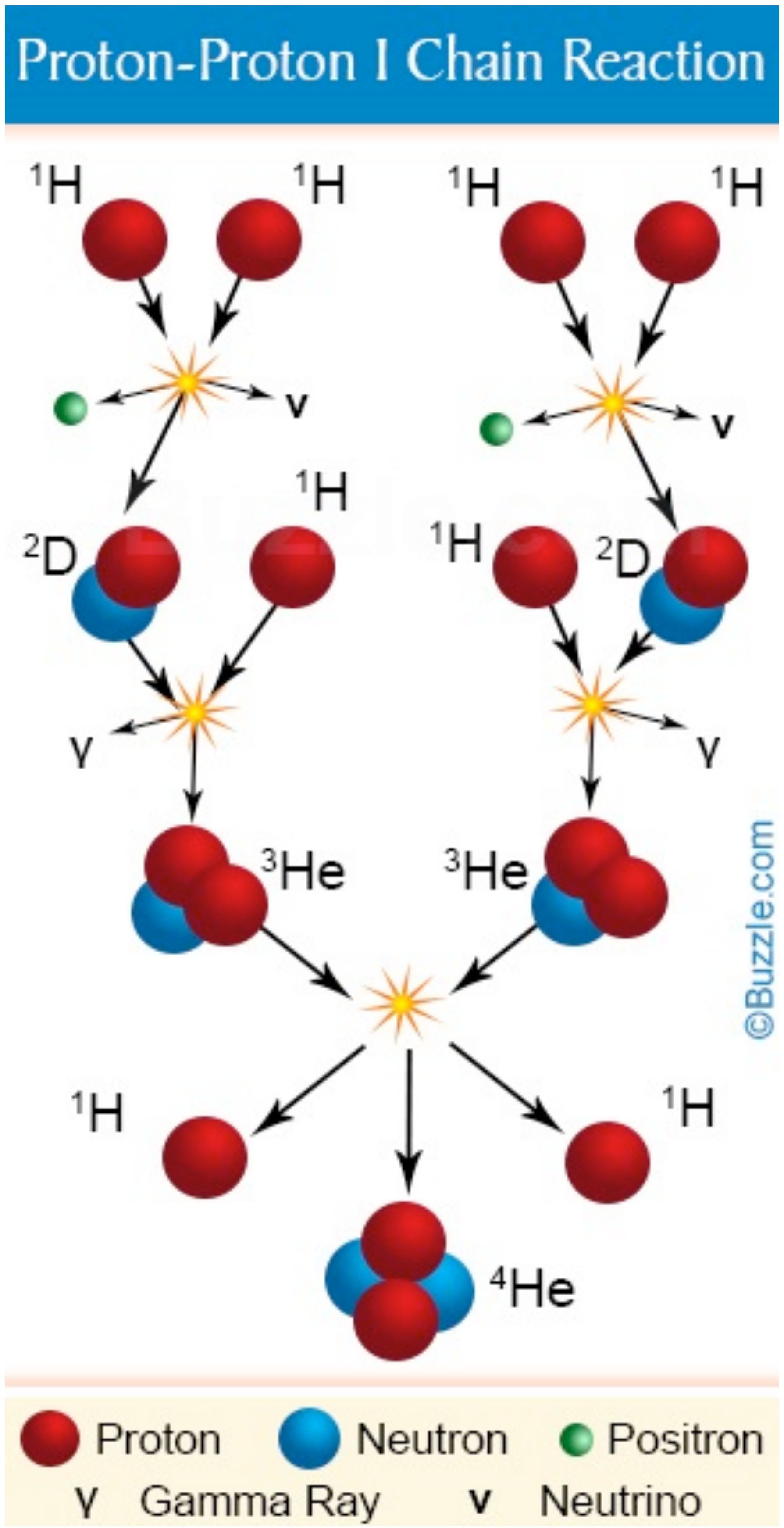
- Energy released = 5.49 MeV.

Step 3:

$${}^3\text{He} + {}^3\text{He} \rightarrow {}^4\text{He} + p + p$$

- This is a strong interaction with a timescale of ~300,000 years in sun's core.

- Energy released = 12.86 MeV.



- Note that each time the process occurs, 4 protons are converted to one ^{4}He nucleus, two neutrinos, photons and kinetic energy.

Net energy released =

$$(4 \times 0.511 + 2 \times 0.425 + 2 \times 5.49 + 12.86)\ \mathrm{MeV} = 26.73\ \mathrm{MeV}$$

step 1 step 2 step 3

- This equals the difference in rest mass energy between 4 protons and one ^{4}He nucleus.
- What is the rest mass of a proton? How does the net released energy compare to the rest mass energy of 4 protons? $m_p = 938$ MeV

This is 0.7% of the rest mass of 4 protons.

