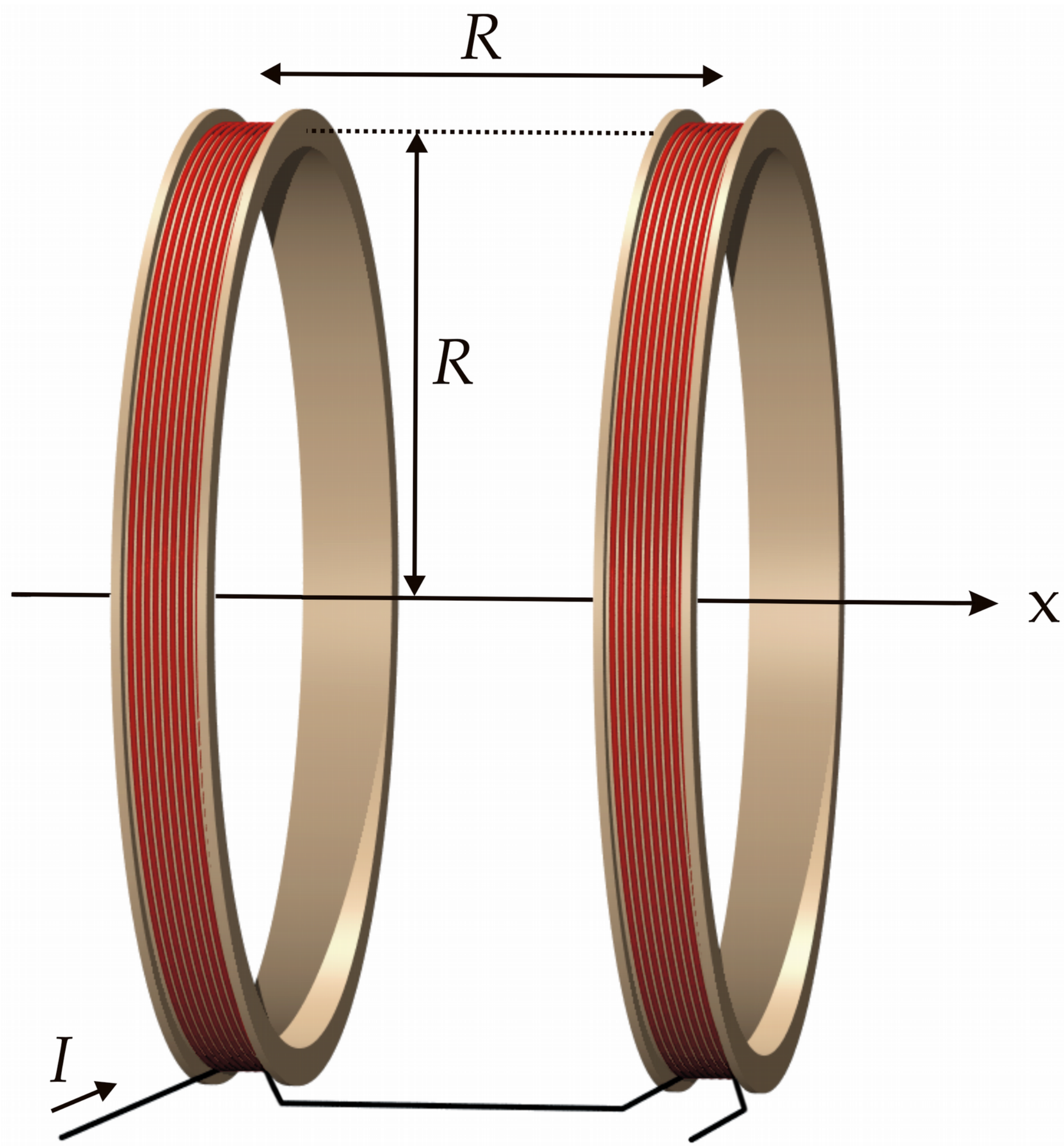
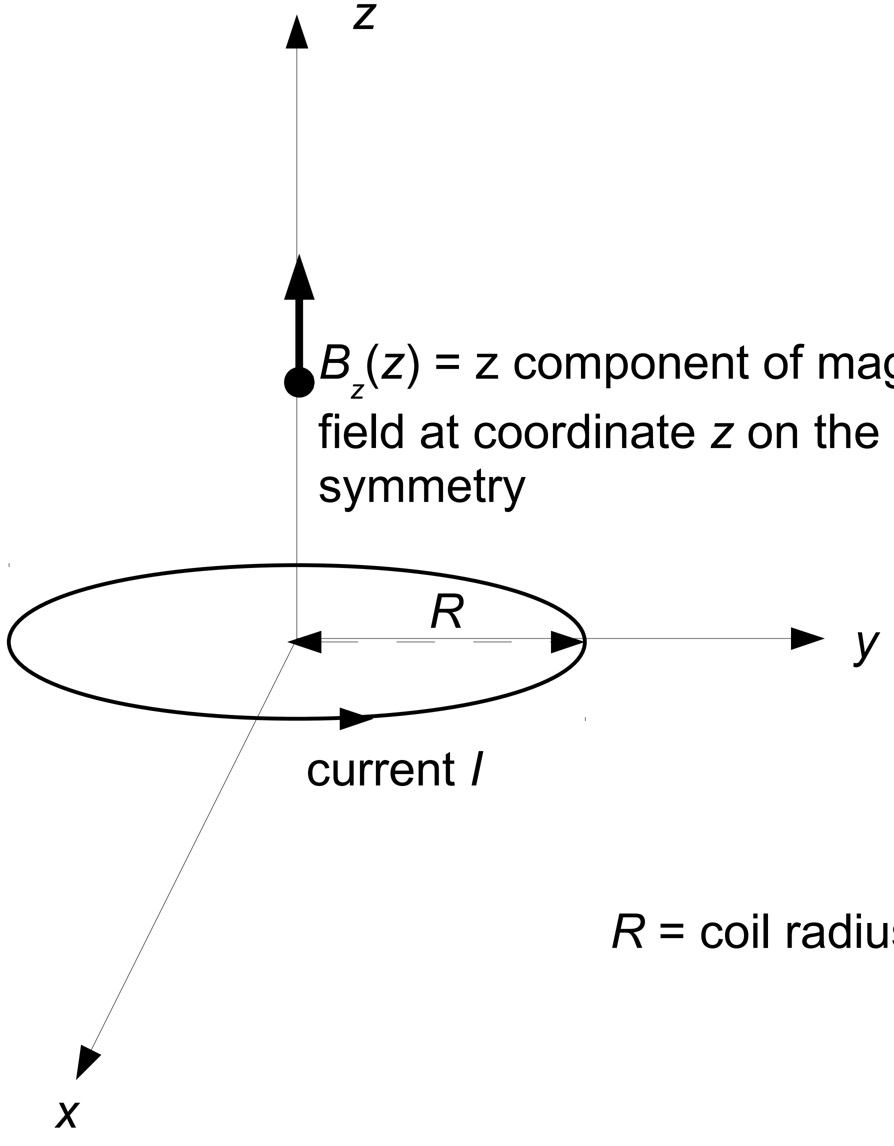
Class Progress

Basics of Linux, gnuplot, C
Visualization of numerical data
Roots of nonlinear equations
  (Midterm 1)
Solutions of systems of linear equations
Solutions of systems of nonlinear equations
Monte Carlo simulation
Interpolation of sparse data points
Numerical integration
  (Midterm 2)
Solutions of ordinary differential equations

Helmholtz Coil for Uniform Magnetic Field



Magnetic Field Along Axis of a Single Coil



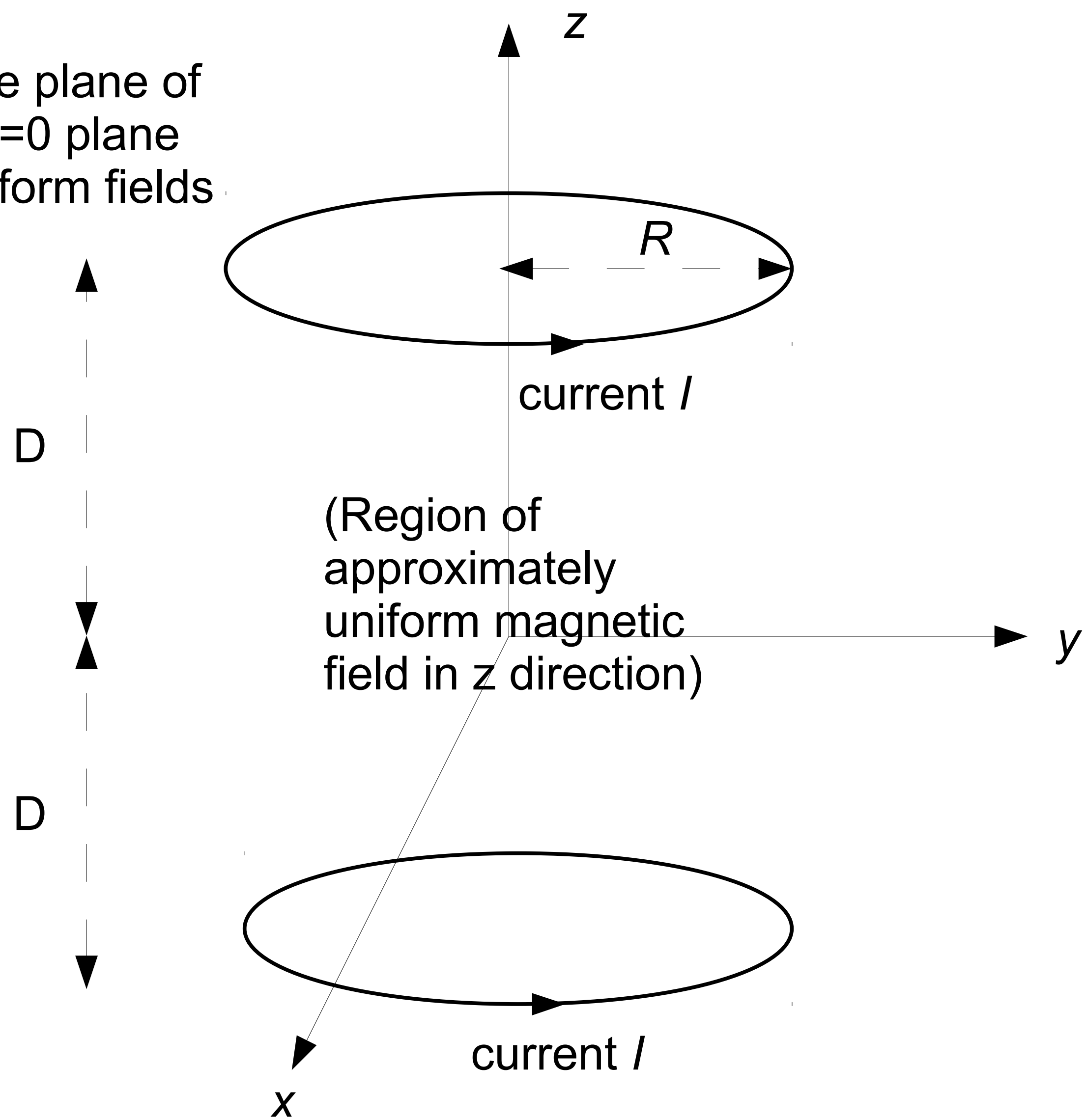
$B_z(z)$ = z component of magnetic field at coordinate z on the axis of symmetry

$$B_z(z) = \frac{\mu_0 I R^2}{2\left(R^2+z^2\right)^{\frac{3}{2}}}$$

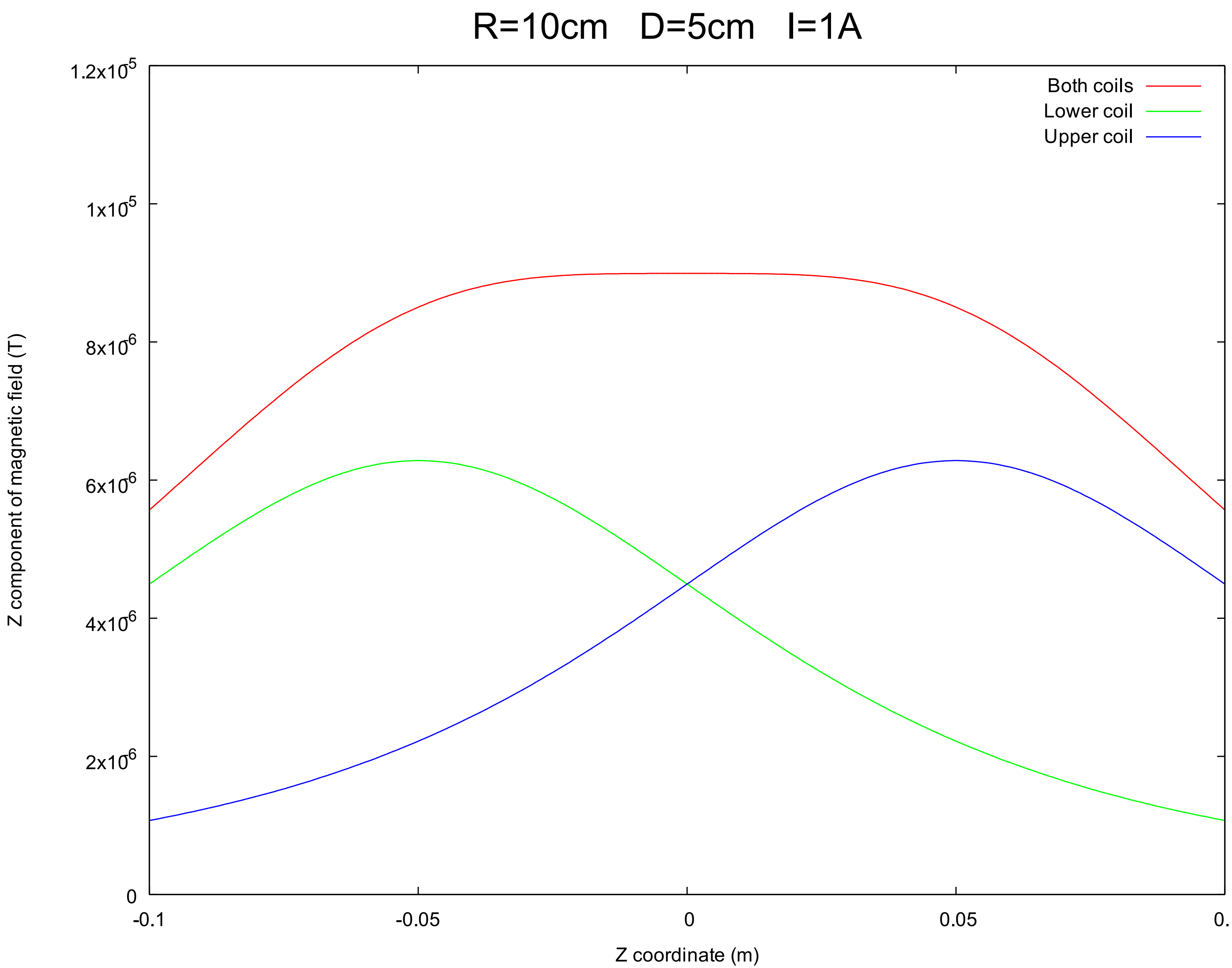
R = coil radius

Magnetic Field on Axis of a Helmholtz Coil

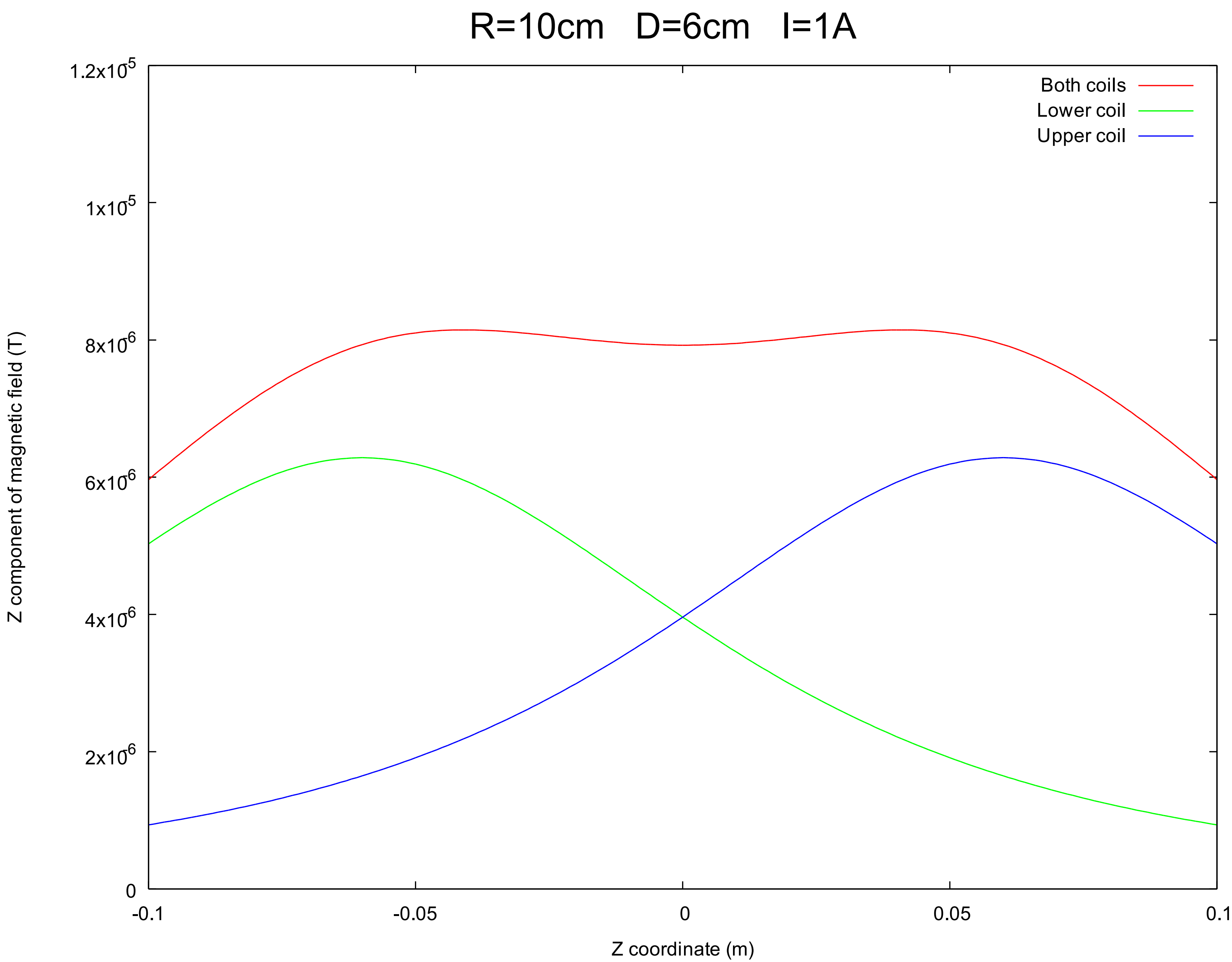
R = coil radius
D = z distance of the plane of each coil from the z=0 plane
D = *R*/2 for best uniform fields



