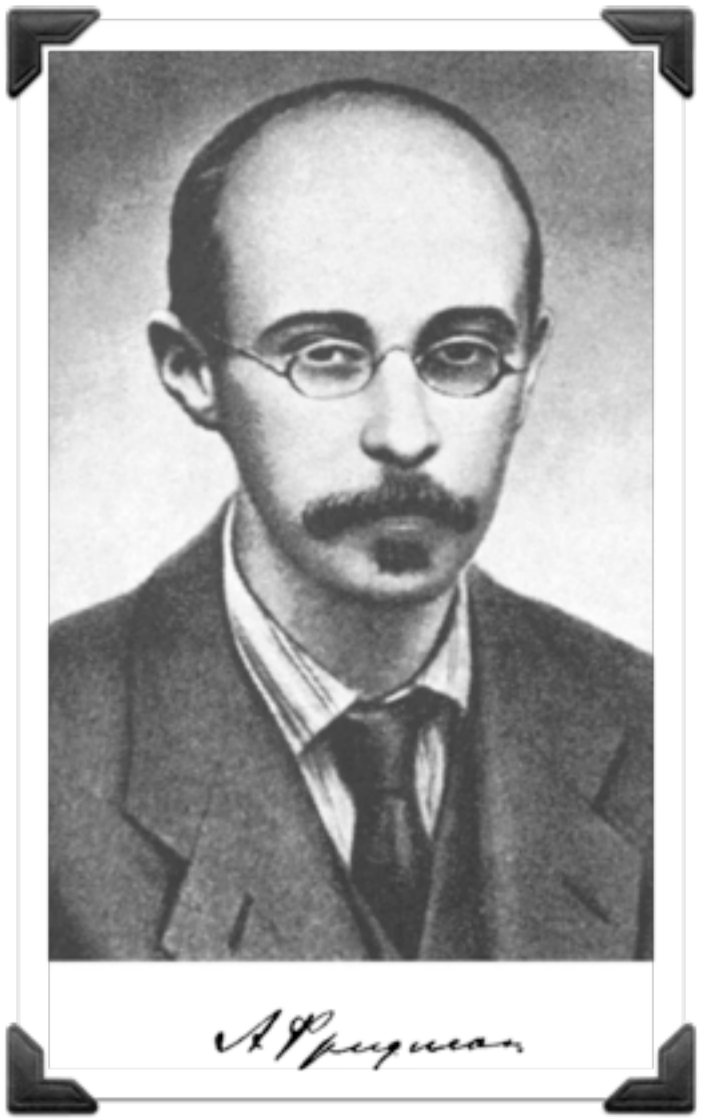
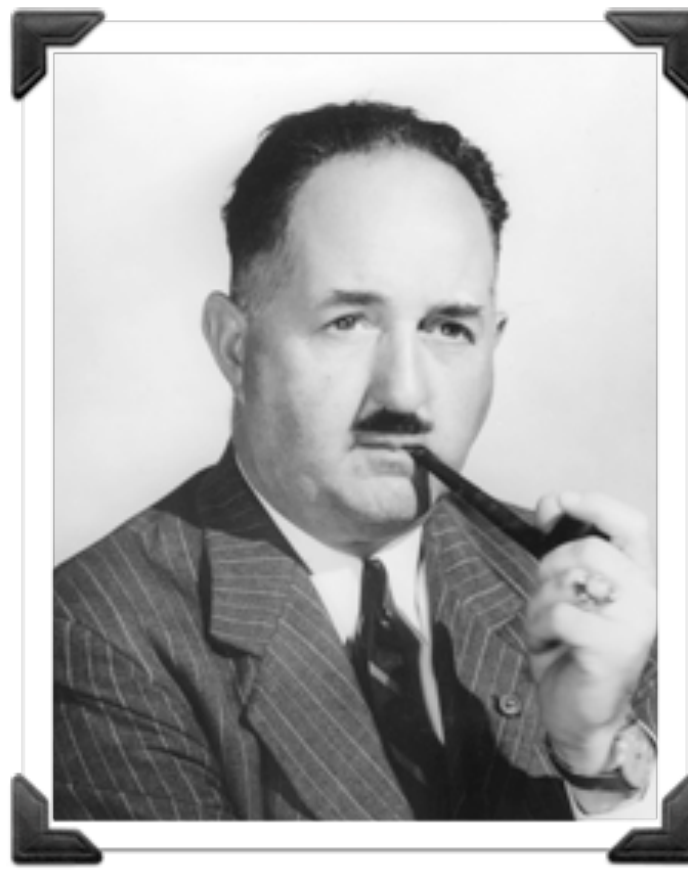
Principles of Astrophysics and Cosmology