The efficiency of mass to energy conversion is 0.7%.

Calculate the time it takes for the sun tor radiate away 10% of the energy available from the p-p chain.

$$\tau = \frac{E}{L} = \frac{0.1 \times 0.007 \times M_\odot c^2}{L_\odot}$$

$$= \frac{0.1 \times 0.007 \times 2 \times 10^{33}\ \mathrm{g} \times (3 \times 10^{10}\ \mathrm{cm\ s^{-1}})^2}{3.8 \times 10^{33}\ \mathrm{erg\ s^{-1}}}$$

$$\tau = 3.3 \times 10^{17}\ s = 10^{10}\ yr$$

Hydrogen fusion can easily produce the solar luminosity over the age of the Solar System.

Question: Are the conditions in the Sun suitable for these type of reactions to take place?

Consider the forces acting on the nuclei.

- The strong force is attractive, but only operates over short distances.

$$r_0 \approx 1.4 \times 10^{-13} \text{ cm}$$

- The Coulomb force is repulsive, acts over large distances.

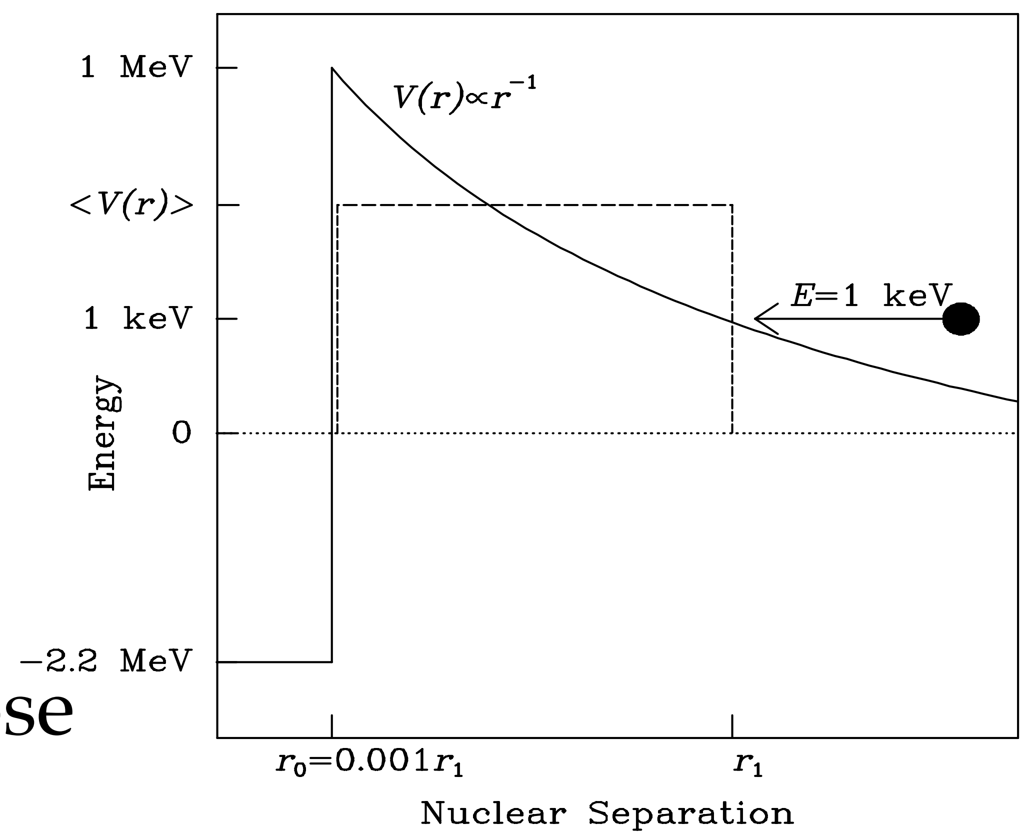
$$E_{\text{coul}} = \frac{Z_A Z_B e^2}{r}$$

- The closest approach is given by

$$r_1 = \frac{Z_A Z_B e^2}{E}$$

This distance is ~ 10^{-10} cm for $E_{kinetic}$ ~1 keV (T~ 10^7 K).

- Need 1000 x average energy to get close enough for the strong force.