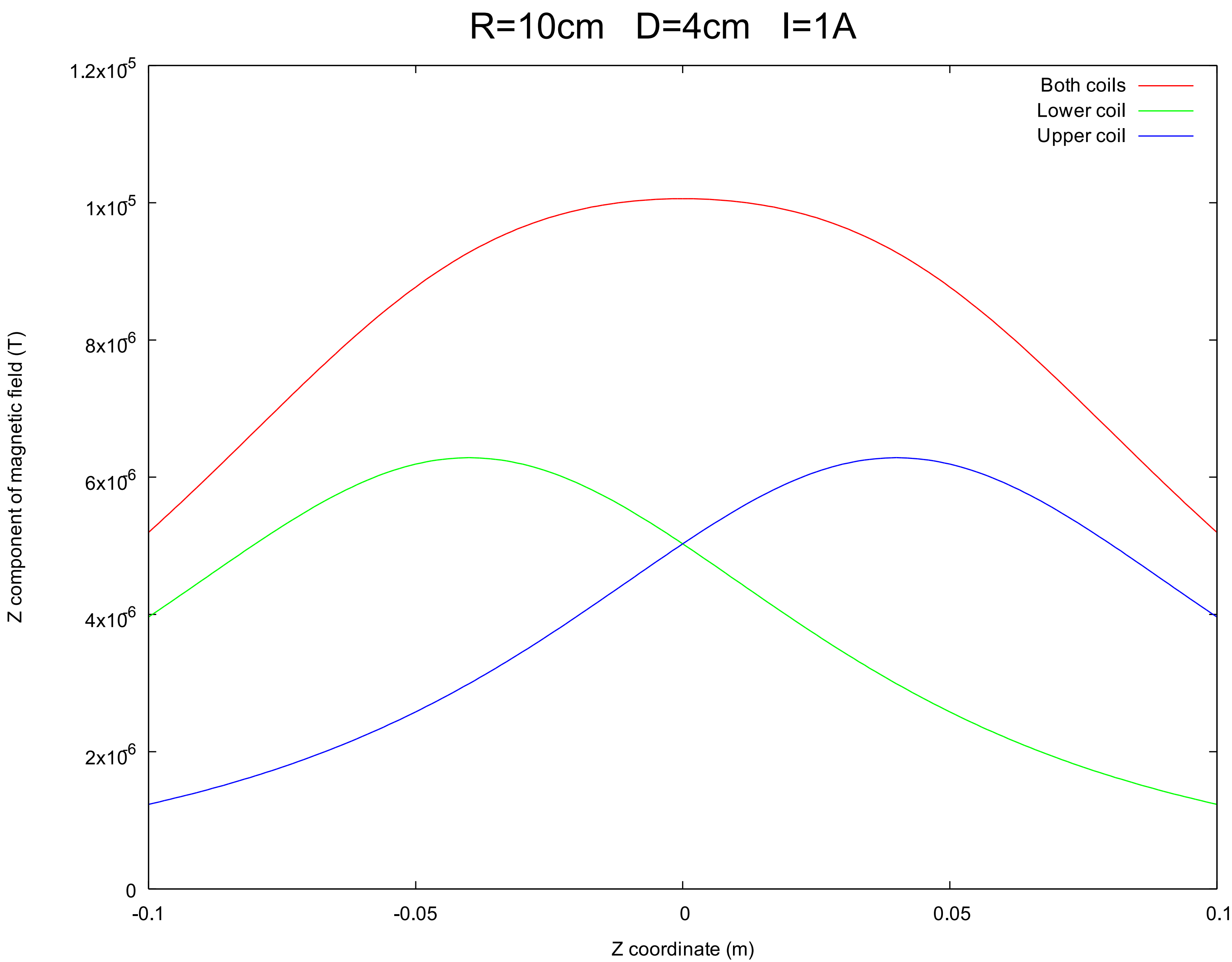
Magnetic Field on Axis of a Helmholtz Coil



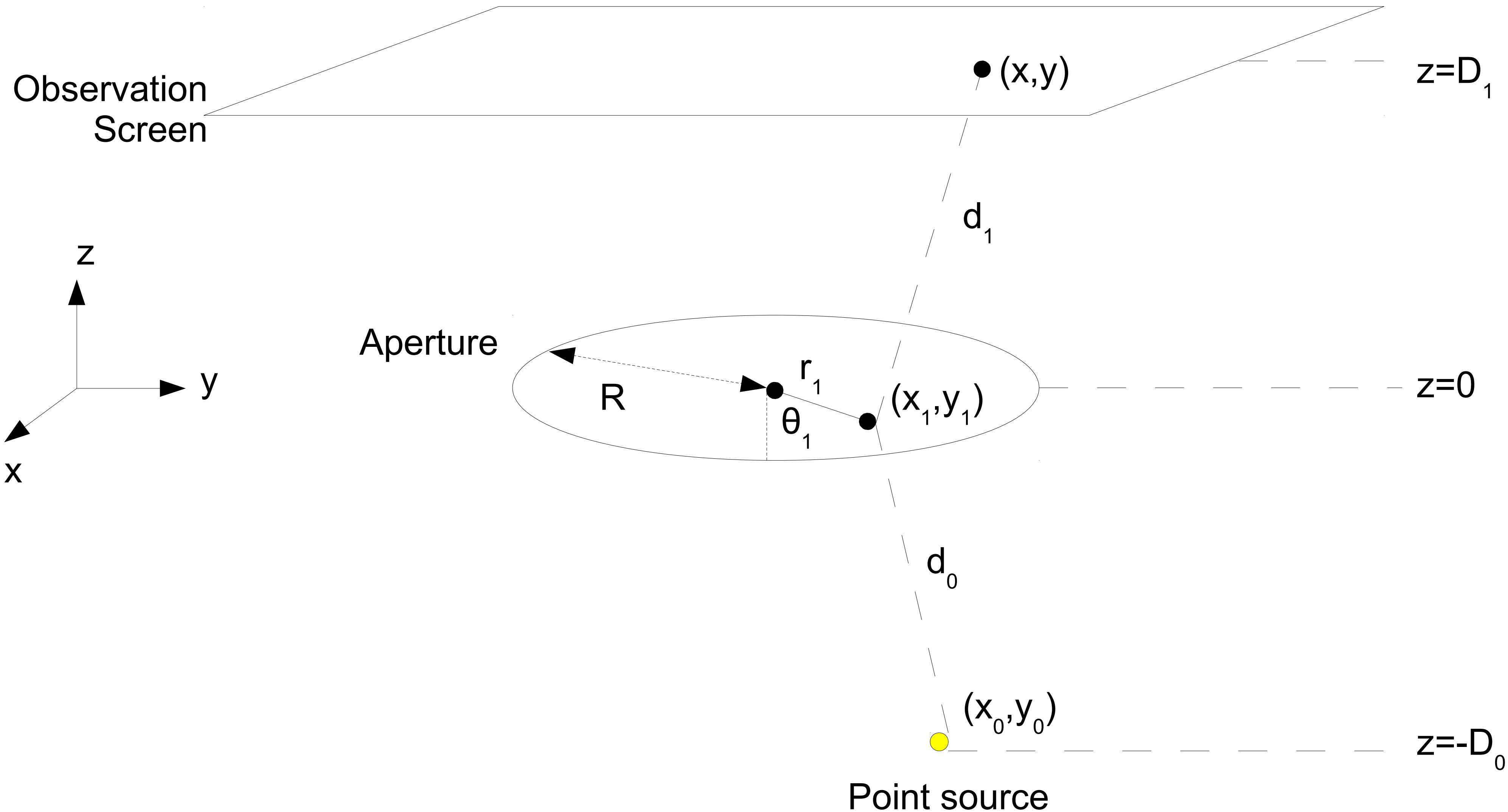
Magnetic Field on Axis of a Helmholtz Coil



Magnetic Field on Axis of a Helmholtz Coil



Two-Dimensional Wave Diffraction Patterns



Review of Phasors

Consider any physical system described by a linear second order ODE with constant coefficients, driven by a sinusoid at a fixed frequency and a reference phase:

$$\frac{dx^2}{dt^2} + a\frac{dx}{dt} + b\,x = c\,e^{i\omega t}$$

Look only for a solution in the form of a steady state sinusoid with unknown amplitude and phase. Form a prototype solution of the form:

$$x = A\,e^{i(\omega t - \phi)} = A\,e^{i\phi}e^{i\omega t}$$

Plug prototype solution back into ODE:

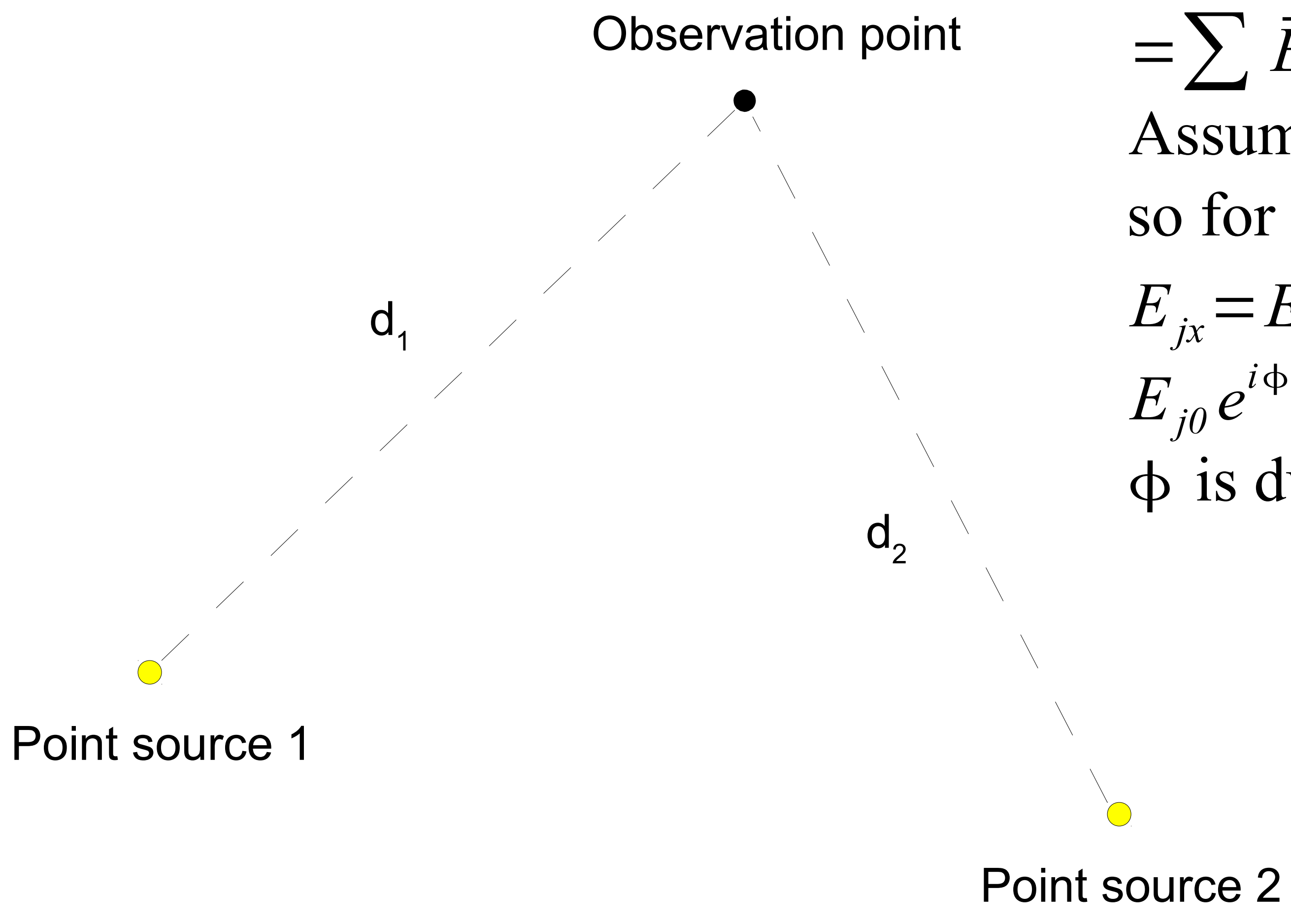
$$-\omega^2 A\,e^{i\phi}e^{i\omega t} + a\,i\,\omega\,A\,e^{i\phi}e^{i\omega t} + b\,A\,e^{i\phi}e^{i\omega t} = c\,e^{i\omega t}$$

Cancel $e^{i\omega t}$ factors:

$$-\omega^2 A\,e^{i\phi} + a\,i\,\omega\,A\,e^{i\phi} + b\,A\,e^{i\phi} = c$$

Second order ODE has been transformed into a complex algebraic equation
$Ae^{i\Phi}$ is called the “phasor” of the solution x

Phasor Method for Wave Interference



$\vec{E}$ field at observation point

$= \sum \vec{E}_j$ from point source j

Assume uniform polarization,

so for instance $\vec{E}_j = E_{jx}\vec{x}$

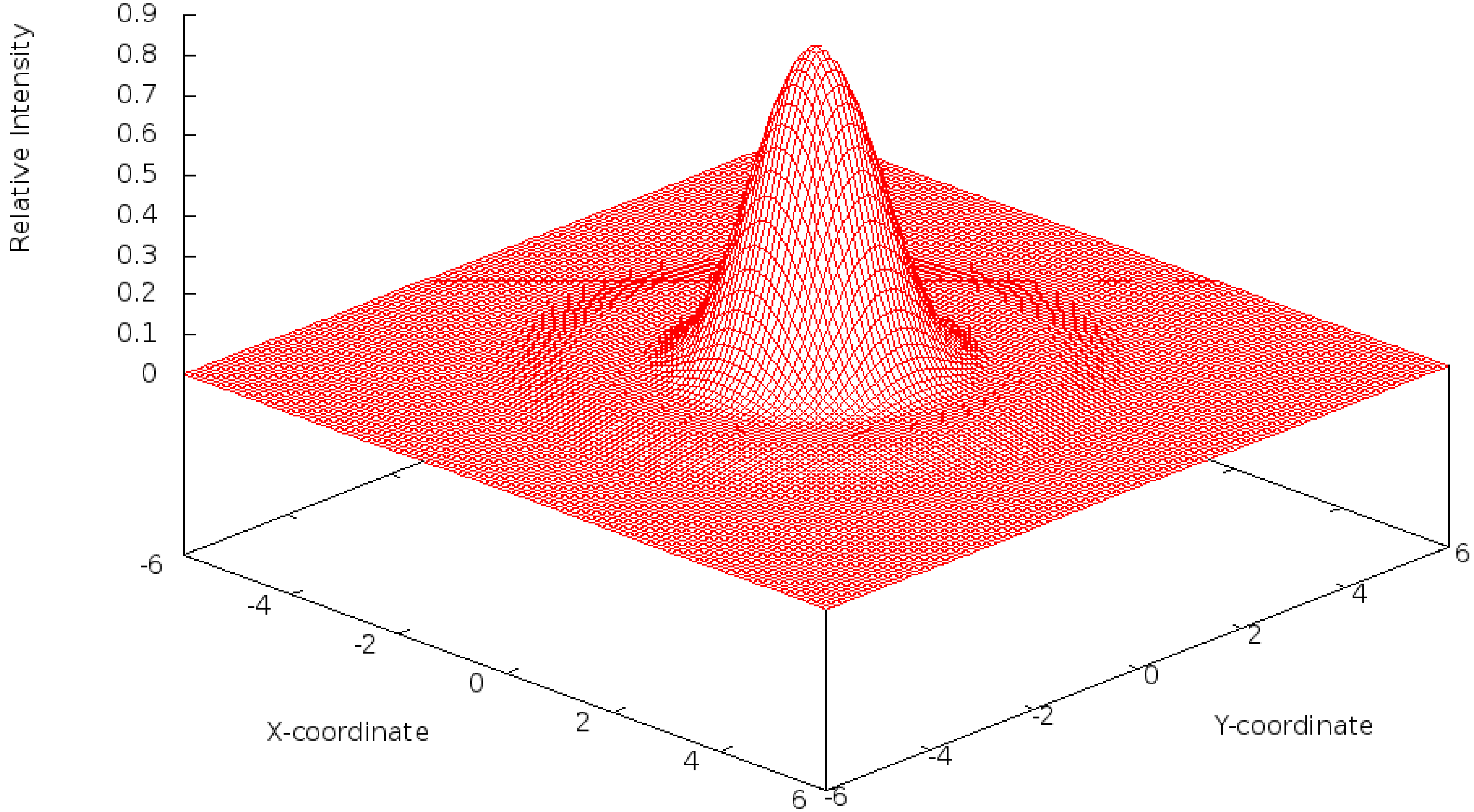
$E_{jx} = E_{j0}\, e^{i(\omega t - \phi)} = E_{j0}\, e^{i\phi}\, e^{i\omega t}$

$E_{j0}\, e^{i\phi}$ is the 'phasor' from source j

ϕ is due to path length differences

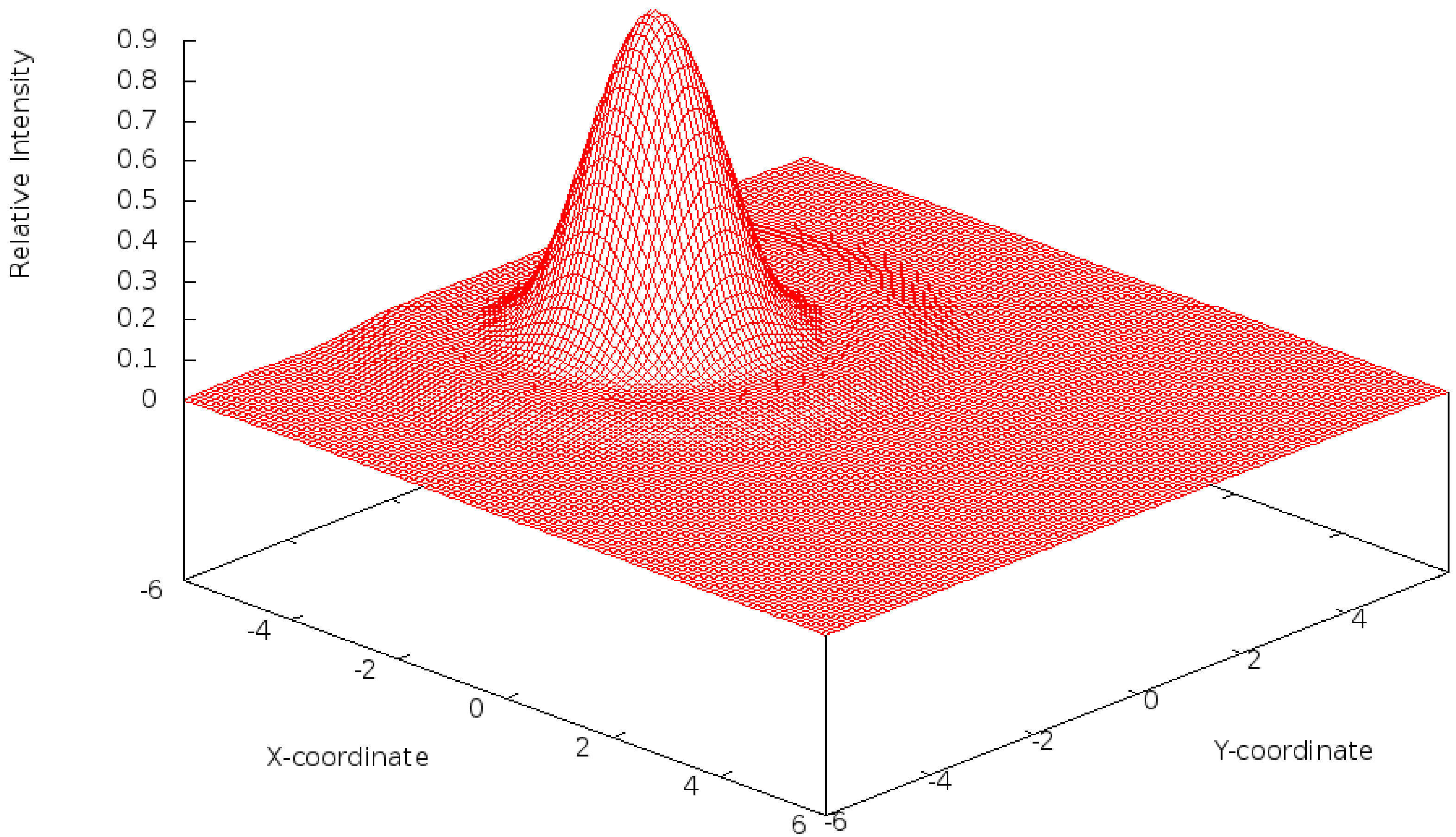
Diffraction Pattern from Single Point Source