Welcome Back to PHYS 3368



Alexander Friedmann
June 16, 1888 -
September 16, 1925



Howard Robertson
January 27, 1903 -
August 26, 1961

Arthur Walker
July 17, 1909 -
March 31, 2001

Announcements

- Reading Assignments: Chapter   4.6, & 8.1 - 8.2 & 8.4.
- Problem Set 10 is due in class on Monday, April 20th.
- Monday, April 13th: Special lecture about something awesome by Matt Stein. If you are participating in Honors Convocation, you are excused from class that day.
- Wednesday, April 15th in class lab. Be to report to FOSC 032 that day.
- Wednesday, April 15th your final paper is due (hard copy and electronic pdf). Be sure to review the paper guidelines.
- Dr. Cooley will be out of town April 14th - April 17th.
- The final exam in this course will be on Wednesday, May 6th from 6:30 - 8:00 pm. It will cover the second half of the course.

Rough Lecture Outline

- April 8: Friedman Equation (8.1 - 8.2, 8.4)
- April 20: History of the Universe and Dark Energy (8.3, 8.5)
- April 22: Redshift and the CMB as tests (9.1 - 9.2)
- April 27: CMB Anisotropy (9.3)
- April 29: Nucleosynthesis & Quasars (9.4 - 9.5)
- May 4: Special Topic Lecture (TBD)

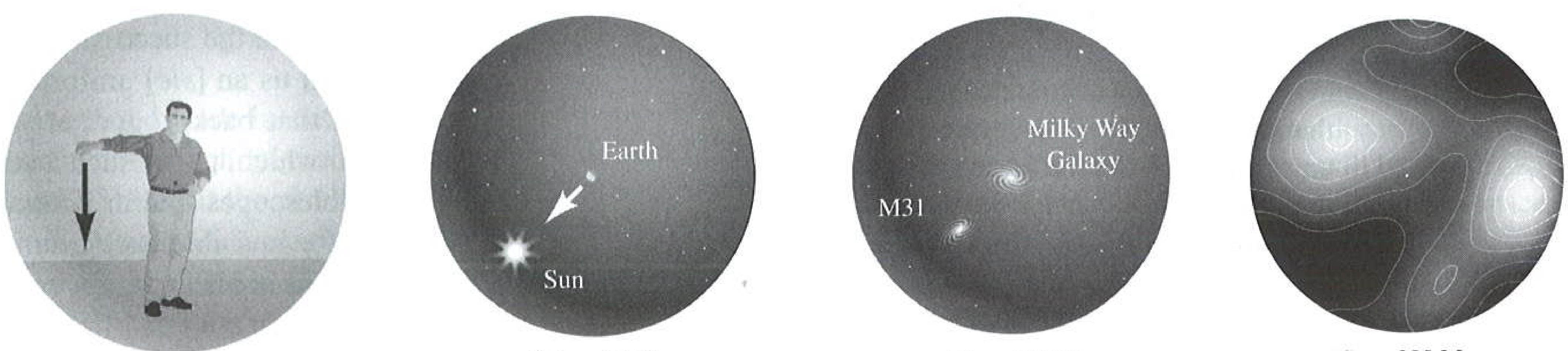
Today's Lecture

- Explore the Cosmological Principle.
- Describe curvature mathematically.
- Derive the Friedmann Equations —
  Newton Edition!

The Cosmological Principle

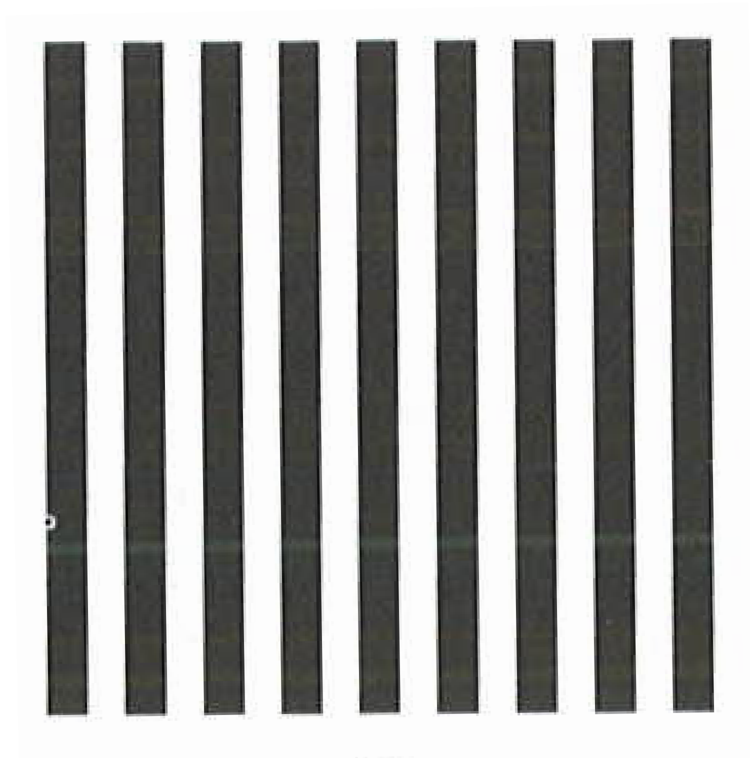
The **Cosmological Principle** postulates that the Universe is **isotropic** and **homogenous**.