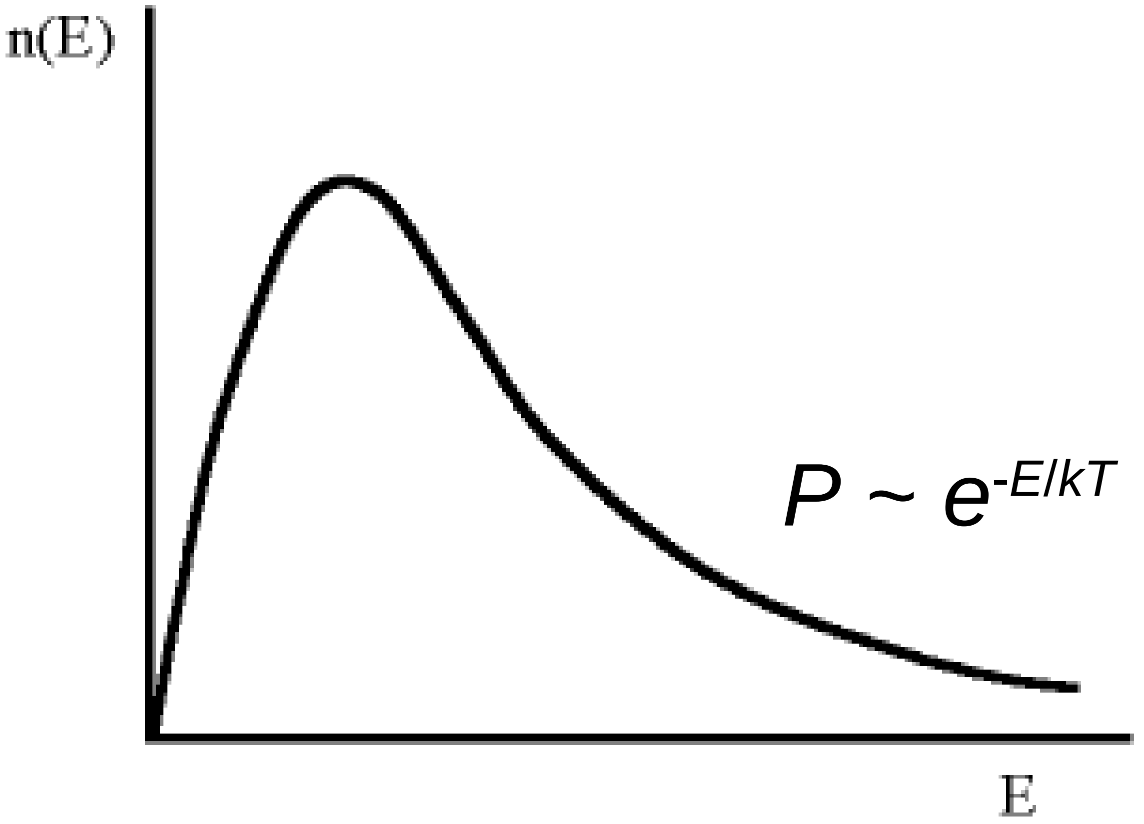


Question: Could particles the high energy tail of the Maxwell-Boltzmann distribution overcome this potential?

- The Maxwell-Boltzman distribution has a tail extending to high energies.

- The probability of state of energy E being occupied at temperature T is

$$P \approx e^{-\frac{E}{kT}}$$
$$\approx e^{-1000} \approx 10^{-434}$$



Maxwell-Boltzmann distribution

- The number of protons in the sun is 10^{57}.

- Hence, there is not a single nucleus in the sun with sufficient energy to undergo fusion.

So, what gives? How do we get fusion?

Quantum Tunneling!

Quantum tunneling through the barrier allows nuclear reactions to take place in the sun.

Recall the 3D Schrodinger equation for the hydrogen atom from PHYS 3305:

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + U\Psi = E\Psi$$

Here we will have a notation change (*U* -> *V*), we consider the case of two nuclei, and we rearrange terms.

$$\frac{\hbar^2}{2\mu}\nabla^2\Psi = [V(r) - E]\Psi$$

where μ is the reduced mass and the potential is given by V(r).

$$\mu \equiv \frac{m_A m_B}{m_A + m_B} \qquad V(r) = \frac{\mathcal{Z}_A \mathcal{Z}_B e^2}{r}$$

Crude Estimate of the Effect of Tunneling:

$$V(r) = \frac{Z_A Z_B e^2}{r}$$

Recall that the wavelength of a massive particle:

$$\lambda = \frac{h}{p}$$

Rewrite the kinetic energy of the particle in terms of momentum

$$\frac{1}{2}\mu v^2 = \frac{p^2}{2\mu}$$

Set the distance of closest approach equal to one wavelength (where potential barrier height is equal to original kinetic energy)

$$\frac{Z_A Z_B e^2}{\lambda} = \frac{p^2}{2\mu} = \frac{(h/\lambda)^2}{2\mu}$$

(Solve for λ).