R=1
D_1=20
D_0=20
λ=0.2
x_0=0
y_0=0



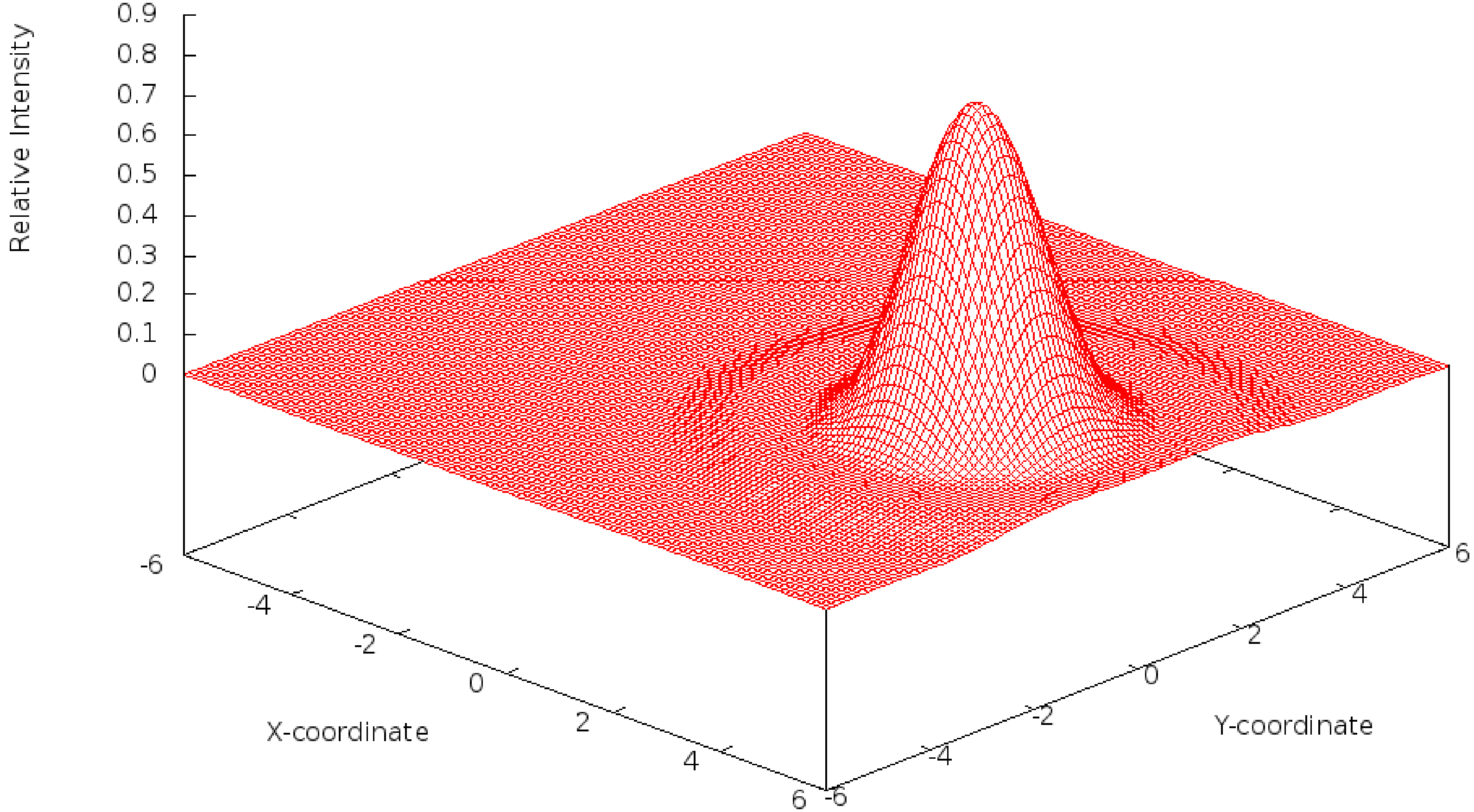
Diffraction Pattern from Single Point Source

R=1
D_1=20
D_0=20
λ=0.2
x_0=3
y_0=0



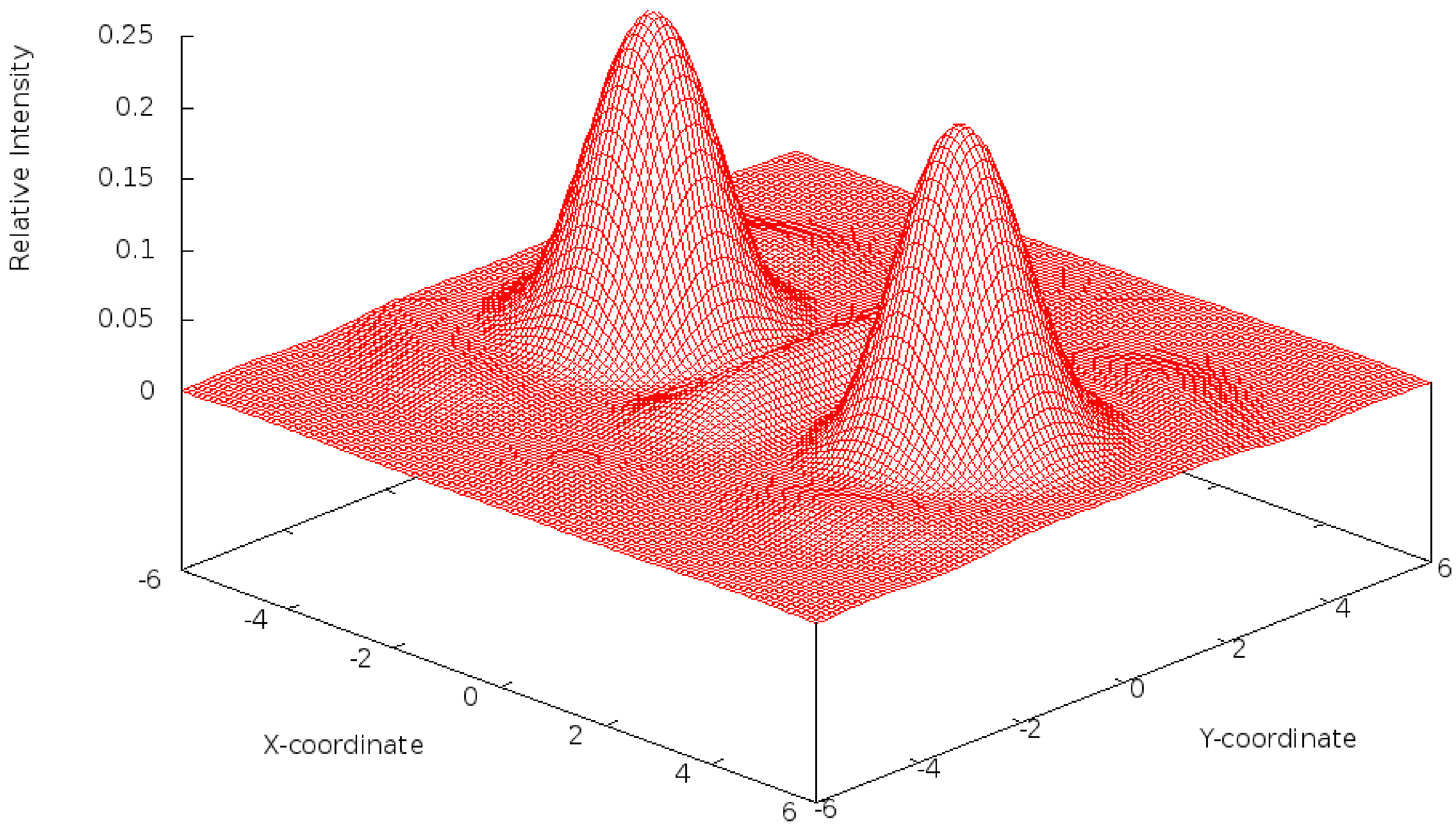
Diffraction Pattern from Single Point Source

$R=1$
$D_1=20$
$D_0=20$
$\lambda=0.2$
$x_0=-3$
$y_0=0$



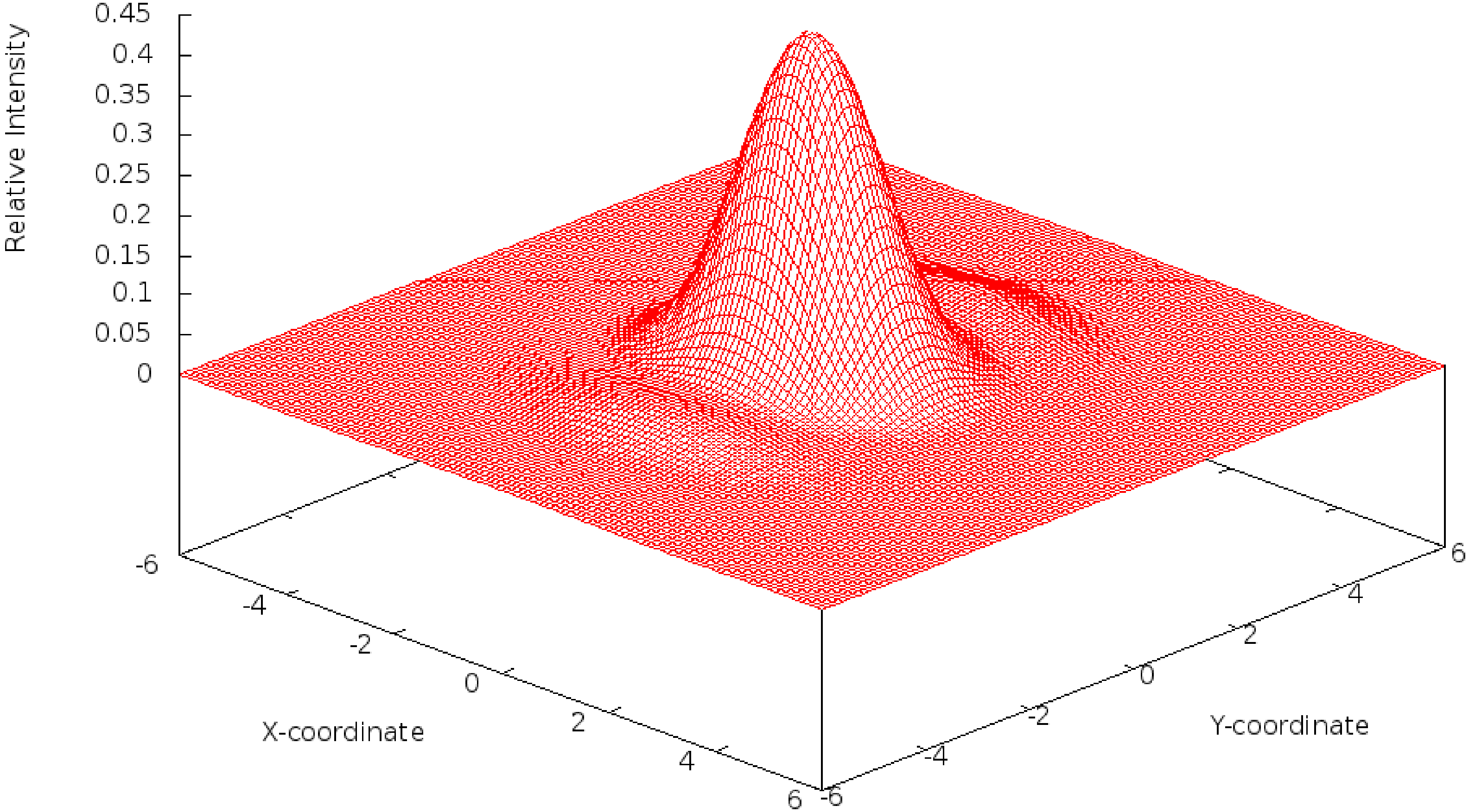
Diffraction Pattern from Two Point Sources

R=1
D_1=20
D_0=20
λ=0.2
x_{01}=-3
y_{01}=0
x_{02}=3
y_{02}=0



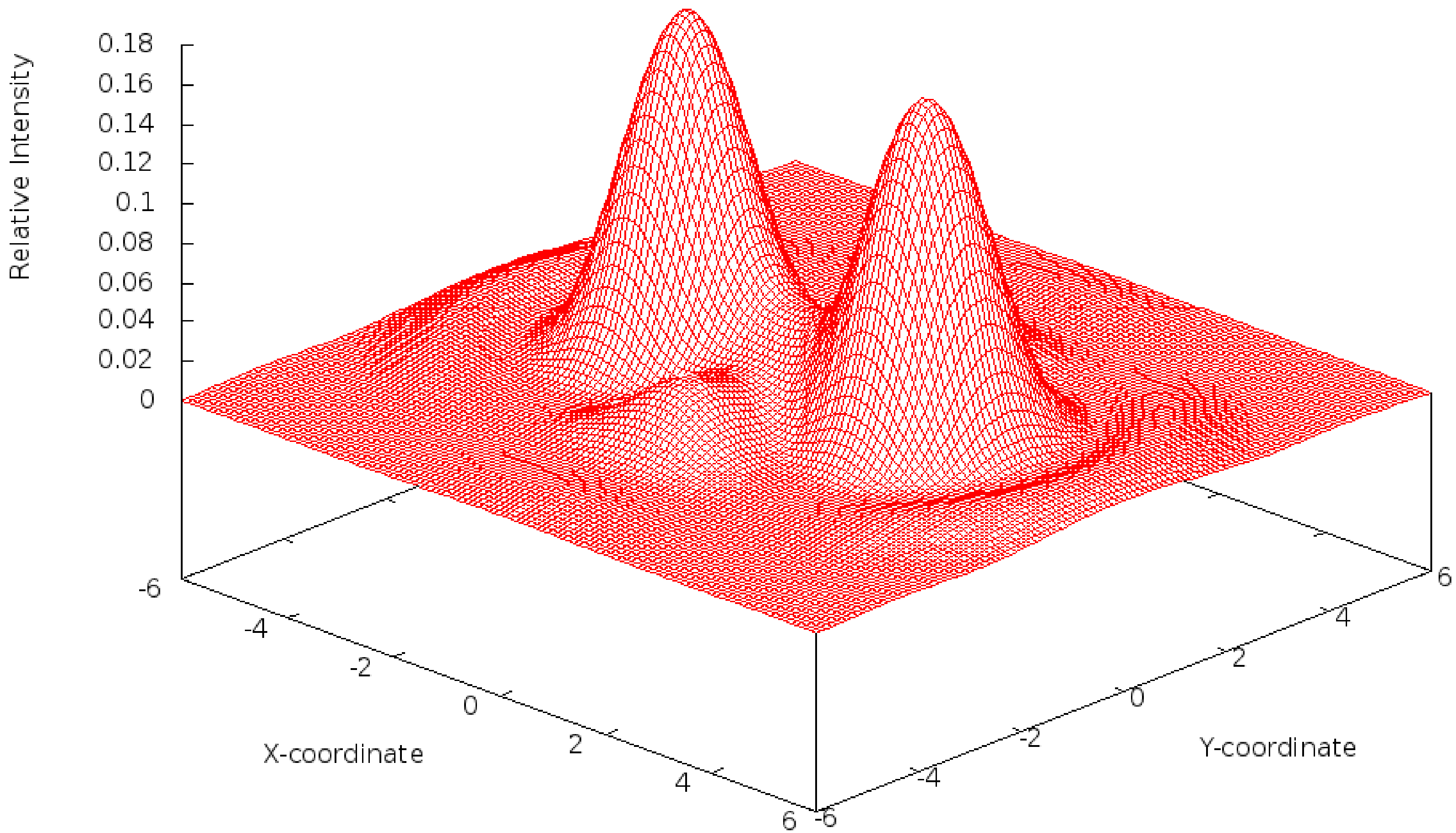
Diffraction Pattern from Two Point Sources

$R=1$
$D_1=20$
$D_0=20$
$\lambda=0.2$
$x_{01}=-1$
$y_{01}=0$
$x_{02}=1$
$y_{02}=0$