- Isotropic: At large enough scales the Universe is the same in all directions. There is no preferred direction in space.
- Homogenous: Every observer sees the same expansion in space. There is no preferred location.



(a) — 3 m (b) — 3 AU (c) — 3 Mpc (d) — 200 Mpc

Is this pattern isotropic and/or homogenous?



Homogeneous on scales larger than stripe width. Not isotropic.

Is this pattern isotropic and/or homogenous?

Isotropic, but not homogenous.

The Cosmological Principle

The **Cosmological Principle** postulates that the Universe is **isotropic** and **homogenous**.

- Isotropic: At large enough scales the Universe is the same in all directions. There is no preferred direction in space.
- Homogenous: Every observer sees the same expansion in space. There is no preferred location.

Does isotropy imply homogeneity or vice versa?

No. Here are some examples to illustrate:

- A spherically symmetric Universe with radially varying density is not homogenous, but will appear isotropic to an observer at the center.
- A rotating Universe may be homogeneous, but is not isotropic (since the preferred direction is along the rotation axis).

General Relativity

- To describe the Universe, we require a relativistic theory of gravity, General Relativity.

- GR relates the density of mass and energy (sources of gravity) to the curvature of spacetime.

- The curvature is described by a metric tensor, which specifies the line element of the curved spacetime.

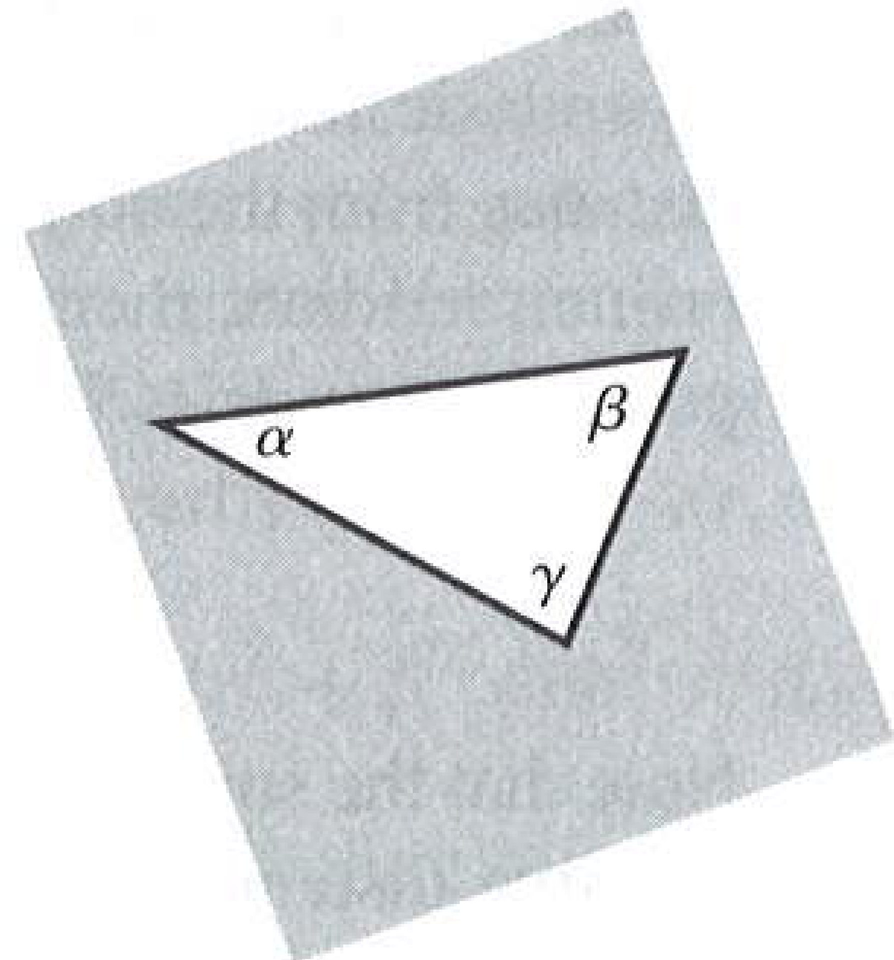
- We need to find the metric of the Universe that corresponds to the cosmological principles of homogeneity and isotropy.

- If space is homogenous, it must have the same curvature everywhere. Three possibilities: flat, positive curved or negative curved.