Recall the relation between T and kinetic energy. Substitute λ and solve.

$$E_{kinetic} = \frac{Z_A Z_B e^2}{r} = \frac{3}{2} k T_{classical} \longrightarrow T_{quantum} \sim 10^7\ K$$

Final Points:

If we had solved the Schrodinger equation, we would have found the energy required to penetrate the Coulomb barrier to be ~500 keV.

The probability to penetrate the barrier is given by the function

$$g(E) = e^{-\sqrt{E_G/E}}$$

where E_G is the **Gamow Energy** and g(E) is the **Gamow Factor**.

The Gamow factor for the sun is

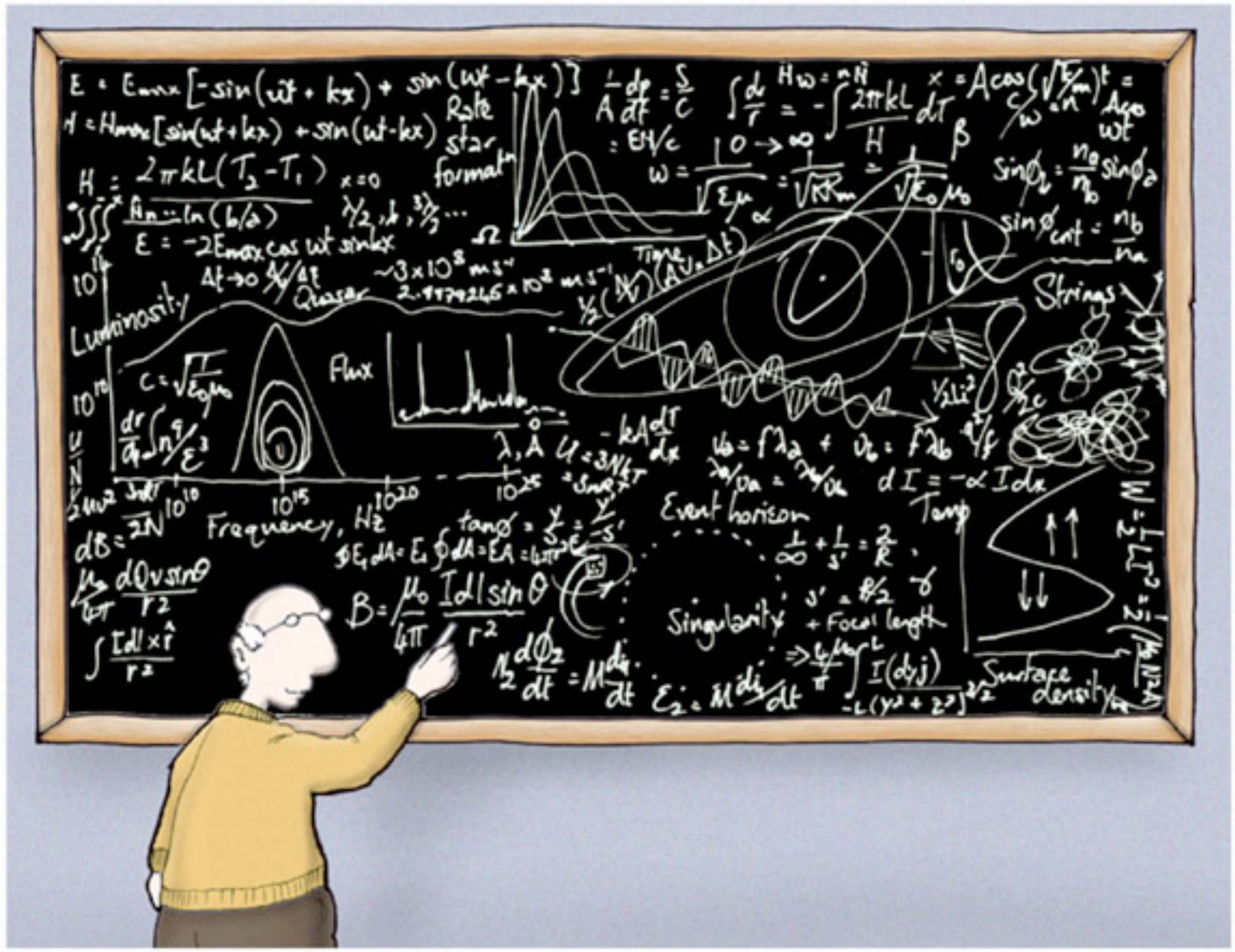
$$g(E) \sim e^{-22} \sim 10^{-10}$$

While this probability is still small, it is much greater than we found for the classical probability.

Next Up

- Nuclear reaction rates
- Convection
- and more awesome stuff ….

Stay Tuned!



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