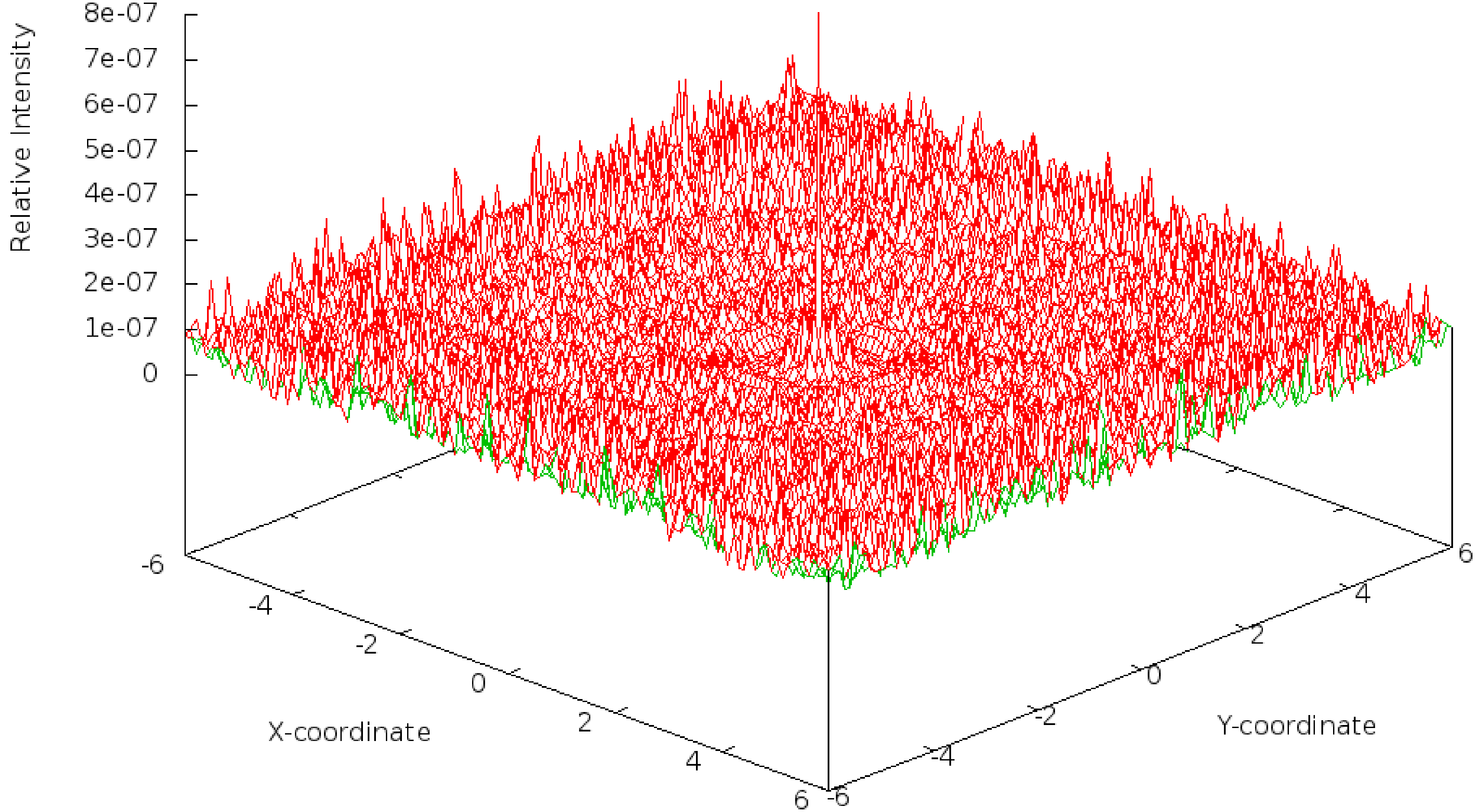


Diffraction Pattern from Two Point Sources

R=1
D_1=20
D_0=20
λ=0.2
x_{01}=-2
y_{01}=0
x_{02}=2
y_{02}=0

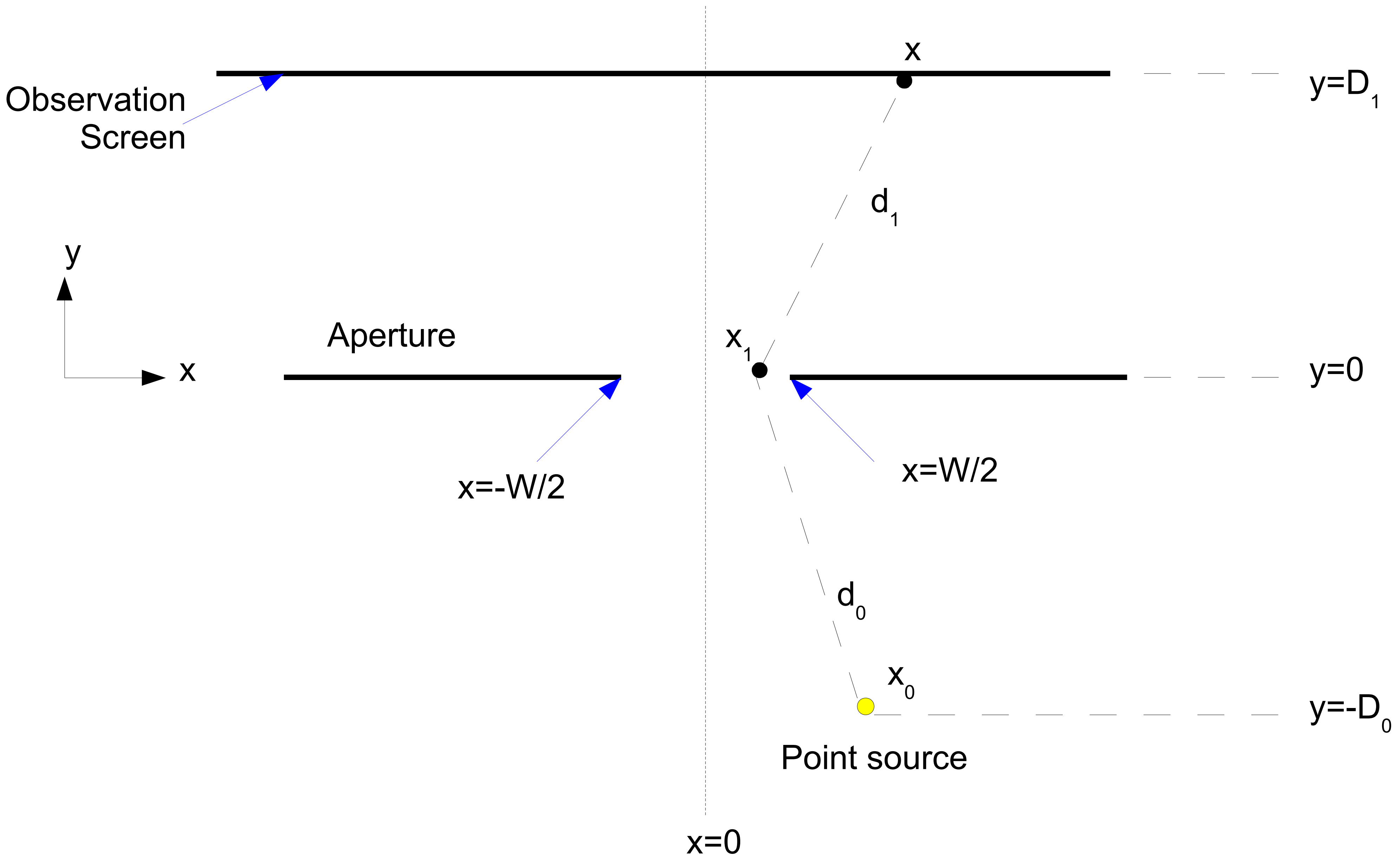


Simulation of the Poisson Spot



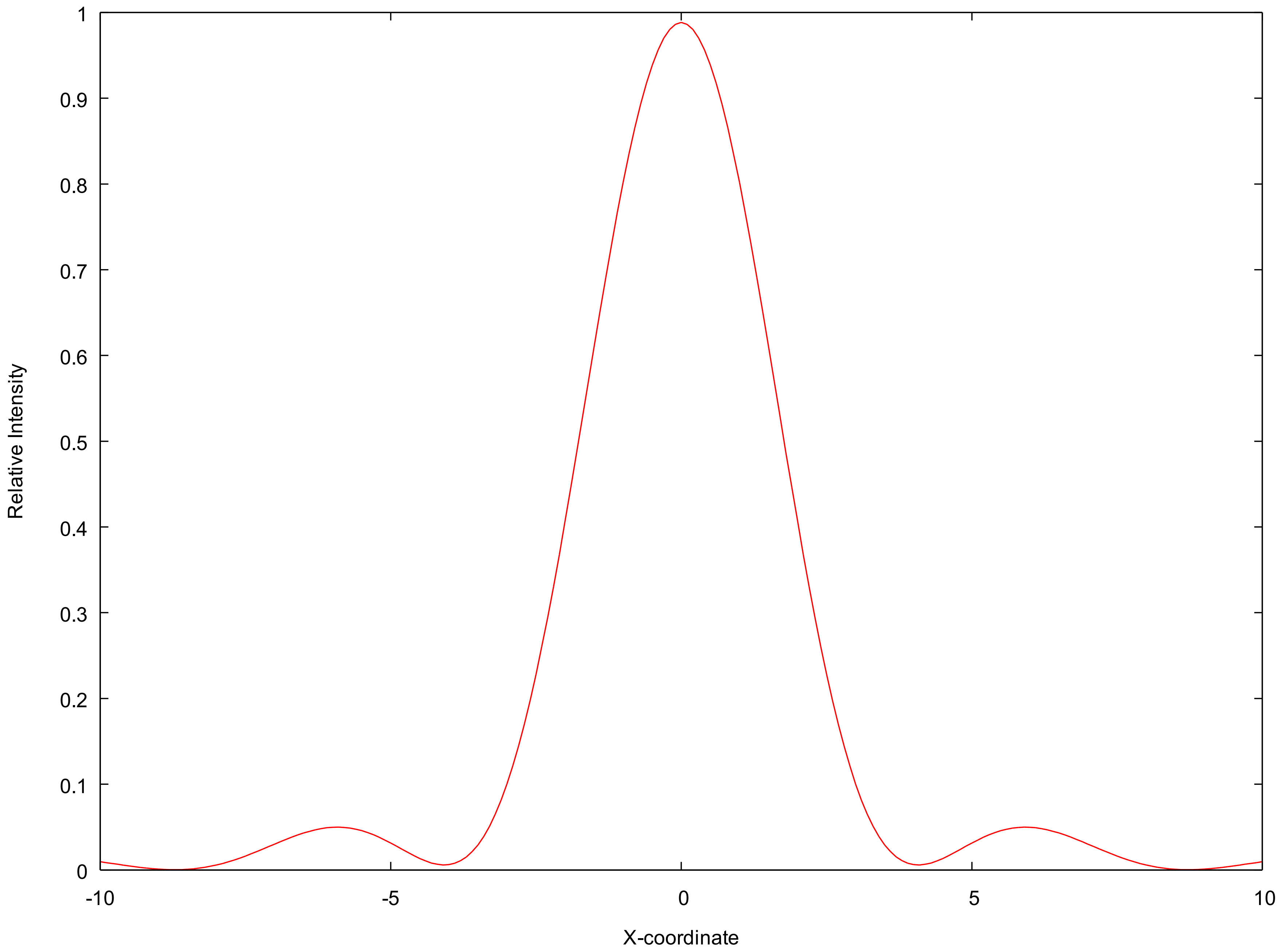
$R_{DISC}=2$
$D_1=20$
$D_0=20$
$\lambda=0.2$
$x_0=0$
$y_0=0$

One-Dimensional Wave Diffraction Patterns



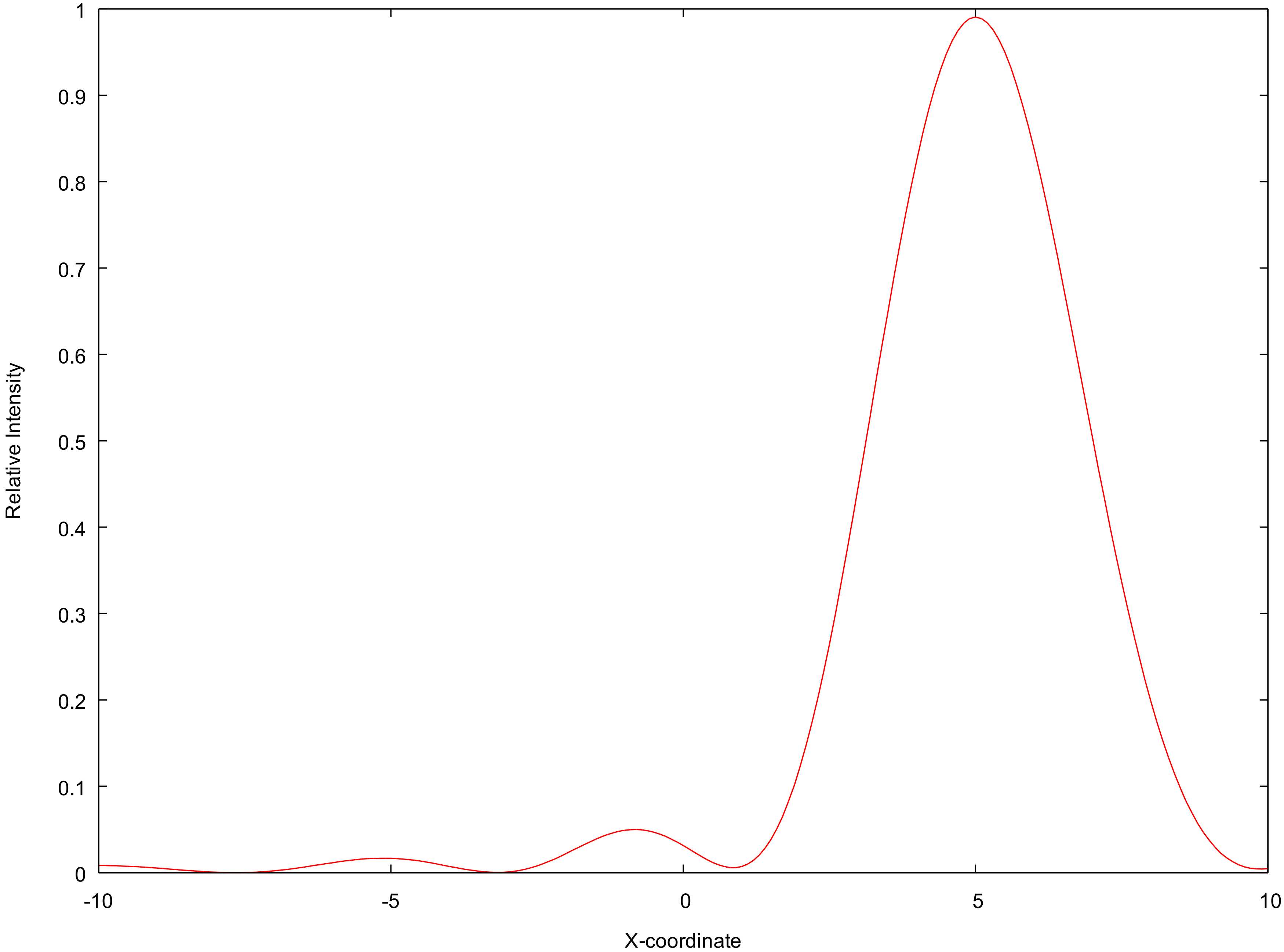
1-Dimensional Diffraction Pattern from Single Point Source

W=1
D_1=20
D_0=20
λ=0.2
x_0=0



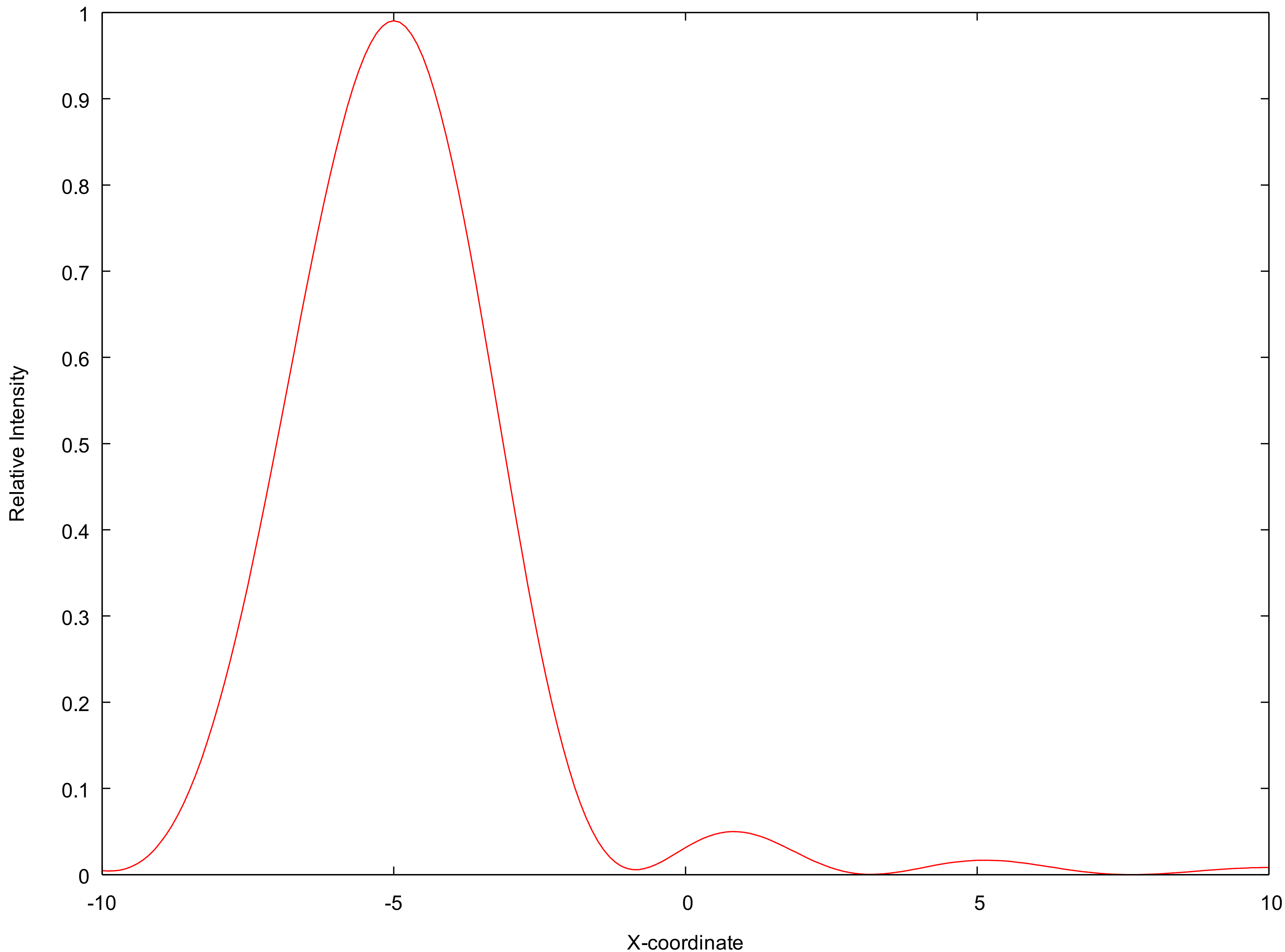
1-Dimensional Diffraction Pattern from Single Point Source

W=1
D_1=20
D_0=20
λ=0.2
x_0=-5



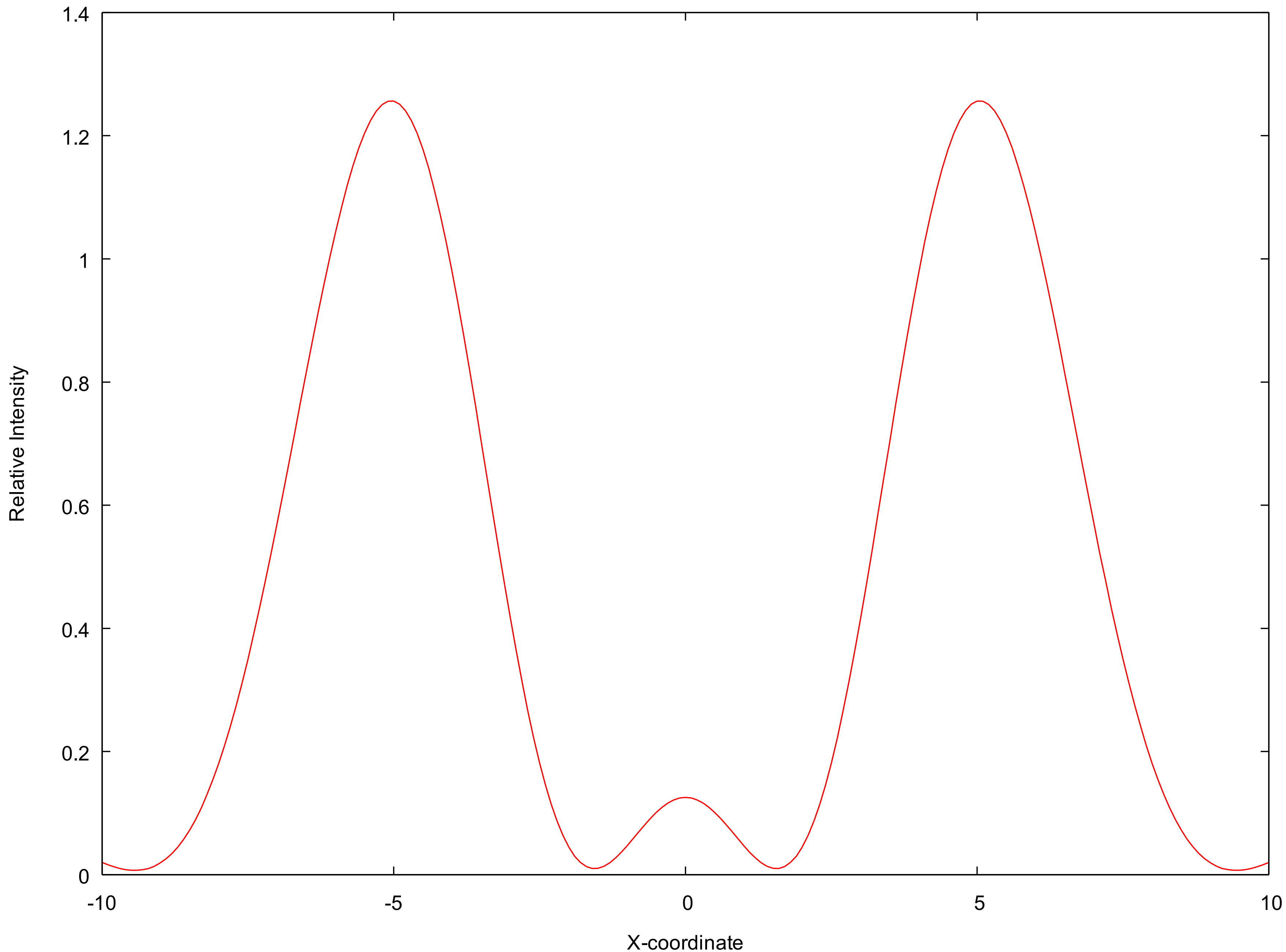
1-Dimensional Diffraction Pattern from Single Point Source