Let's begin by considering the simplest case: two-dimensional spaces on a plane. Here Euclidean geometry holds. A geodesic is a straight line.

If a triangle is constructed on a plane, the angles of the vertices obey the relation

$$\alpha + \beta + \gamma = \pi$$



On a plane, the Pythagorean theorem holds, so

$$dl^2 = dx^2 + dy^2$$

or in polar coordinates

$$dl^2 = dr^2 + r^2 d\theta^2$$

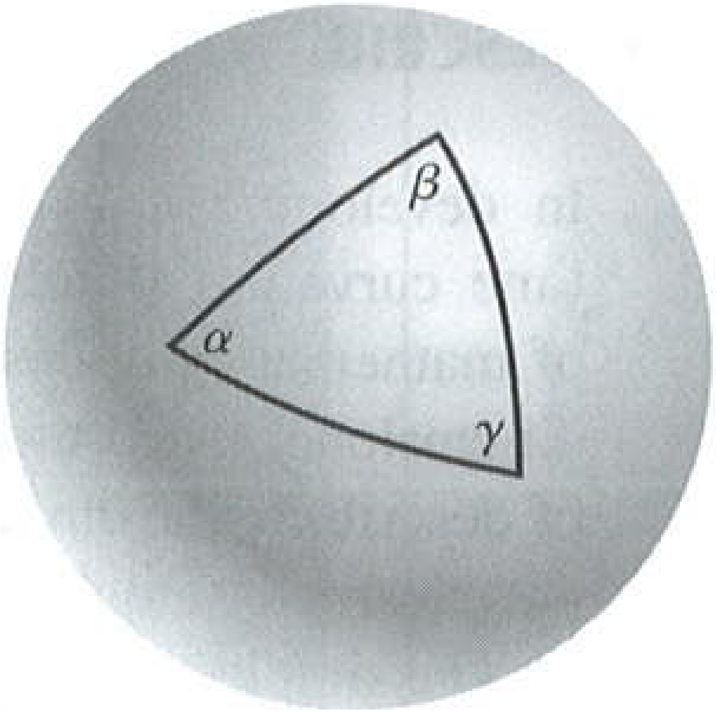
Note: A plane has infinite area and no upper limit on the distance between points.

Now consider a 2D space on the surface of a sphere.

If a triangle is constructed on a plane, the angles of the vertices obey the relation

$$\alpha + \beta + \gamma = \pi + \frac{A}{R^2}$$

where A = area of triangle and R = radius of sphere.



"positively curved" space

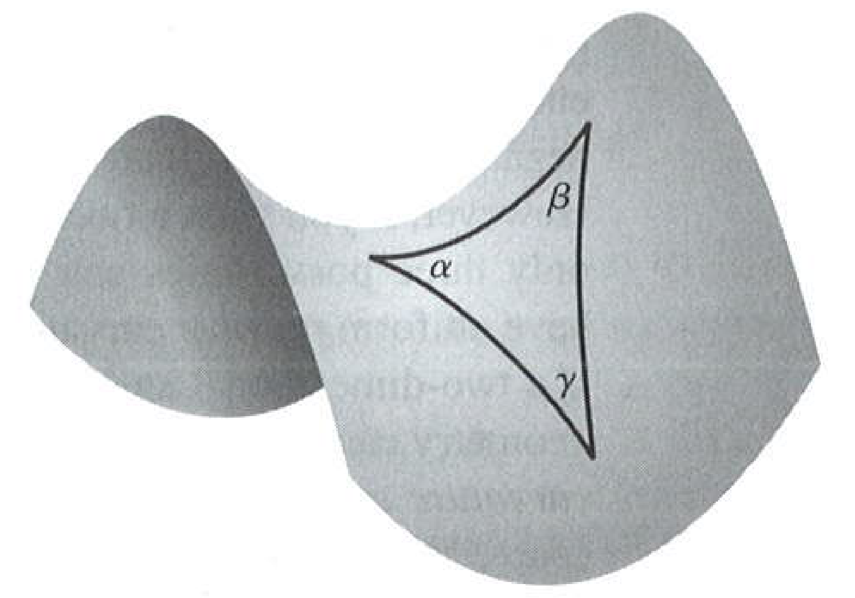
In polar coordinates, the distance between two points is

$$dl^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\theta^2$$

Note: A sphere has finite area and hence a maximum value on the distance between two points. The distance between two antipodal points, at maximum separation is πR.

Now consider a 2D space on the surface of constant negative curvature.

If a triangle is constructed on a plane, the angles of the vertices obey the relation



$$\alpha + \beta + \gamma = \pi - \frac{A}{R^2}$$

where A = area of triangle and R = radius of sphere.

