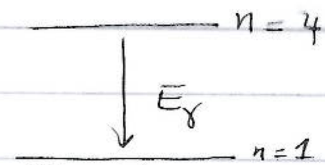
Chapter 5

Exercise 24



$$E_n = \frac{n^2 \pi^2 \hbar^2}{2 m L^2}$$

$$E_\gamma = E_4 - E_1 = \frac{\pi^2 \hbar^2}{2 m L^2}\left(4^2 - 1^2\right) = \frac{\pi^2 \left(1.055 \times 10^{-34} \mathrm{J \cdot s}\right)^2}{2(9.11\ 10^{-31} \mathrm{kg})(5 \times 10^{-9} \mathrm{m})} (15)$$

$$\boxed{E_\gamma = 3.6 \times 10^{-20} \mathrm{J}}$$

$$E_\gamma = h\nu = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{E_\gamma} = \frac{6.63\ 10^{-34} \times 3\ 10^{8}}{3.6 \times 10^{-20}}$$

$$\boxed{\lambda = 5.5 \times 10^{-6}\ \mathrm{m}}$$

Exercise 25

The energy levels get further apart as n increases.
The lowest energy transition will be from the n=2 level to the n=1.

$$E_\gamma = h\nu = \frac{hc}{\lambda} = \frac{(6.63 \times 10^{-34} \mathrm{J\cancel{s}}) \times 3\ 10^{8} \mathrm{\cancel{m}/\cancel{s}}}{450\ 10^{-9} \mathrm{\cancel{m}}} =$$

Also,

$$E_\gamma = E_2 - E_1 = \frac{\pi^2 \hbar^2}{2 m L^2}\left(2^2 - 1^2\right) = \frac{3}{2}\frac{\pi^2 \hbar^2}{m L^2}$$

$$L^2 = \frac{3}{2}\frac{\pi^2 \hbar^2}{m E_\gamma}$$

$$L = \sqrt{\frac{3}{2}\frac{\pi^2 \hbar^2}{m E_\gamma}} = \sqrt{\frac{3}{2}\frac{\pi^2 (1.055\ 10^{-34})^2}{9.11\ 10^{-31}}}$$

Exercise 28.

$$\psi_n(x) = \sqrt{\frac{2}{L}}\, \sin\left(\frac{n\pi x}{L}\right)$$

So,

$$\psi_2(x) = \sqrt{\frac{2}{L}}\, \sin\left(\frac{2\pi x}{L}\right)$$

$$p = \int |\psi_2(x)|^2 dx = \frac{2}{L}\int_{\frac{1}{3}L}^{\frac{2L}{3}} \sin^2 \frac{2\pi x}{L}\, dx$$

To do the integral we need to use:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\int \sin^2 x\, dx = \frac{1}{2}\int (1 - \cos 2x)\, dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{x}{2} - \frac{1}{2}\left[\frac{1}{2}\sin 2x\right] = \frac{x}{2} - \frac{1}{4}\sin 2x$$

Thus:

$$p = \frac{2}{L}\int_{\frac{1}{3}L}^{\frac{2}{3}L} \sin^2 \frac{2\pi x}{L}\, dx = \frac{2}{L}\left(\frac{x}{2} - L\frac{\sin \frac{4\pi x}{L}}{8\pi}\right]_{\frac{1}{3}L}^{\frac{2}{3}L}$$

$$= \frac{2}{L}\left(\frac{L}{6} - L\frac{\sin\frac{8\pi}{3} - \sin\frac{4\pi}{3}}{8\pi}\right) = \frac{1}{3} - 0.138 = 0.196$$

Classicaly, it should be one third ($p = 1/3$)