W=1
D_1=20
D_0=20
λ=0.2
x_0=5



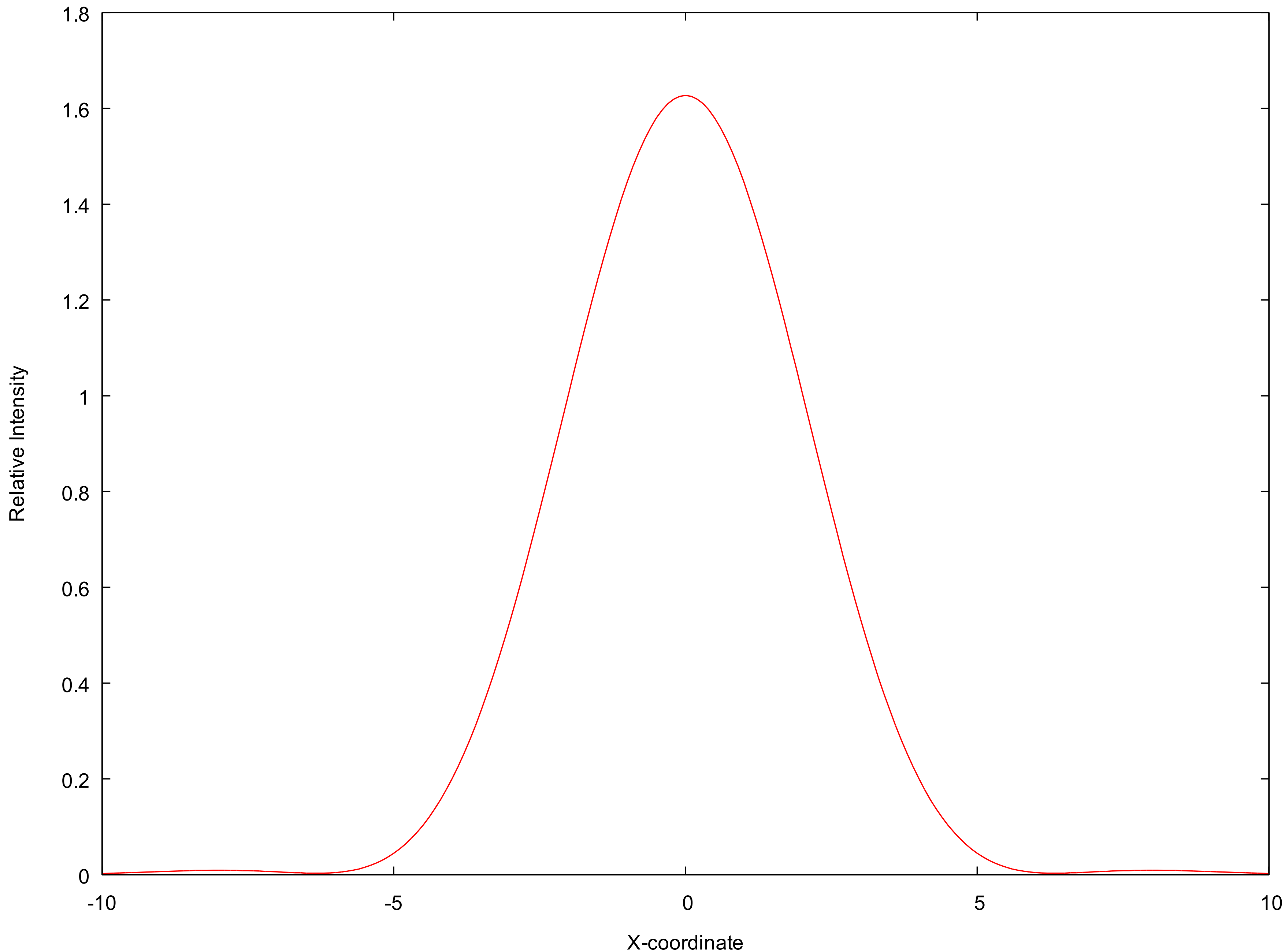
1-Dimensional Diffraction Pattern from Two Point Sources

W=1
D_1=20
D_0=20
λ=0.2
x_{01}=5
x_{02}=-5



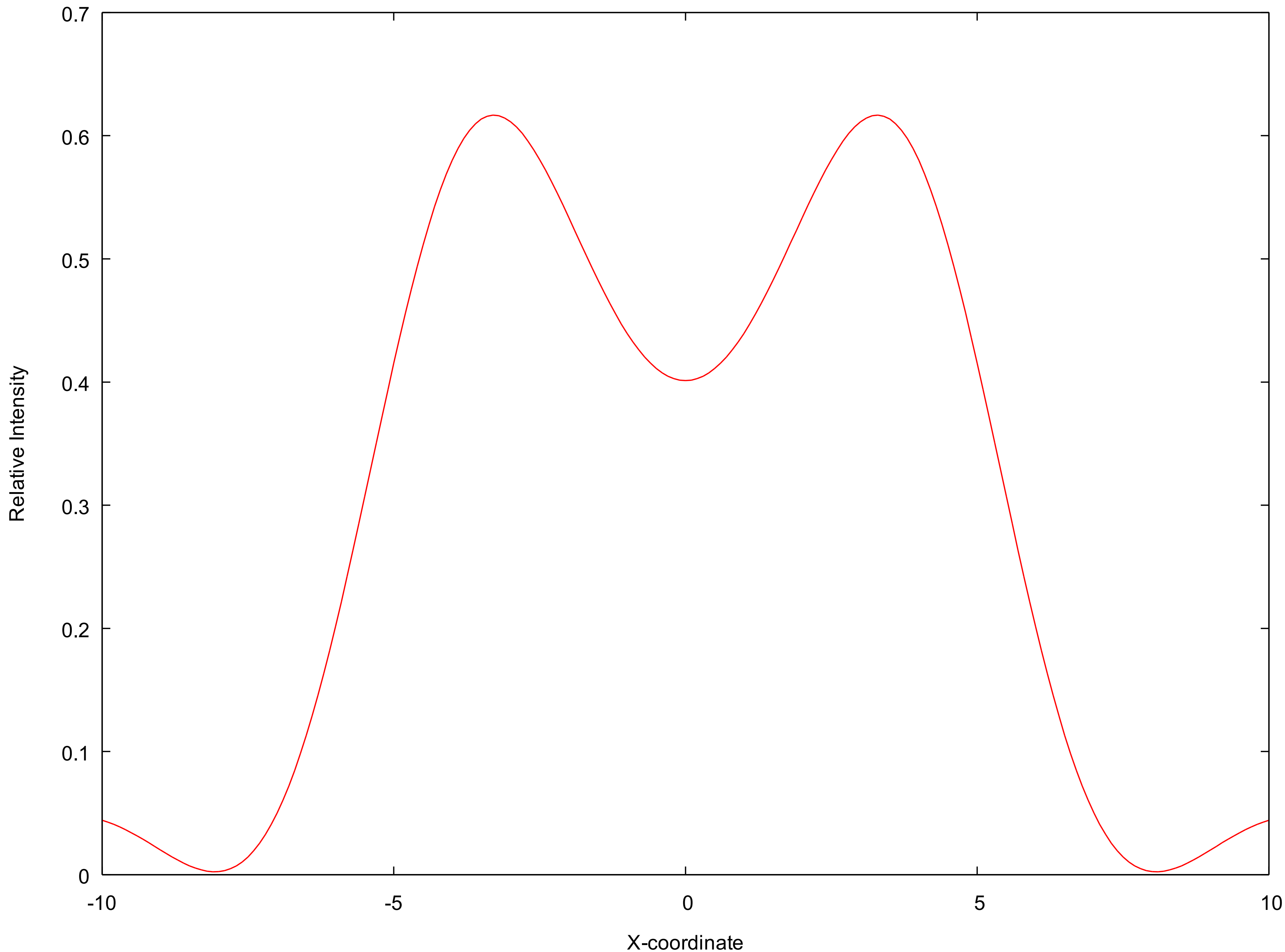
1-Dimensional Diffraction Pattern from Two Point Sources

W=1
D_1=20
D_0=20
λ=0.2
x_{01}=2
x_{02}=-2



1-Dimensional Diffraction Pattern from Two Point Sources

$W=1$
$D_1=20$
$D_0=20$
$\lambda=0.2$
$x_{01}=3$
$x_{02}=-3$



Plotting Vectors with gnuplot

We have used:

```
plot 'efield1.dat' using 1:2 with lines

plot 'efield1.dat' using 1:2 with points
```