In polar coordinates, the distance between two points is

$$ds^2 = dr^2 + R^2 \sinh^2\left(\frac{r}{R}\right) d\theta^2$$

Note: A surface of constant negative curvature has infinite area and no upper limit on the distance between points.

These results can be extend to a three dimensional space quite easily.

Flat 3D space:

$$dl^2 = dx^2 + dy^2 + dz^2$$

line element dl gives distance between two points

or in spherical coordinates

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2$$

Uniform positive curvature:

$$dl^2 = dr^2 + R^2 sin^2(r/R)[d\theta^2 + \sin^2\theta d\phi^2]$$

Uniform negative curvature:

$$dl^2 = dr^2 + R^2 sinh^2(r/R)[d\theta^2 + \sin^2\theta d\phi^2]$$

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \qquad dl^2 = dr^2 + R^2 sin^2(r/R)[d\theta^2 + \sin^2\theta d\phi^2]$$

$$dl^2 = dr^2 + R^2 sinh^2(r/R)[d\theta^2 + \sin^2\theta d\phi^2]$$

We can write this more compactly as

$$dl^2 = dr^2 + S_k(r)^2 d\Omega^2$$

where

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$$

and

$$S_\kappa(r) = \begin{cases} R\sin(r/R) & \text{k = +1} \quad \text{positive curvature} \\ r & \text{k = 0} \quad \text{flat} \\ R\sinh(r/R) & \text{k = -1} \quad \text{negative curvature} \end{cases}$$

Friedmann-Robertson-Walker Metric

So, far we have considered metrics for 2D and 3D spaces. Relativity tells us that space and time together comprise 4D spacetime.

When we add the time dimension to the line element, the spacetime interval between two events becomes

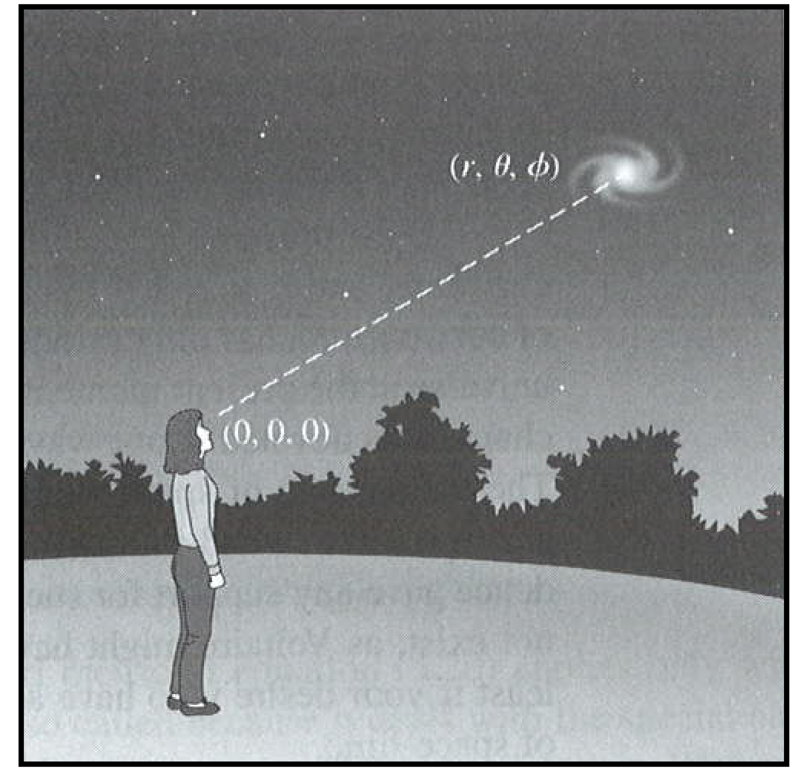
$$\boxed{ds^2 = c^2 dt^2 - dl^2 = c^2 dt^2 - R^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right)}$$

The coefficients of this interval constitute the **Friedmann-Robertson-Walker (FRW) metric**.

Notes:

- *R* is a scale factor that multiplies the dimensionless spatial part of the FRW metric.
- The coordinates *(r, θ, φ)* are **comoving coordinates**. Thus, a galaxy at coordinates *(r, θ, φ)* remains at those coordinates, even as *R* grows with time.

The **proper distance** between two points is equal to the length of the spacial geodesic between them when the scale factor *R* is fixed at *R(t)*.