This is lower because the region is centered on a node

Exercise 33

Eq 5.12 $\frac{d^2\psi(x)}{dx^2} = -k^2\psi(x)$ verify that $A\sin(kx) + B\cos(kx)$ is a solution

$$\frac{d\psi(x)}{dx} = \frac{d}{dx}\left(A\sin(kx) + B\cos(kx)\right) = Ak\cos kx - Bk\sin kx$$

$$\frac{d^2\psi(x)}{dx} = \frac{d^2}{dx^2}\left(A\sin(kx) + B\cos(kx)\right) = -Ak^2\sin kx - Bk^2\cos(kx)$$

$$= -k^2\left(A\sin(kx) + B\cos(kx)\right) = -k^2\psi(x)$$

Exercise 34

$E_{e^-} = 50\,eV$, Electroctrostic walls of 200 eV high.
How far does its wave function extend beyond the walls?
This distance is known as the penetration depth (Eq 5-24)

$$\delta = \frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(U_0 - E)}} = \frac{1.055\times10^{-34}\,J.S}{\sqrt{2(9.11\,10^{-31}kg)(200_{eV} - 50_{eV})1.6\times10^{-19}\,J/_{eV}}}$$

$$\boxed{\delta = 1.6\,10^{-11}m}$$

This says that the penetration should be deeper as the energy E ~~nears~~ nears the value of the Confining potential U_0.

Exercise 41

We have $\sqrt{E}\cot\left(\frac{\sqrt{2mE}}{\hbar}L\right) = -\sqrt{U_0 - E}$

It is Convenient to multiply it by $\frac{\sqrt{2m}}{\hbar}L$

So

$$\frac{\sqrt{2m}}{\hbar}L\cot\left(\frac{\sqrt{2m}L}{\hbar}\right) = -\frac{\sqrt{2mU_0L^2 - 2mEL^2}}{\hbar}$$

Now Let $x = \frac{\sqrt{2mE}}{\hbar}L$

Thus:

$$x\cot x = -\sqrt{\frac{2mU_0L^2}{\hbar^2} - x^2}$$, Given that $U_0 = 4\frac{\pi^2\hbar^2}{2mL^2}$

So

$$x\cot x = -\sqrt{\frac{\cancel{2mL^2}}{\cancel{\hbar^2}}\,4\frac{\pi^2\cancel{\hbar^2}}{\cancel{2mL^2}} - x^2} = -\sqrt{4\pi^2 - x^2}$$

Thus We need to solve $\boxed{x\cot x = -\sqrt{4\pi^2 - x^2}}$

Here, I used Mathematica (please see the following page)

here, I got two solution $\begin{cases} x = 2.698 \\ \text{or} \\ x = 5.284 \end{cases}$

Problem 34

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Here Mathematica was used to solve this transcendental equation.
FindRoot is the option to use in that
case as the Solve command is reserved for polynomial equations.
you will have to give a value of x that you may think that
the solutions is at the vicinity of that value.
As you see below there is only two reasonable solutions for
that transcendental equation.i.e 2.6978 and 5.284

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In[18]:= FindRoot$\left[x \operatorname{Cot}[x] == -\sqrt{4\pi^2 - x^2}, \{x, 1\}\right]$

Out[18]= $\{x \to 5.28408\}$

FindRoot$\left[x \operatorname{Cot}[x] == -\sqrt{4\pi^2 - x^2}, \{x, 2\}\right]$

Out[19]= $\{x \to 2.6978\}$

In[20]:= FindRoot$\left[x \operatorname{Cot}[x] == -\sqrt{4\pi^2 - x^2}, \{x, 3\}\right]$

Out[20]= $\{x \to 2.6978\}$

In[21]:= FindRoot$\left[x \operatorname{Cot}[x] == -\sqrt{4\pi^2 - x^2}, \{x, 4\}\right]$

Out[21]= $\{x \to 5.28408\}$

In[22]:= FindRoot$\left[x \operatorname{Cot}[x] == -\sqrt{4\pi^2 - x^2}, \{x, 5\}\right]$

Out[22]= $\{x \to 5.28408\}$

In[23]:= FindRoot$\left[x \operatorname{Cot}[x] == -\sqrt{4\pi^2 - x^2}, \{x, 6\}\right]$

Out[23]= $\{x \to 5.28408\}$

In[24]:= FindRoot$\left[x \operatorname{Cot}[x] == -\sqrt{4\pi^2 - x^2}, \{x, 7\}\right]$