Column with X coordinate

Column with Y coordinate

Another plot mode:

```
plot 'efield1.dat' using 1:2:3:4 with vectors
```

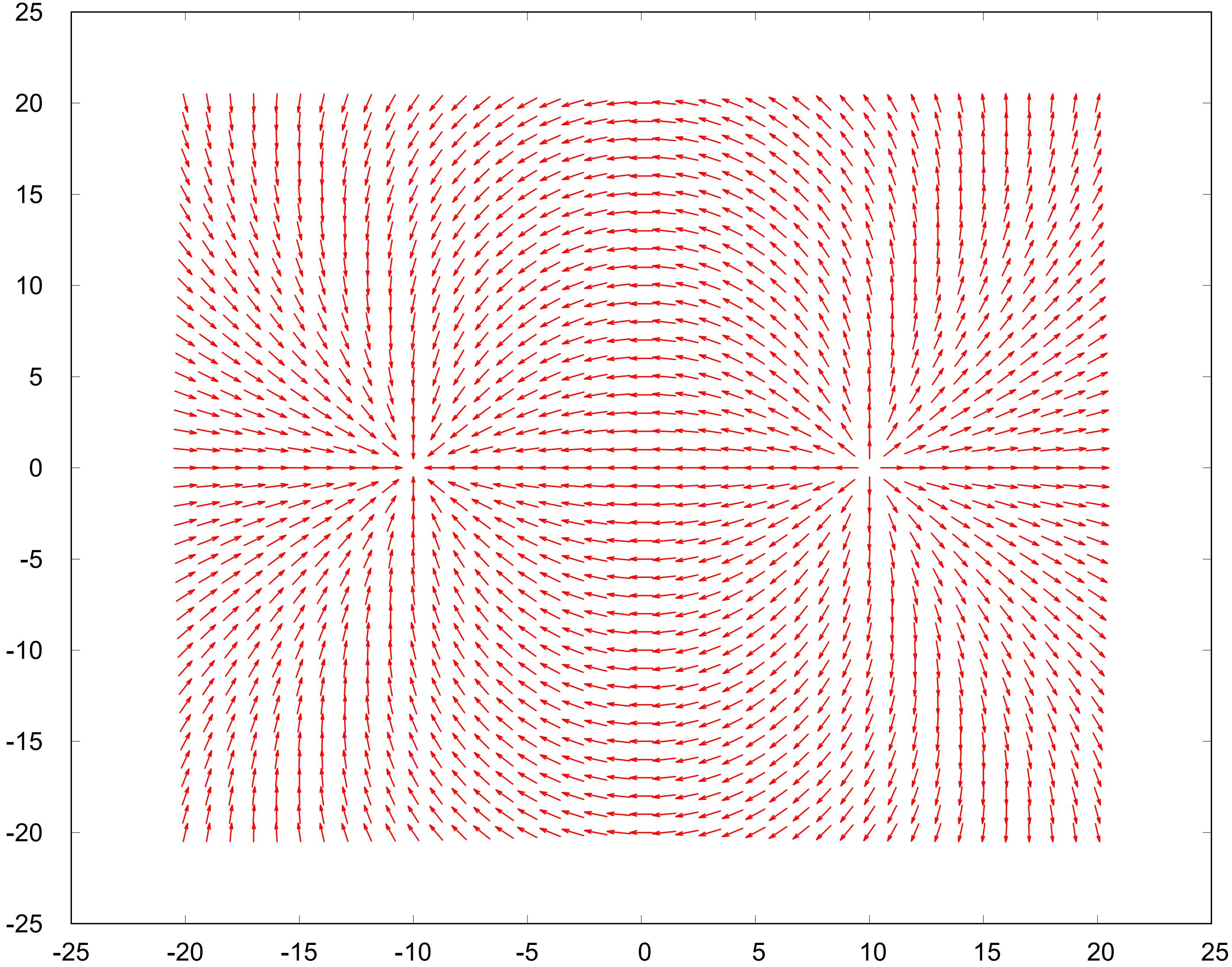
Column with starting X coordinate

Column with starting Y coordinate

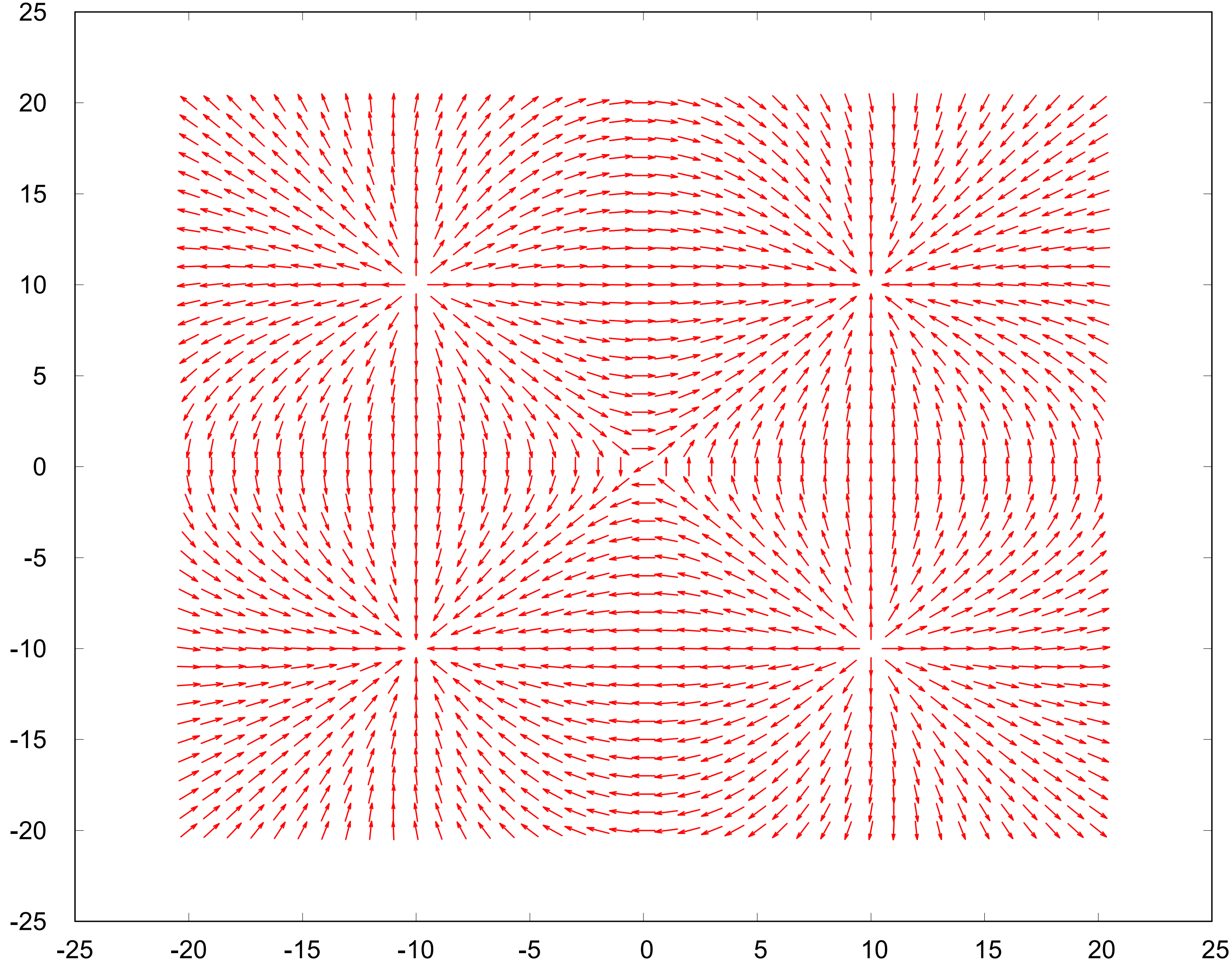
Column with X distance

Column with Y distance

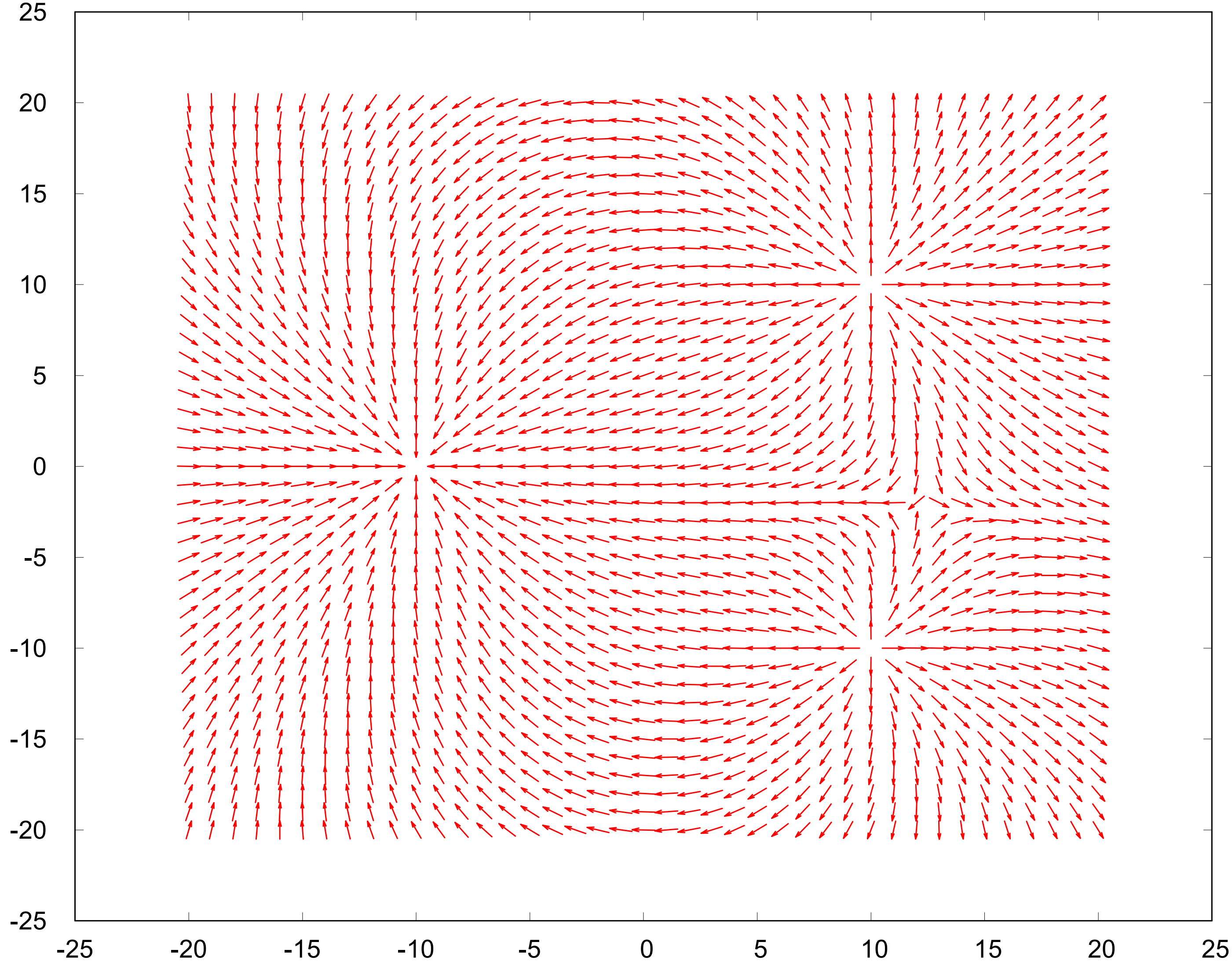
Electric Field Vectors for Dipole



Electric Field Vectors for Quadrapole



Electric Field Vectors for Example Arbitrary Charges



C Code for Mapping Electric Field Vectors

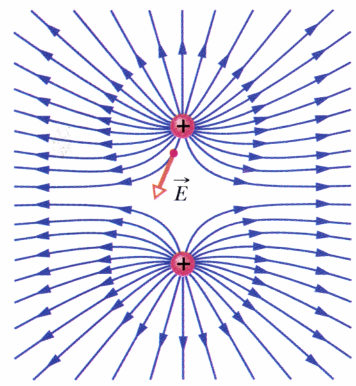
```
xstop = xstop + xinc * 0.5;
ystop = ystop + yinc * 0.5;
rtest = 1.0e-6 * (xinc*xinc + yinc*yinc);
x = xstart;
while (((xinc > 0.0) && (x < xstop)) || ((xinc < 0.0) && (x > xstop))) {
  y = ystart;
  while (((yinc > 0.0) && (y < ystop)) || ((yinc < 0.0) && (y > ystop))) {
    ex = 0.0;
    ey = 0.0;
    i = 0;
    while (i < n) {
      rsq = (x-xi[i])*(x-xi[i]) + (y-yi[i])*(y-yi[i]);
      if (rsq < rtest) break;
      r = sqrt(rsq);
      ex += qi[i] * (x-xi[i]) / (r * rsq);
      ey += qi[i] * (y-yi[i]) / (r * rsq);
      i++;
    }
    if (i >= n) {
      emag = sqrt(ex*ex + ey*ey);
      vx = xinc * ex / emag;
      vy = yinc * ey / emag;
      printf("%.8g %.8g %.8g %.8g\n",x-0.5*vx,y-0.5*vy,vx,vy);
    }
    y = y + yinc;
  }
  x = x + xinc;
}
```

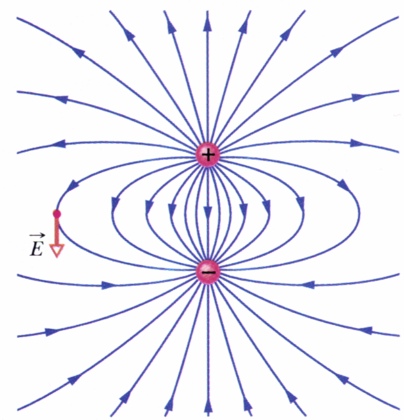
Loop over n charges.
double array qi[] hold each charge q_i and

double arrays xi[] and yi[] hold x_i and y_i coordinates

Classical Electric Field Lines

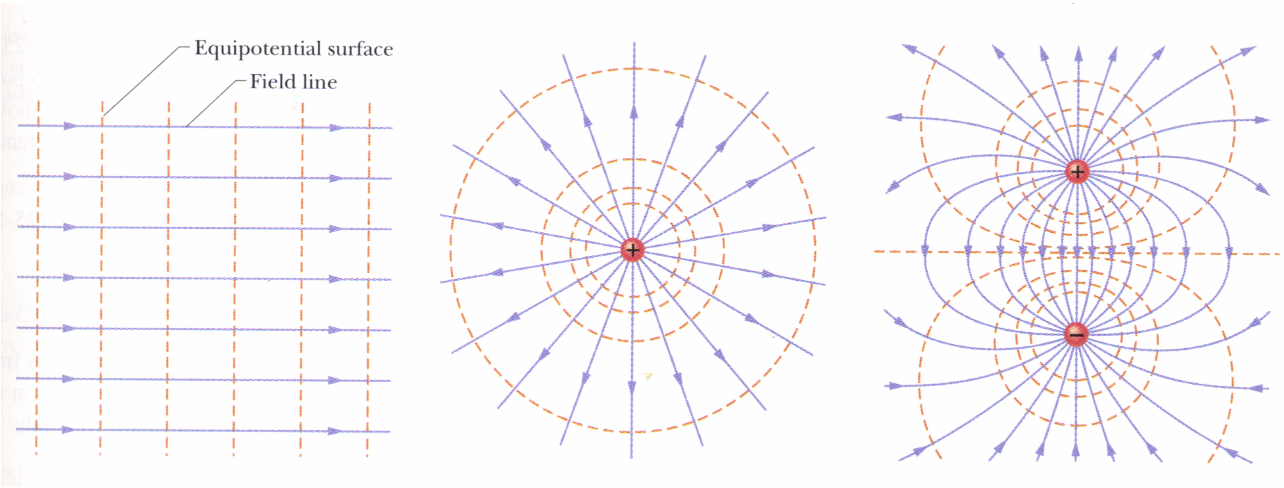
Note that the vectors plotted by this method only record the direction of the E field vector at an array of sampled points. These are not quite equivalent to the classical field lines, which convey both the direction of the field and its magnitude from the density of drawn lines.



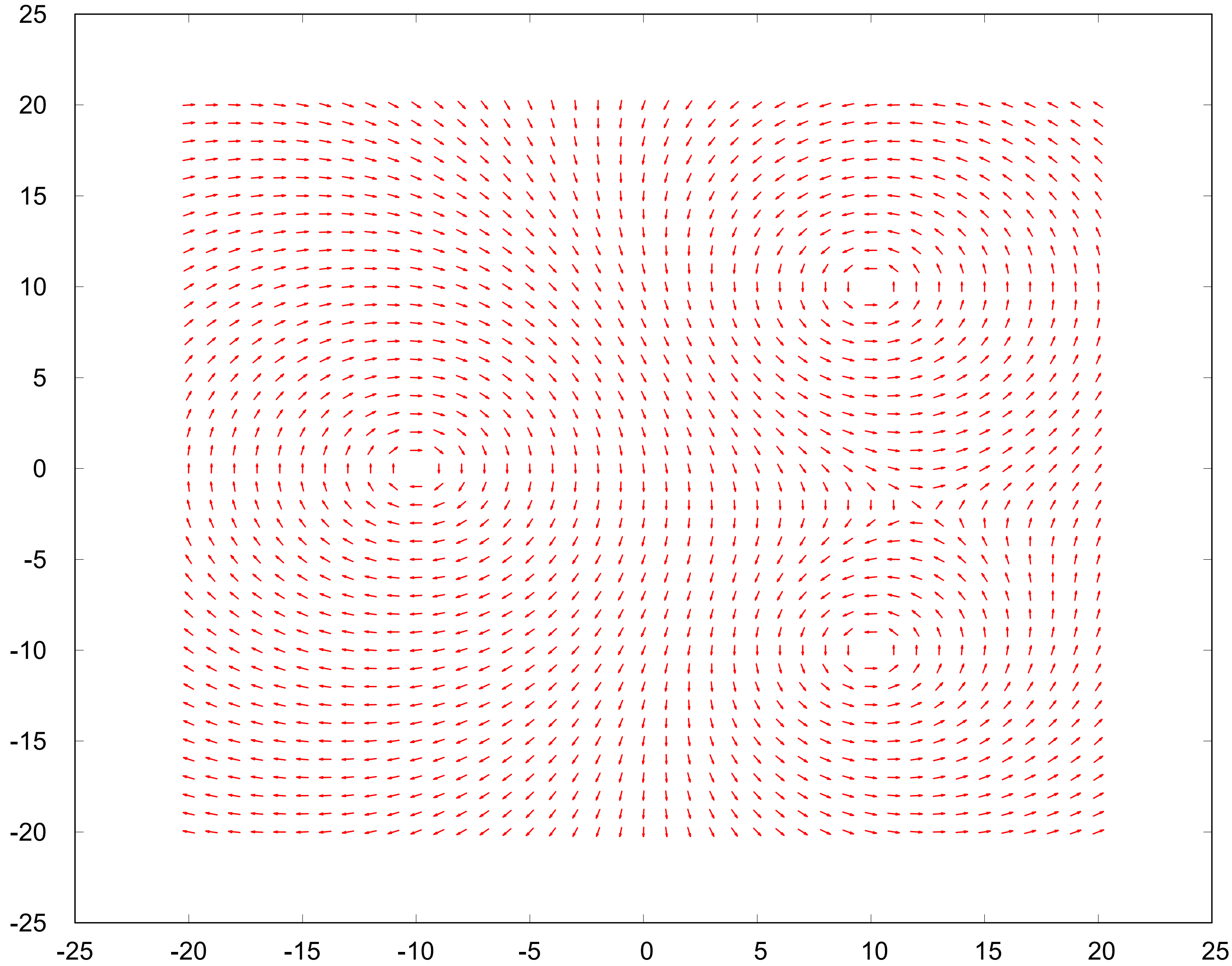


Classical Electric Equipotential Lines

A simple modification of this algorithm can be used to plot a sampling of the direction of the electric equipotential lines. At a given point, field lines and equipotential lines will have slopes that are negative reciprocals of each other. Again, no magnitude information is present, only direction.



Equipotential Vectors for Example Arbitrary Charges



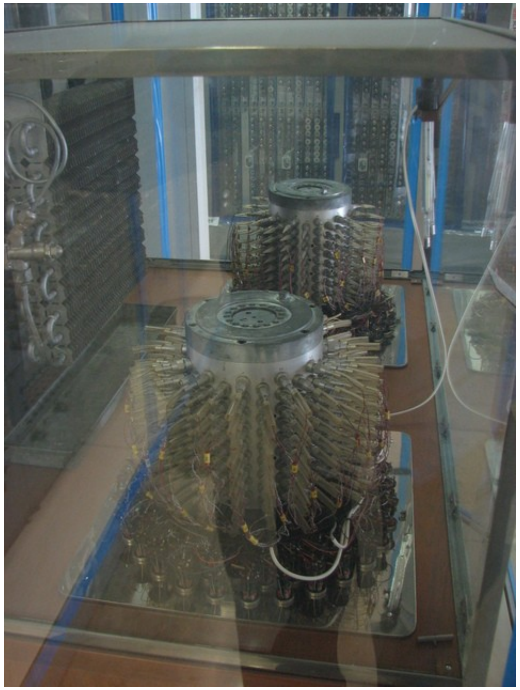
C Code for Mapping Equipotential Vectors

```
xstop = xstop + xinc * 0.5;
ystop = ystop + yinc * 0.5;
rtest = 1.0e-6 * (xinc*xinc + yinc*yinc);
x = xstart;
while (((xinc > 0.0) && (x < xstop)) || ((xinc < 0.0) && (x > xstop))) {
  y = ystart;
  while (((yinc > 0.0) && (y < ystop)) || ((yinc < 0.0) && (y > ystop))) {
    ex = 0.0;
    ey = 0.0;
    i = 0;
    while (i < n) {
      rsq = (x-xi[i])*(x-xi[i]) + (y-yi[i])*(y-yi[i]);
      if (rsq < rtest) break;
      r = sqrt(rsq);
      ex += qi[i] * (x-xi[i]) / (r * rsq);
      ey += qi[i] * (y-yi[i]) / (r * rsq);
      i++;
    }
    if (i >= n) {
      emag = sqrt(ex*ex + ey*ey);
      vx = -0.5 * xinc * ey / emag;
      vy = 0.5 * yinc * ex / emag;
      printf("%.8g %.8g %.8g %.8g\n",x-0.5*vx,y-0.5*vy,vx,vy);
    }
    y = y + yinc;
  }
  x = x + xinc;
}
```

Only these two statements change!

Examples of Optical Glare

Glare is the reflected room light off of an intervening plexiglass protective sheet. Reflections occur at the interface between two materials of different refractive indices.



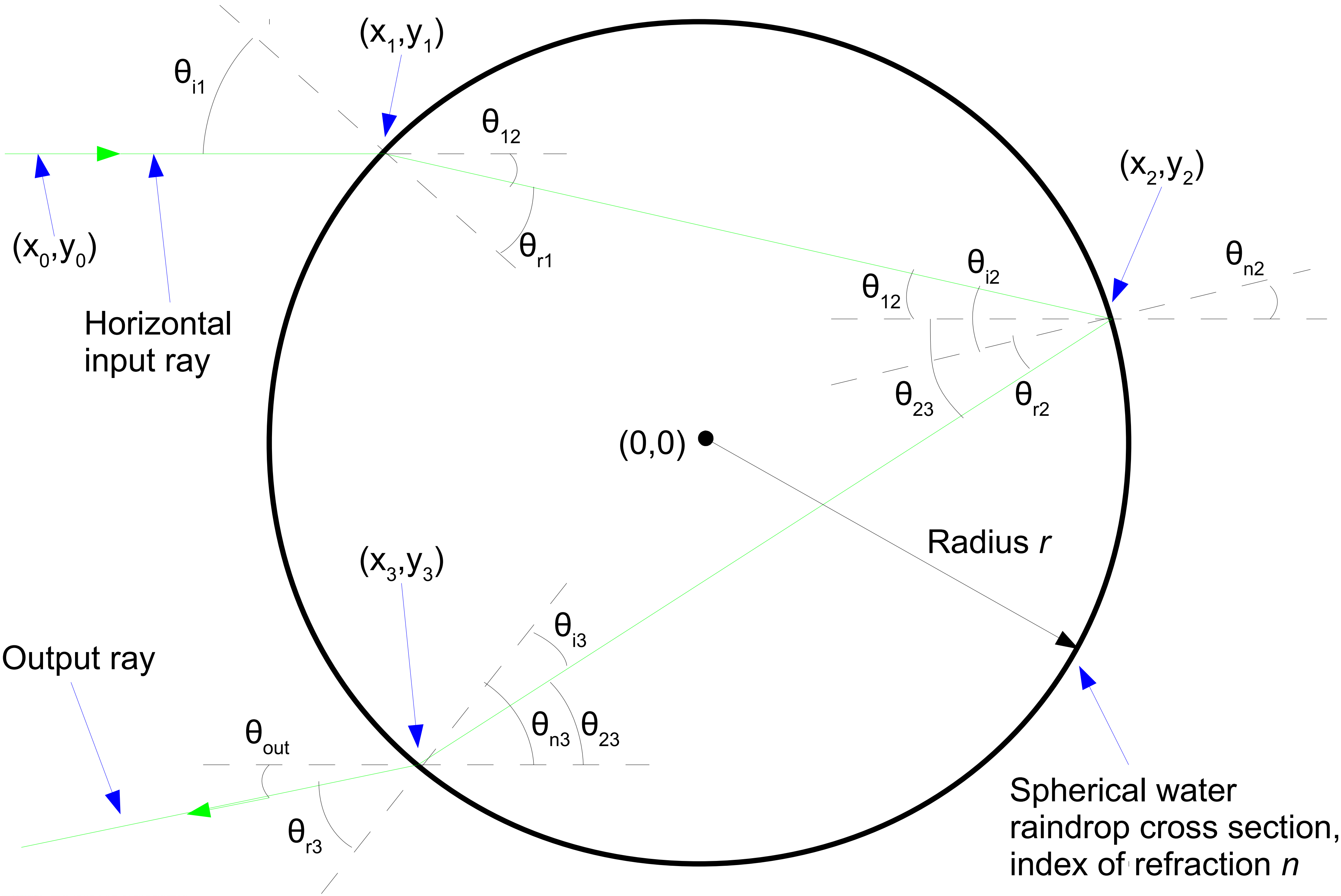
(Vacuum tube arithmetic unit and magnetic drum memory from the Univac 1, on display at the Deutches Museum of Technology, Munich, Germany)

Computing Another Kind of Glare: a Rainbow

Rene Descartes Graphical Derivation

“I took my pen and made an accurate calculation of the paths of the rays which fall on the different points of a globe of water to determine at which angles, after two refractions and one or two reflections they will come to the eye, and I then found that after one reflection and two refractions there are many more rays which can be seen at an angle of from forty-one to forty-two degrees than at any smaller angle; and that there are none which can be seen at a larger angle"

Calculating Reflection and Refraction Angles



Calculating Reflection and Refraction Angles

$$x_1 = -\sqrt{r^2 - y_0^2} \qquad y_1 = y_0 \qquad \theta_{i1} = \arctan\left(\left|\frac{y_1}{x_1}\right|\right)$$

$$\text{Snell's law: } \theta_{r1} = \arcsin\left(\frac{1}{n}\sin(\theta_{i1})\right)$$

$$\theta_{12} = \theta_{i1} - \theta_{r1} \qquad m_{12} = \tan(-\theta_{12})$$

$$x_2 = \frac{x_1(m_{12}^2 - 1) - 2m_{12}y_1}{m_{12}^2 + 1} \qquad y_2 = y_1 + m_{12}(x_2 - x_1)$$

$$\theta_{n2} = \arctan\left(\frac{y_2}{x_2}\right) \qquad \theta_{i2} = \theta_{n2} + \theta_{12} \qquad \theta_{23} = \theta_{n2} + \theta_{i2} = 2\theta_{n2} + \theta_{12} \qquad m_{23} = \tan(\theta_{23})$$

$$x_3 = \frac{x_2(m_{23}^2 - 1) - 2m_{23}y_2}{m_{23}^2 + 1} \qquad y_3 = y_2 + m_{23}(x_3 - x_2)$$