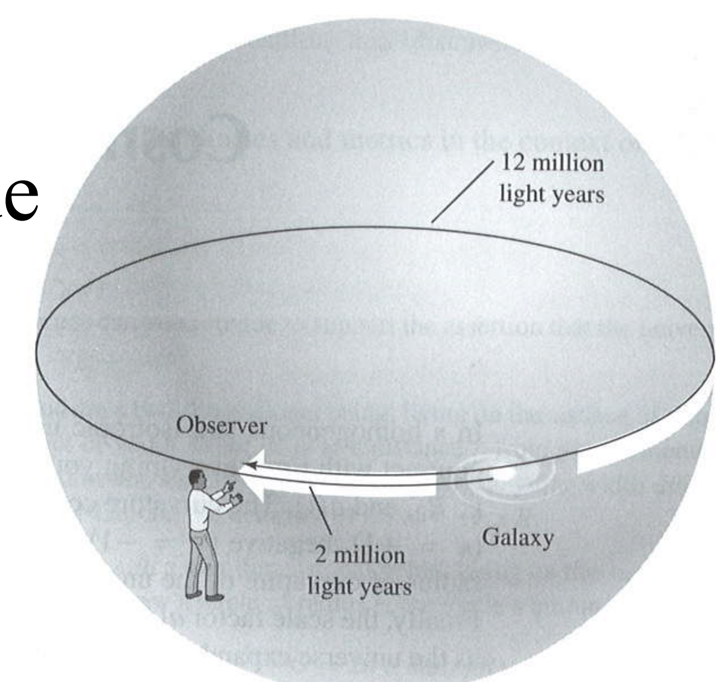
$$l = \int_{r=0}^{r} dl \quad = R(t) \int_0^r \frac{dr}{\sqrt{1 - kr^2}}$$

$$= \begin{cases} R \sin^{-1} r & \text{if } k = +1 \\ Rr & \text{if } k = 0 \\ R \sinh^{-1} r & \text{if } k = -1 \end{cases}$$

Consider the case of positive curvature.

$$l = R \sin^{-1} r \quad \longrightarrow r = \sin \frac{l}{R}$$

What is the maximum value of *r* and what is the value of *l* at that point?



r reaches maximum value of 1 at proper distance

$$l = \frac{\pi R}{2}$$

Consider the area of a sphere centered on us and passing through a galaxy at coordinate *r*, which corresponds to a physical coordinate *r'* = *rR*.

$$A = 4\pi r'^2 = 4\pi R^2 r^2 = 4\pi R^2 \sin^2 \frac{l}{R}$$

Notes:

- Beyond $l = \pi R/2$, the area of the sphere decreases.
- At the antipode $l = \pi R$, the sphere is centered on us and enclosing all previous spheres has ZERO area.

The velocity of the galaxy at *r* is the time derivative of the distance

$$v = \dot{l} = \dot{R}(t)\int_0^r \frac{dr}{\sqrt{1-kr^2}} = \frac{\dot{R}}{R} l = H(t) l$$

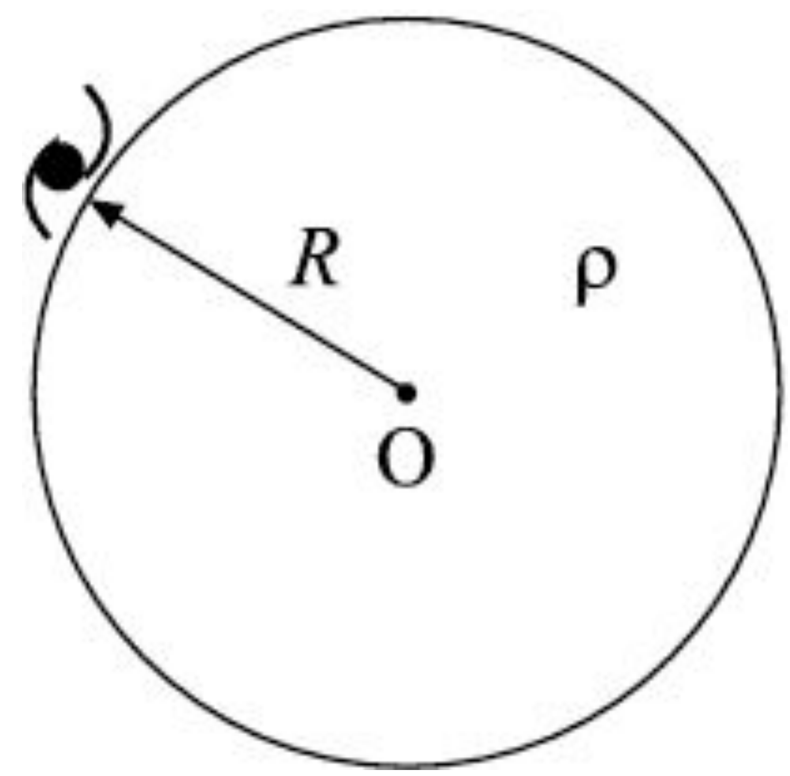
This ratio is the Hubble parameter.

Friedmann Equations - Newtonian Edition

Consider a spherical region of radius *R*, total mass *M*, and constant density ρ. A galaxy of mass *m* is at the edge of the region at a radius *R* from an observer at the center.