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FindRoot::lstol :
 The line search decreased the step size to within tolerance specified by AccuracyGoal and PrecisionGoal
   but was unable to find a sufficient decrease in the merit function.  You may need more
   than MachinePrecision digits of working precision to meet these tolerances. More...
```

Out[24]= $\{x \to 7.64152 + 1.58728 \times 10^{-14}\, i\}$

In[25]:= FindRoot$\left[x \operatorname{Cot}[x] == -\sqrt{4\pi^2 - x^2}, \{x, 8\}\right]$

```
FindRoot::lstol :
 The line search decreased the step size to within tolerance specified by AccuracyGoal and PrecisionGoal
   but was unable to find a sufficient decrease in the merit function.  You may need more
   than MachinePrecision digits of working precision to meet these tolerances. More...
```

Out[25]= $\{x \to 8. + 1.60235 \times 10^{-14}\, i\}$

We have $x = \frac{\sqrt{2mE}}{\hbar} L \Rightarrow x^2 = \frac{2mE}{\hbar^2} L^2$

$$E = x^2 \frac{\hbar^2}{2mL^2} \quad , \quad x = 2.698 \quad \text{or } x = 5.28$$

$$E_1 = 7.28 \frac{\hbar^2}{2mL^2} \qquad E_2 = 27.9 \frac{\hbar^2}{2mL^2}$$

To compare with the infinite well $\left(E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}\right)$ we need to multiply E_1 & E_2 by $\frac{\pi^2}{\pi^2}$

$$\text{Thus } E_1 = 0.74 \frac{\pi^2\hbar^2}{2mL^2} \text{ \& } E_2 = 2.83 \frac{\pi^2\hbar^2}{2mL^2}$$

The height of the well here is $4 \frac{\pi^2\hbar^2}{2mL^2}$, is at the level $n = 2$ state of the infinite well.

Because the penetration of the classically forbidden region, the wavelengths here, just as in the finite well, should be longer and the energies lower than in the infinite well

$0.74 < 1^2$ and $2.83 < 2^2 \Rightarrow$ At least two energies.

Exercise 50

Amplitude $A = 10\,cm = 0.1\,m \quad k = 120\ N/m$

The maximum potential Energy which is equal to its total mechanical energy.

(a) $$PE_{max} = E_{total} = \frac{1}{2} k x^2 = \frac{1}{2}(120)(0.1)^2 = 0.6\,J$$

We know that $E_n = (n + 1/2)\hbar\omega_0$ wher $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{120}{2}} = 7.75\,s^{-1}$

$$n = \frac{E_n}{\hbar\omega_0} - \frac{1}{2}$$

$$\boxed{n = 7.33 \; 10^{32}}$$

(b) Minimum $\Delta E = \hbar\omega_0 = (1.055\ 10^{-34}\ J s)(7.75\ s^{-1})$
$= 8.2\ 10^{-34}\ J.$

The corresponding fractional change in Energy is

$$\frac{8.2\ 10^{-34} J}{0.6\ J} = 1.4\ 10^{-33}$$

Exercise 60

Show that the uncertainty in the position of a ground-state harmonic oscillator (HO) is $\frac{1}{\sqrt{2}}\left(\frac{\hbar^2}{mk}\right)^{1/4}$

$$\psi_0(x) = \left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-\frac{1}{2}b^2x^2}$$

$$b = \left(\frac{mk}{\hbar^2}\right)^{1/4}$$

$$\Delta X = \sqrt{\overline{x^2} - \overline{x}^2} = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

$$\overline{x} = \int_{-\infty}^{+\infty} \psi^*(x)\, x\, \psi(x)\, dx = \int_{-\infty}^{+\infty} \left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-(\frac{1}{2})b^2x^2}\, x \left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-\frac{1}{2}b^2x^2}$$