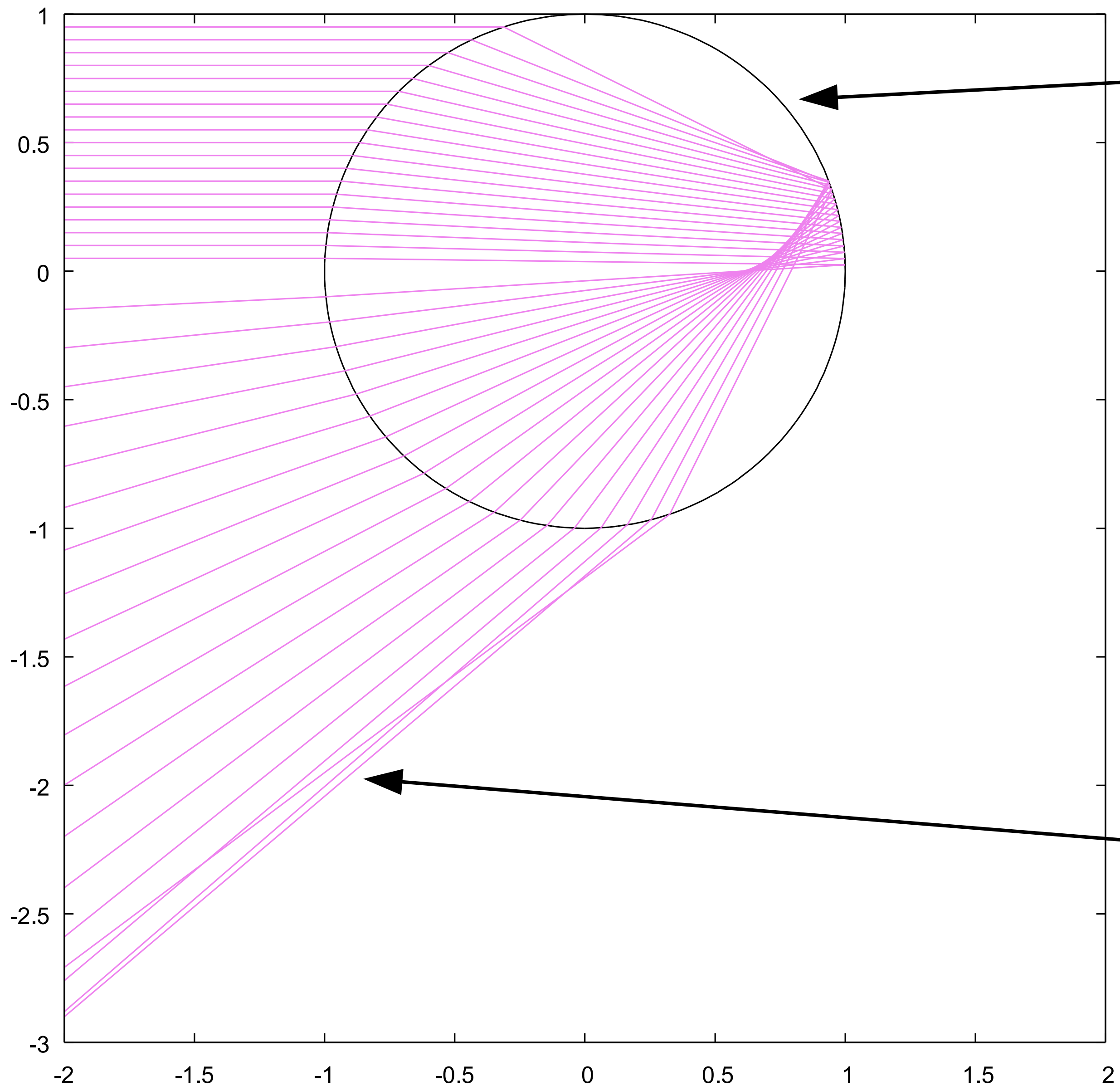
$$\theta_{n3} = \arctan\left(\frac{y_3}{x_3}\right) \qquad \theta_{i3} = \theta_{n3} - \theta_{23}$$

$$\text{Snell's law: } \theta_{r3} = \arcsin(n \cdot \sin(\theta_{i3}))$$

$$\theta_{out} = \theta_{n3} - \theta_{r3}$$

Reflection and Refractions through a Raindrop

Incoming violet light
λ=400nm



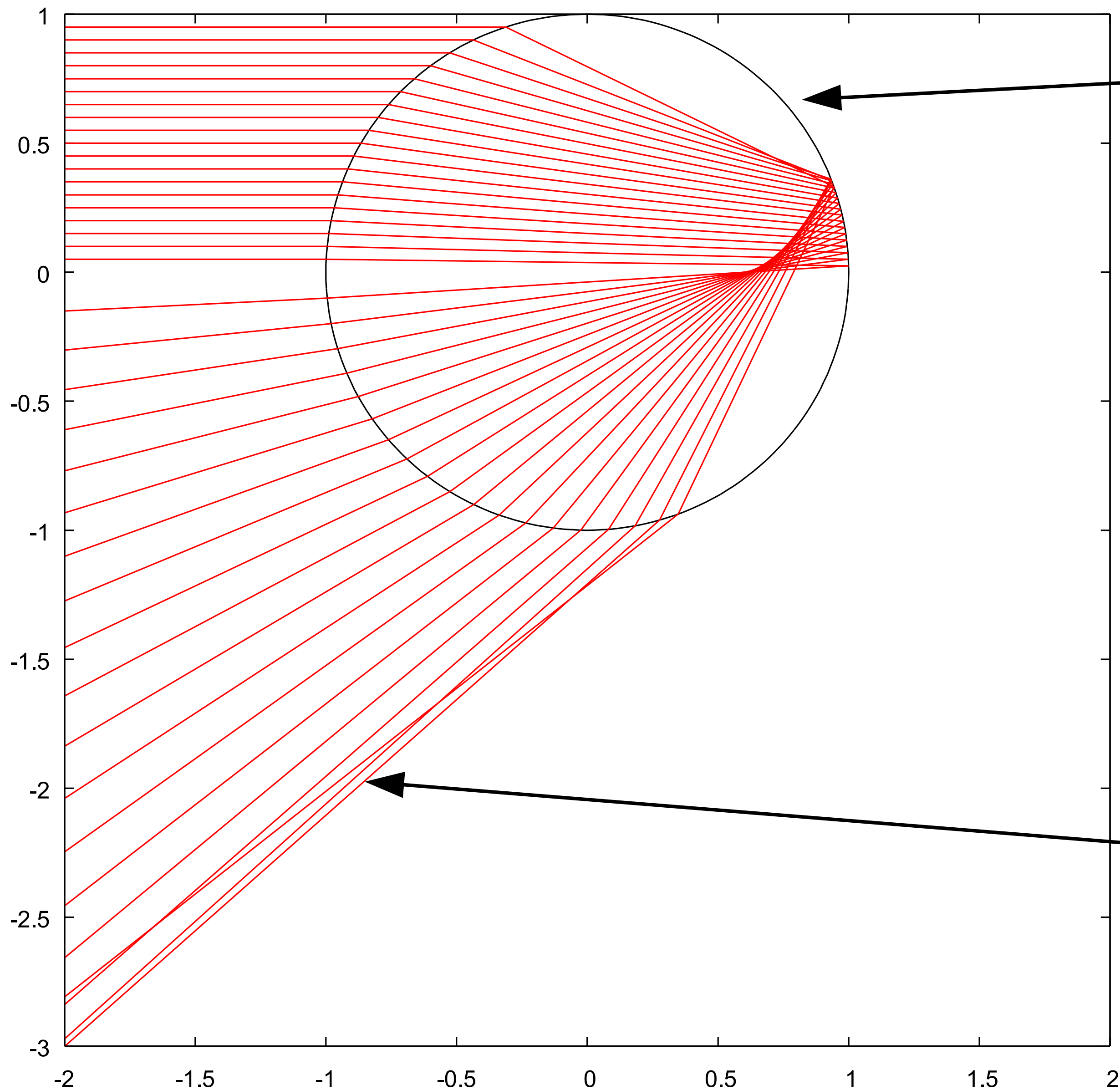
Spherical raindrop of water, n=1.339 at λ=400nm

Reflected light rays to observer on ground

Many reflection paths bundled at an angle of about 40˚ below horizontal

Reflection and Refractions through a Raindrop

Incoming red light
λ=700nm

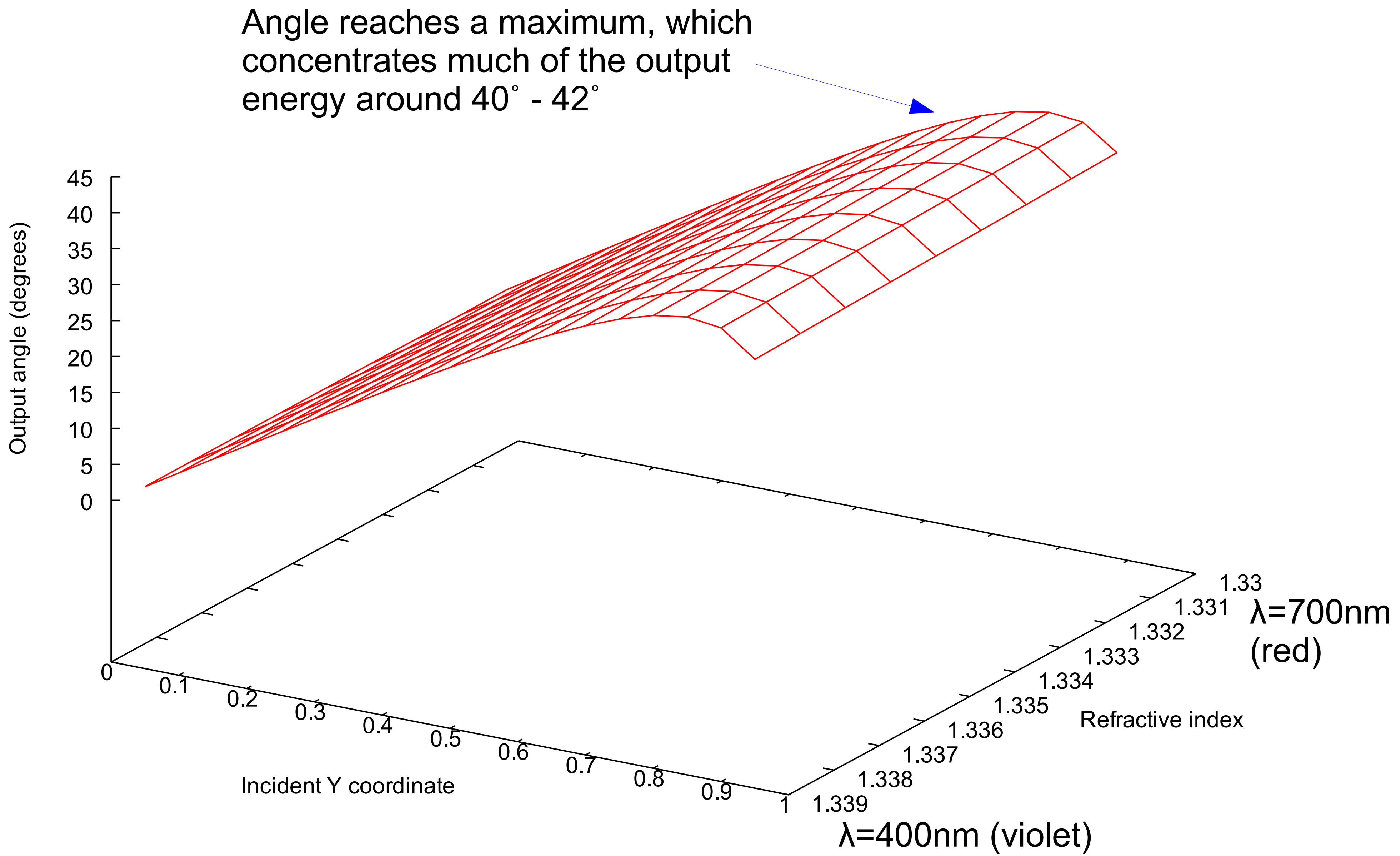


Spherical raindrop of water, n=1.331 at λ=700nm

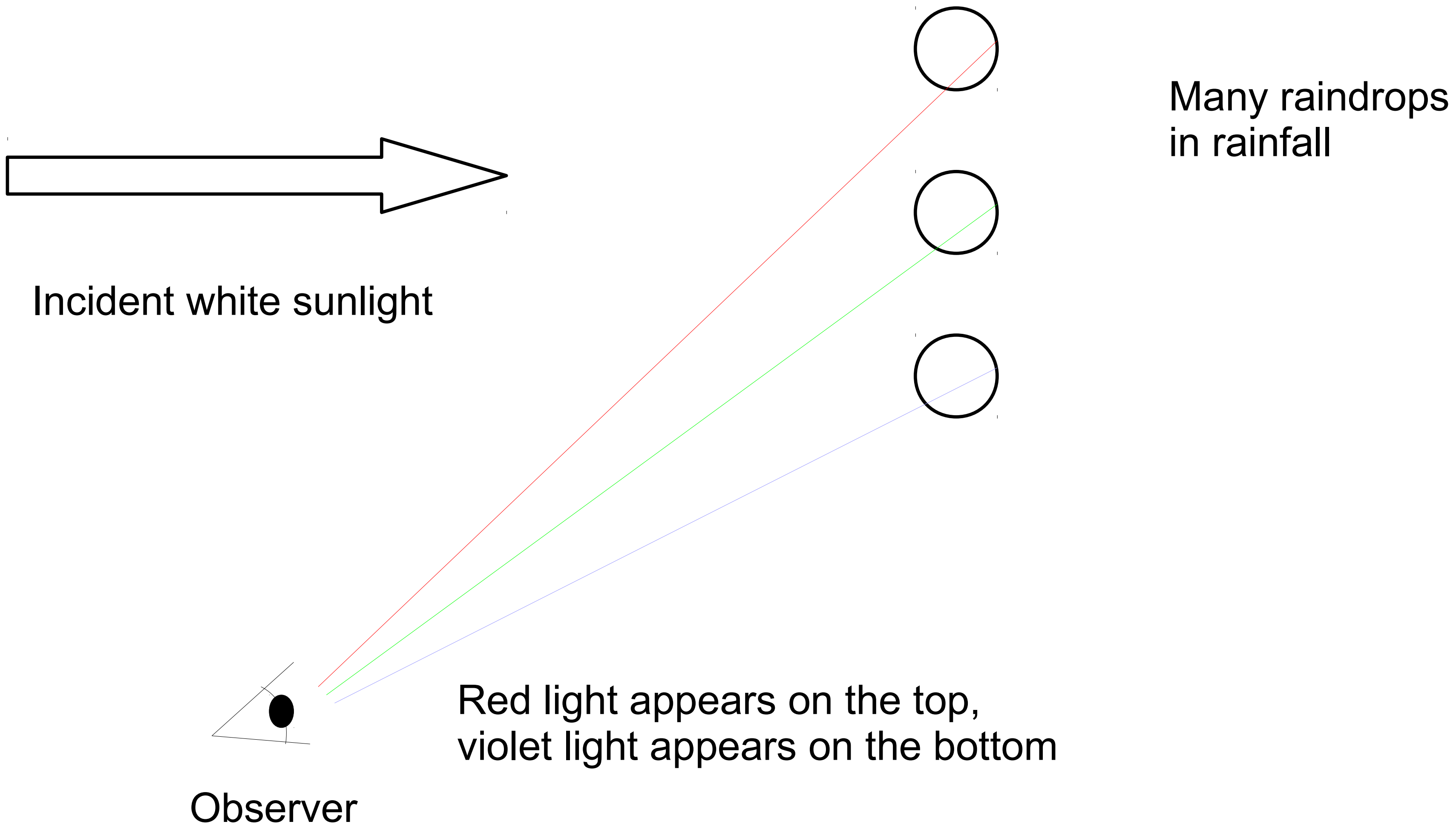
Reflected light rays to observer on ground

Many reflection paths bundled at an angle of about 42˚ below horizontal

Output Angle Relative to Horizontal



Total Effect of Many Raindrops is a Rainbow



Wave Packets for Quantum Mechanics

Solutions of the time-independent Schrödinger equation in one dimension

$$\frac{d^2\psi(x)}{dx^2} = -\frac{2m}{\hbar^2}\left[E - V(x)\right]\psi(x)$$

in regions where particles are free to move and not subject to forces, or

$$V(x) = 0 \quad \text{and} \quad E > 0$$

are of the form

$$\psi(x) = A e^{+ikx} + B e^{-ikx}$$

which makes the complete solution including the time dependence

$$\psi(x) = A e^{+i(kx - \omega t)} + B e^{-i(kx + \omega t)}$$

$$\text{where} \quad k = \frac{p}{\hbar} \text{ and } \omega = \frac{E}{\hbar}$$

so the relationship between ω and k is

$$\omega = \frac{\hbar}{2m} k^2$$

Wave Packets for Quantum Mechanics

A single wave solution is associated with a continuous flux of particles in the positive or negative *x* direction, with momentum *p* and energy *E*.

Solutions that are linear combinations of multiple waves form wave packets that are associated with single particles.