What is the total energy?

$$\frac{1}{2}m\dot{R}^2 - \frac{GMm}{R} = E$$



Note that we can write *M* as

$$M = \frac{4\pi}{3}R^3\rho$$

$$\frac{1}{2}m\dot{R}^2 - \frac{GMm}{R} = E \qquad\qquad M = \frac{4\pi}{3}R^3\rho$$

Substitute:

$$\frac{m\dot{R}^2}{2} = E + \frac{4\pi Gm}{3}\rho R^2$$

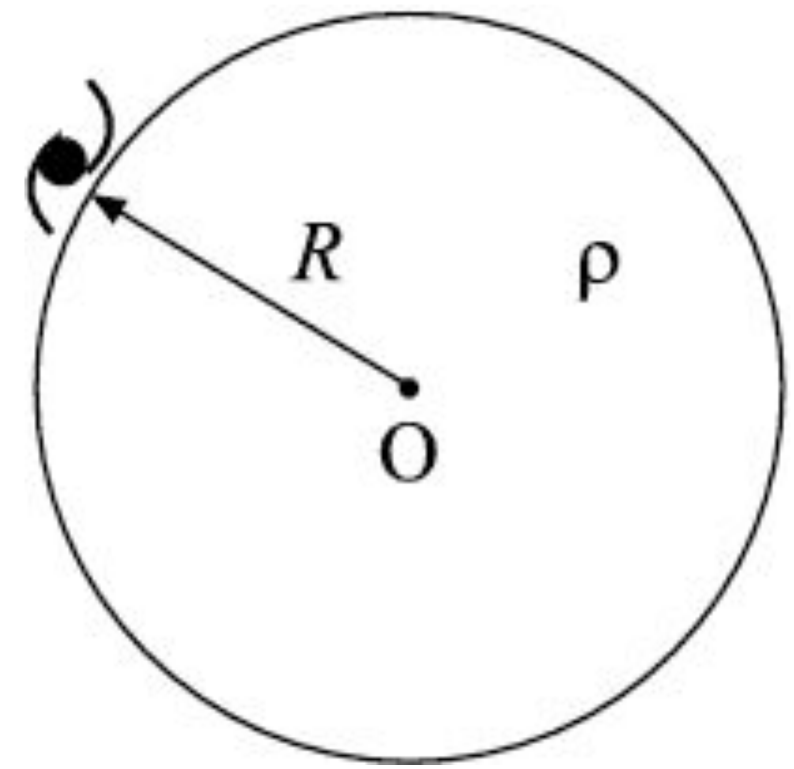
$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\rho + \frac{2E}{mR^2}$$

If we had used GR to calculate the correct form of this equation we would have found

$$\boxed{\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}}$$

This is the **First Friedmann Equation** which tells us that the sum of the potential and kinetic energies of the Universe is locally conserved.

Now let's calculate the equation of motion for this galaxy.



$$F = -\frac{GMm}{R^2} = ma = m\ddot{R}$$

Use same substitution

$$M = \frac{4\pi}{3} R^3 \rho$$

Yields:

$$m\ddot{R} = -\frac{Gm}{R^3}\frac{4\pi R^3}{3}\rho \quad \longrightarrow \frac{\ddot{R}}{R} = \frac{4\pi G}{3}\rho$$

If we had used GR to calculate the correct form of this equation we would have found

$$\boxed{\frac{\ddot{R}}{R} = -\frac{4\pi G}{3c^2}(\rho c^2 + 3P)}$$

This is the **Second Friedmann Equation** which is just an equation of motion under the influence of gravity.

Conservation of energy implies that, in a system undergoing adiabatic compression or expansion the energy U, pressure P, and volume V obey

$$dU = -PdV$$

Is this condition consistent with the Cosmological Principle?

Yes, in a homogenous and isotropic Universe there can be no net energy flow from one region to another.

The energy U is given by

$$U = \rho c^2 V$$

Substitute and take the time derivative of both sides

$$\frac{d(\rho c^2 V)}{dt} = -P\frac{dV}{dt}$$

$$\frac{d(\rho c^2 V)}{dt} = -P\frac{dV}{dt}$$

Simplify

$$\dot{\rho}c^2 V + \rho c^2 \frac{dV}{dt} = -P\frac{dV}{dt}$$

$$\dot{\rho}c^2 V = -\rho c^2 \dot{V} - P\dot{V}$$

$$\dot{\rho}c^2 = -\frac{\dot{V}}{V}(\rho c^2 + P)$$

Note that $V \propto R^3$

$$\frac{dV}{V} = 3\frac{dR}{R}$$

Substituting yields

$$\boxed{\dot{\rho}c^2 = -3\frac{\dot{R}}{R}(\rho c^2 + P)}$$

This is the **Third Friedmann Equation** which is also known as the fluid equation or the energy conservation equation.

