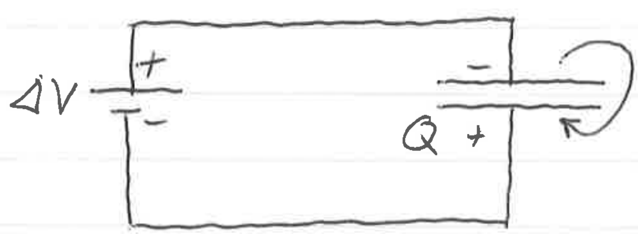
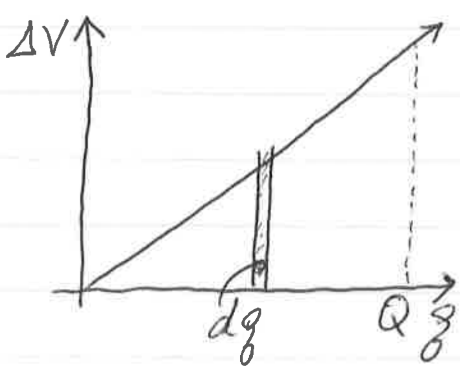
Energy stored in a charged Capacitor



To understand the energy stored, need to consider work done to put charge on capacitor

Move positive charge dq to 'positive' plate. (assme mass = 0)

- initially, no work to move it
- work required once some charge on plate
  - higher potential



$$dW = \Delta V dq = \frac{q}{C} dq$$

To get total work, need to sum (integrate) over all charges as ΔV increases

$$W = \int_0^Q \frac{q}{C}\,dq = \frac{1}{C}\int_0^Q q\,dq$$

TOTAL WORK

$$= \boxed{\frac{Q^2}{2C}}$$

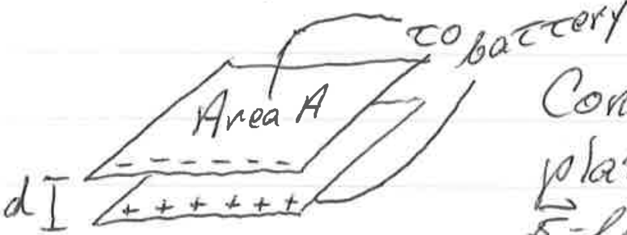
This is essentially what a battery does to a capacitor when it forces charge to build up there.

Considering $C = Q/V$, we have

$$W = \frac{(CV)^2}{2C} = \frac{C^{\cancel{2}}V^2}{2\cancel{C}}$$

$$\boxed{W = \frac{CV^2}{2}}$$

Example:



Consider our // plates with uniform $\vec{E}$-field

Disconnect the battery & increase separation to d'

What happens to:

$C \Rightarrow \epsilon_0 A/d$ so decreases
$Q \Rightarrow$ stays same
$\vec{E} \Rightarrow \sigma/\epsilon_0$ so stay same
$V \Rightarrow q/c$ so increases as $C \downarrow$
$W \Rightarrow \frac{1}{2}\frac{q^2}{C}$ so increases

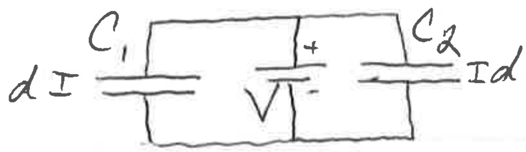
The dependence is coming from $C \propto d^{-1}$ dependence.

$\therefore$ W increases linearly with increasing separation

Stored Energy

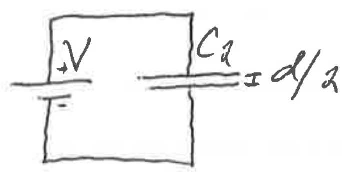
Let's consider a circuit with two $\parallel$ plate capacitors

a)



What happens if we remove C_1 and reduce separation by half?

b)



a) $C_{eff} = C_1 + C_2$
$= 2\epsilon_0 A/d$

Work needed $= \dfrac{2\epsilon_0 A}{2d} V^2$

This is equal to the potential energy difference between having charge + not having charge on $C_1 + C_2$.