$$\psi(x) = \sum_k A_k \psi_k(x)$$

Or in the continuous limit

$$\psi(x) = \int A(k) \psi_k(x) dk$$

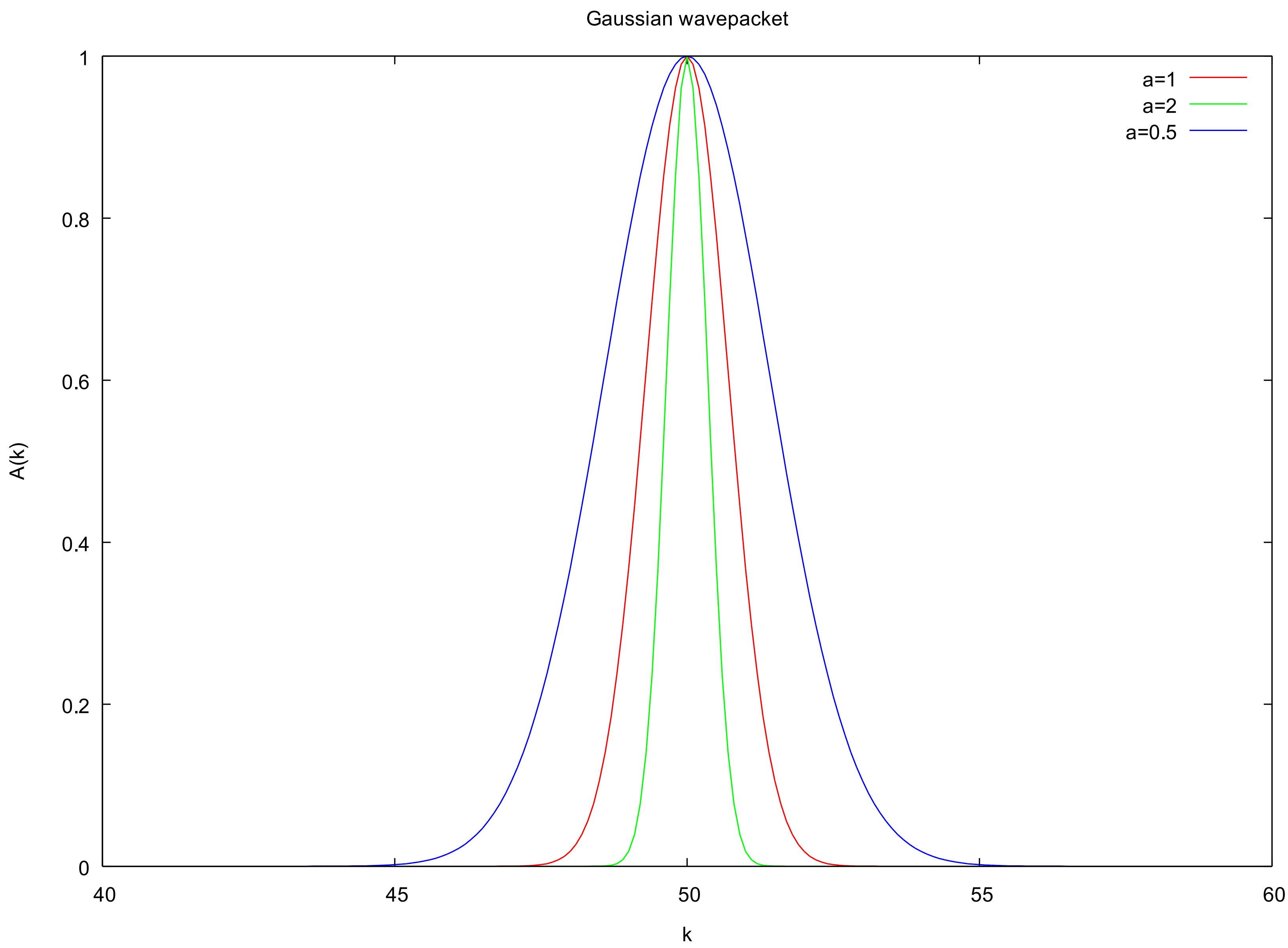
Look at Gaussian wave packets where

$$A_k = e^{-a^2 (k - k_0)^2}$$

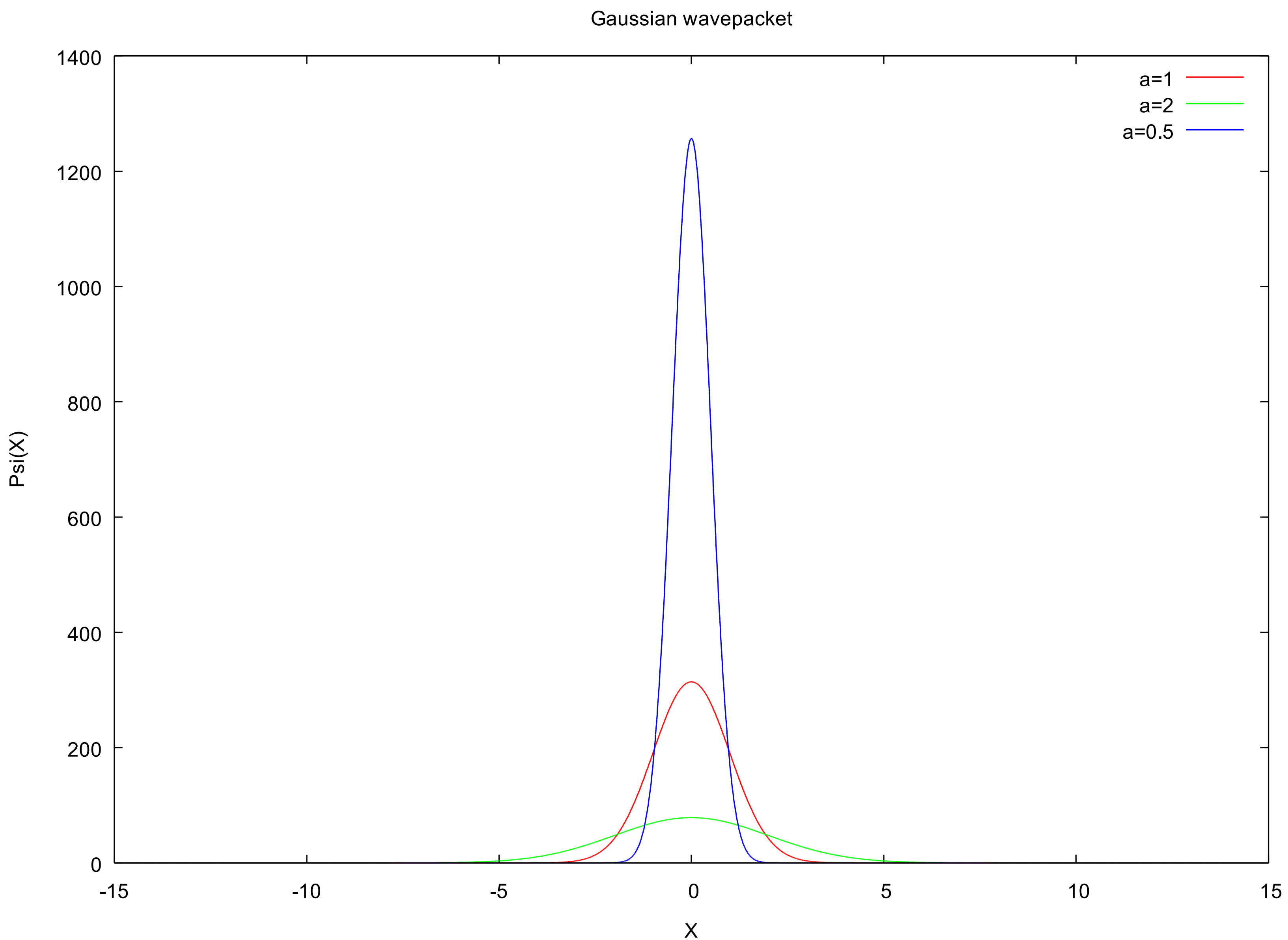
Or in the continuous limit

$$A(k) = e^{-a^2 (k - k_0)^2}$$

Distribution of Wave Numbers



Corresponding Spatial Wave Distribution



Phase and Group Velocities

The phase velocity v_p of a single wave with particular values of *k* and ω is the velocity of the point in space that maintains a constant sinusoidal phase

$$v_p = \frac{\omega}{k} = \frac{\hbar}{2m} k$$

The group velocity v_g of a wave packet is the velocity of the peak of spatial distribution

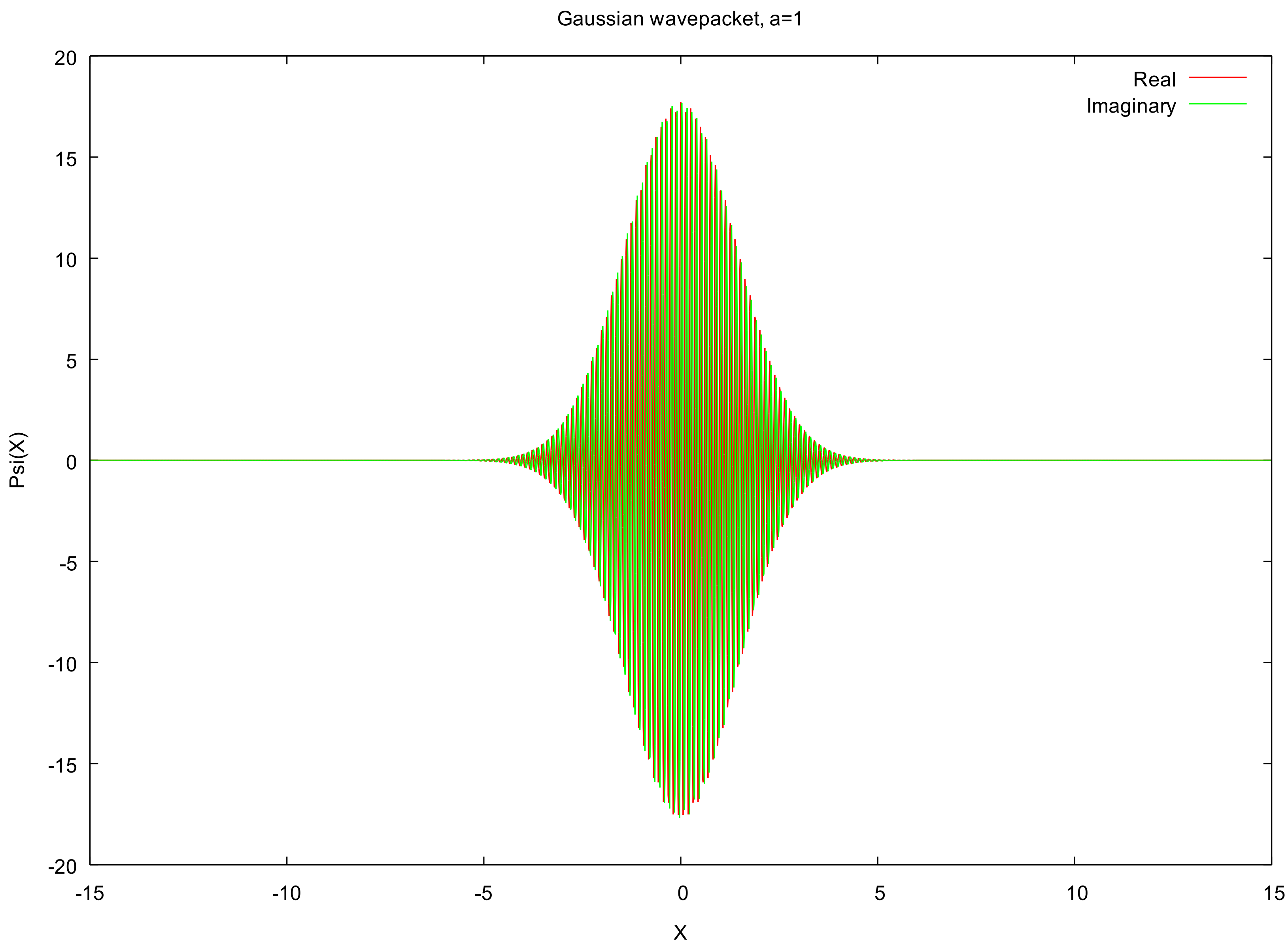
$$v_g = \frac{d\omega}{dk}\Big|_{(k=k_0)} = \frac{2\hbar}{2m} k_0 = 2 v_{p0}$$

Where v_{p0} is the phase velocity of the single wave **at the center of the *k* distribution**

Note: $$v_g = \frac{2\hbar}{2m} k_0 = \frac{\hbar}{m}\frac{p_0}{\hbar} = \frac{p_0}{m}$$

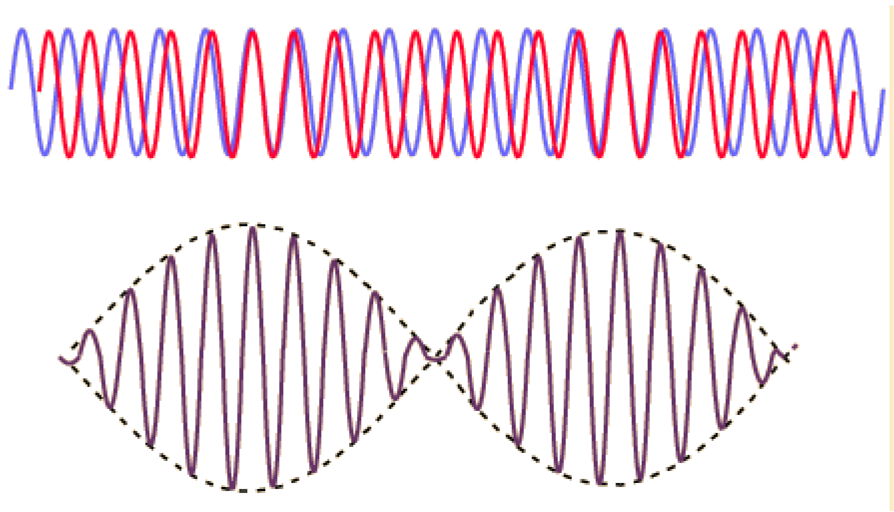
So it is the wave packet group velocity that corresponds to the velocity p/m in the classical limit

Discrete Approximation to Gaussian Packet



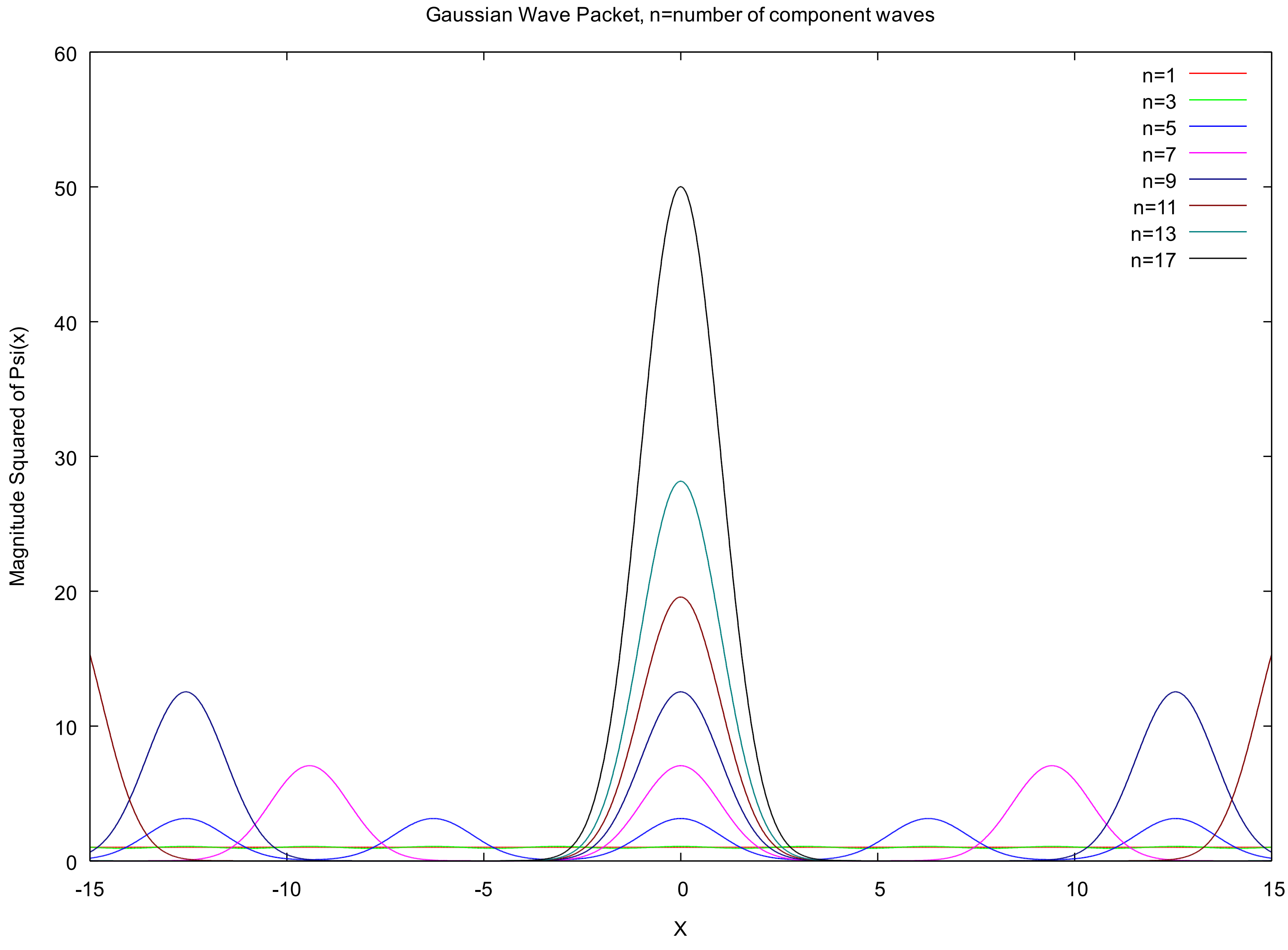
Why Do Wave Packets Form a Pulse?

Consider the superposition of just two sine waves of different frequencies. They form a "beat" frequency where constructive and destructive interference alternate:

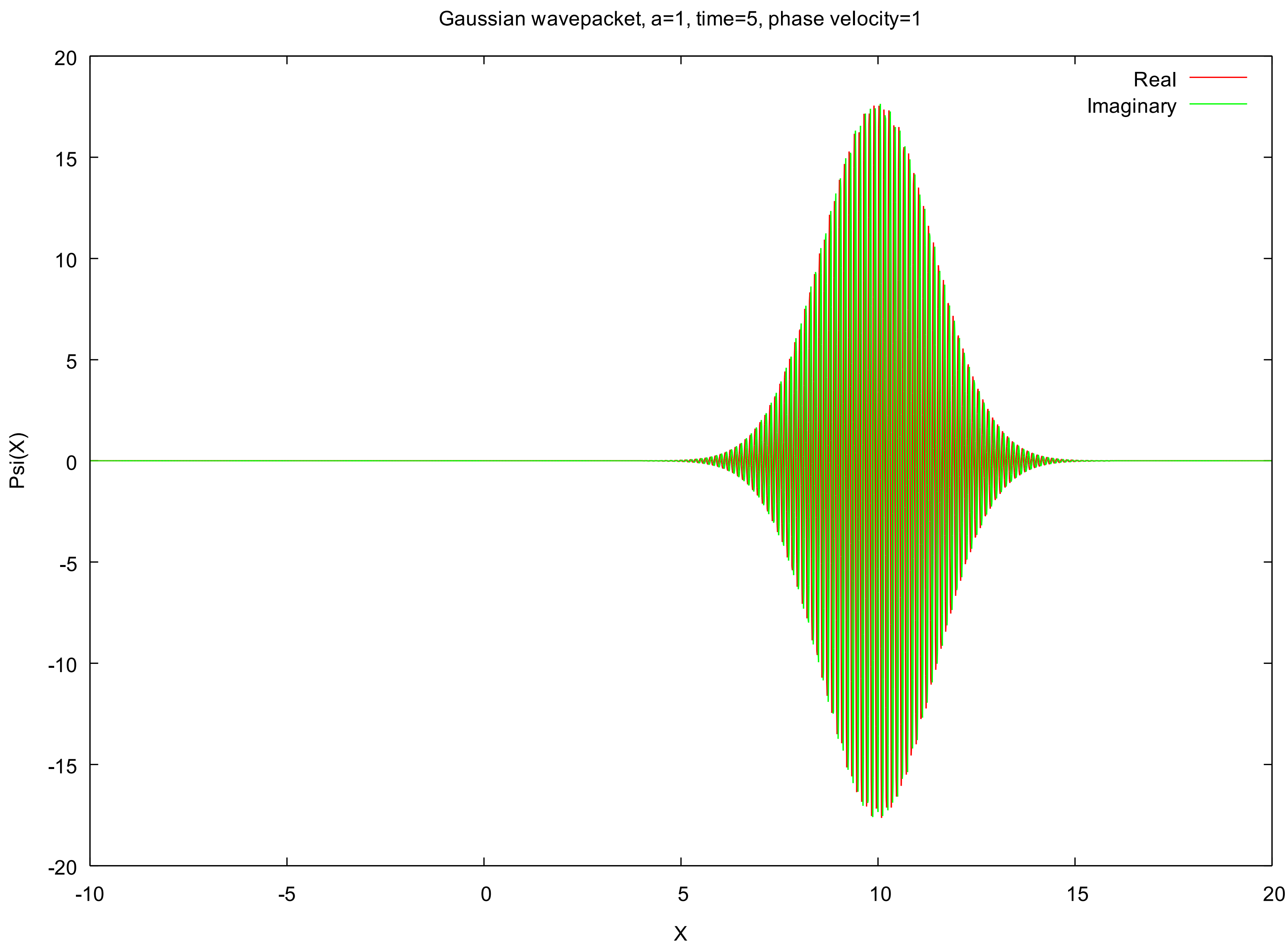


Wave packets are extensions of this property, with the sum over many frequencies. The region where all the waves line up with constructive interference becomes isolated to a pulse.

Quality of Discrete Approximations



Group Velocity versus Phase Velocity



Uncertainty Principle of Wave Packets

$$\overline{\Delta x^2} = \overline{[x-\bar{x}]^2} = \frac{\int \psi(x)^2 [x-\bar{x}]^2 dx}{\int \psi(x)^2 dx}$$

$$\overline{\Delta k^2} = \overline{[k-\bar{k}]^2} = \frac{\int A(k)^2 [k-\bar{k}]^2 dk}{\int A(k)^2 dk}$$

$$\text{then } \overline{\Delta x^2} \cdot \overline{\Delta k^2} \geq \frac{1}{4}$$