Summary

We have the FRW metric:

$$ds^2 = c^2 dt^2 - R^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right)$$

We have 3 Friedmann Equations:

$$\left(\frac{\dot{R}}{R} \right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

sum of potential and kinetic energy is conserved locally

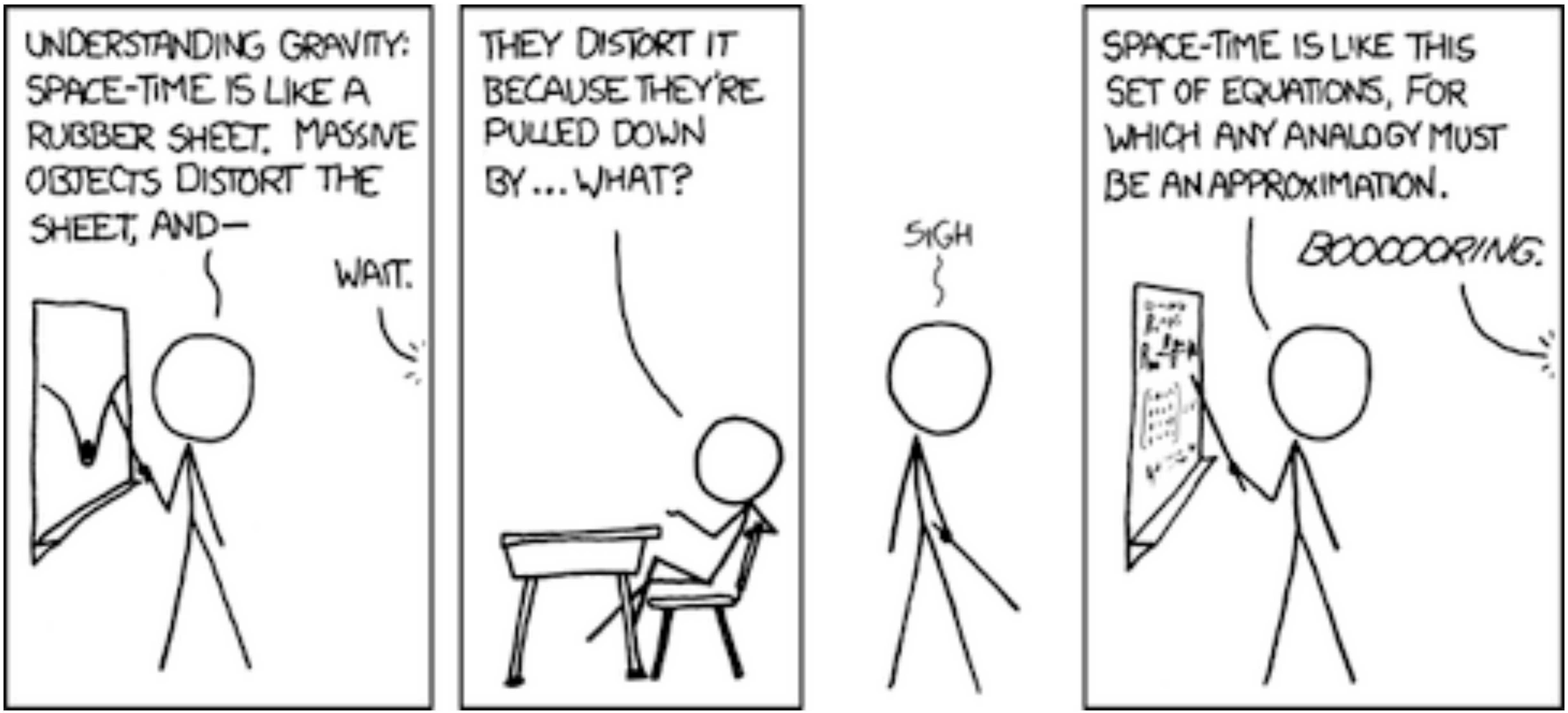
$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3c^2} (\rho c^2 + 3P)$$

equation of motion under the influence of gravity

$$\dot{\rho} c^2 = -3 \frac{\dot{R}}{R} (\rho c^2 + P)$$

the fluid equation

Stay Tuned!


UNDERSTANDING GRAVITY: SPACE-TIME IS LIKE A RUBBER SHEET. MASSIVE OBJECTS DISTORT THE SHEET, AND—
WAIT.
THEY DISTORT IT BECAUSE THEY'RE PULLED DOWN BY... WHAT?
SIGH
SPACE-TIME IS LIKE THIS SET OF EQUATIONS, FOR WHICH ANY ANALOGY MUST BE AN APPROXIMATION.
BOOOOORING.