$$U = \frac{\epsilon_0 A}{d} V^2 \;\left(\boxed{= \frac{1}{2} C V^2}\right)$$

Now consider b)

$$C_2 = \epsilon_0 A/(d/2) = 2\epsilon_0 A/d$$

the energy stored is

$$\underline{\underline{U}} = \frac{1}{2} C V^2 = \underline{\underline{\frac{\epsilon_0 A}{d} V^2}}$$

which is the same as in a)

So we see adding in parallel acts like re-engineered C_2

- and we get same benefit of energy stored

Energy Density

The potential energy difference between charged & uncharged capacitors

- means an equivalent energy 'stored' by capacitor
- think of it as kinetic energy give to charges to allow the E-field to move them back

Sometimes want to know energy per volume, 'energy density'. Consider constant field case

- parallel plate capacitor

$$W = \frac{1}{2}\frac{Q^2}{C} = \frac{Q^2}{2\epsilon_0 A/d}$$

The electric field is

$$\mathcal{E} = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A} \quad \therefore Q = \epsilon_0 A \mathcal{E}$$

So that

$$W = \frac{1}{2} \frac{(\epsilon_0 A \mathcal{E})^2}{\epsilon_0 A/d}$$

$$= \boxed{\frac{\epsilon_0 A d}{2} \mathcal{E}^2}$$

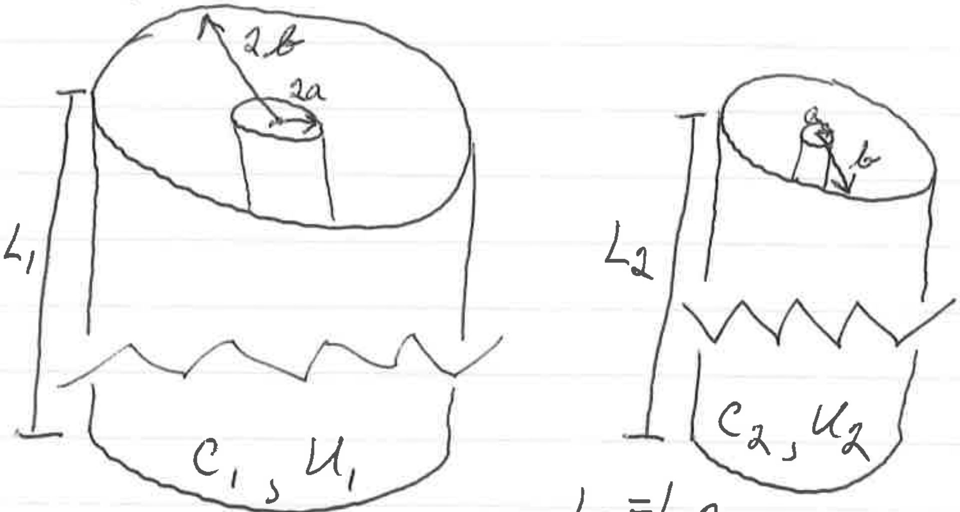
To calculate the energy density

$$u = \frac{W}{\text{volume}} = \frac{W}{Ad}$$

$$\boxed{u = \frac{1}{2} \epsilon_0 \mathcal{E}^2} \quad \text{Joules}/m^3$$

<u>Example</u>

Consider two cylindrical capacitors:



$L_1 = L_2 = L$

Both capacitors given same amount of charge.

Is energy stored, U_1, greater less than or equal to U_2?

10.9

Capacitance for a cylinder,

$$C = L/(2k\ln(b/a))$$

which gives energy

$$U_i = \frac{1}{2}C_iV^2 = \frac{Q^2}{2C_i}$$

$$U_1/U_2 = \frac{\cancel{\frac{1}{2}}\, Q^2/C_1}{\cancel{\frac{1}{2}}\, Q^2/C_2} = \frac{C_2}{C_1} = \frac{\cancel{L}/(2k\ln(b/a))}{\cancel{L}/(2k\ln(2b/2a))}$$

$$= \frac{\cancel{k}\ln(\cancel{2}b/\cancel{2}a)}{\cancel{k}\ln(b/a)} = \underline{\underline{1}}$$

<u>So capacitors store same amount of energy $U_1 = U_2$</u>