$= 0$ because we have an odd function integrated over a symetric interval.

$$\overline{x^2} = \langle x^2\rangle = \int_{-\infty}^{+\infty} \psi^*(x)\, x^2\, \psi(x)\, dx = \int_{-\infty} \left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-\frac{1}{2}b^2x^2}\, x^2 \left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-\frac{1}{2}b^2x^2}$$

$$= \frac{b}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{+\infty} x^2 e^{-b^2x^2}\, dx}_{\text{Gaussian Integral}}$$

$$\int_{-\infty}^{+\infty} x^2 e^{-\alpha x^2}\, dx = \frac{1}{2\alpha}\left(\frac{\pi}{\alpha}\right)^{1/2}$$

So $\int_{-\infty}^{+\infty} x^2 e^{-b^2x^2}\, dx = \frac{1}{2b^2}\left(\frac{\pi}{b^2}\right)^{1/2} = \frac{\sqrt{\pi}}{2b^3}$

Thus $\overline{x^2} = \frac{b}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2b^3} = \frac{1}{2b^2} \Rightarrow \Delta X = \sqrt{\langle x^2\rangle} = \sqrt{\frac{1}{2b^2}} = \frac{1}{\sqrt{2}\, b}$

$$\boxed{\Delta X = \frac{1}{\sqrt{2}}\left(\frac{\hbar^2}{mk}\right)^{1/4}}$$

Exercise 5.61. Simple Harmonic Oscillator (SHO) in ground state

Uncertainty in Momentum

$$\Delta p = \sqrt{\overline{p^2} - \bar{p}^2} \qquad \psi_0(x) = \left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-\frac{1}{2}b^2x^2}$$

$$b = \left(\frac{mk}{\hbar^2}\right)^{1/4}$$

$$\bar{p} = 0 \quad (\text{symmetry}) \qquad p = \frac{\hbar}{i}\frac{\partial}{\partial x} = -i\hbar\frac{\partial}{\partial x}, \; i^2 = -1$$

$$\overline{p^2} = \int_{-\infty}^{+\infty} \psi(x)\left(-i\hbar\frac{\partial}{\partial x}\right)^2 \psi(x)\,dx = \int_{-\infty}^{+\infty} \left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-\frac{1}{2}b^2x^2}\left(-i\hbar\frac{\partial}{\partial x}\right)^2\left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-\frac{1}{2}b^2x^2}\,dx$$

$$\overline{p^2} = -\hbar^2\left(\frac{b}{\sqrt{\pi}}\right)\int_{-\infty}^{+\infty} e^{-\frac{1}{2}b^2x^2}\frac{\partial^2}{\partial x^2} e^{-\frac{1}{2}b^2x^2}\,dx$$

$$\frac{\partial^2}{\partial x^2}\left(e^{-\frac{1}{2}b^2x^2}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(e^{-\frac{1}{2}b^2x^2}\right)\right) = \frac{\partial}{\partial x}\left(-\frac{1}{\cancel{2}}b^2\cancel{2}x\, e^{-\frac{1}{2}b^2x^2}\right)$$

$$= \frac{\partial}{\partial x}\left(-b^2x\, e^{-\frac{1}{2}b^2x^2}\right)$$

$$= -b^2 e^{-\frac{1}{2}b^2x^2} + b^2x\,\frac{1}{\cancel{2}}b^2\cancel{2}x\,e^{-\frac{1}{2}b^2x^2} = -b^2 e^{-\frac{1}{2}b^2x^2} + b^4x^2 e^{-\frac{1}{2}b^2x^2}$$

$$= \left(-b^2 + b^4x^2\right) e^{-\frac{1}{2}b^2x^2}$$

Thus,

$$\overline{p^2} = -\hbar^2\left(\frac{b}{\sqrt{\pi}}\right)\int_{-\infty}^{+\infty} e^{-\frac{1}{2}b^2x^2}\left(-b^2 + b^4x^2\right) e^{-\frac{1}{2}b^2x^2}\,dx$$

$$= -\hbar^2\left(\frac{b}{\sqrt{\pi}}\right)\left[-b^2\int_{-\infty}^{+\infty} e^{-b^2x^2}\,dx + b^4\int_{-\infty}^{+\infty} x^2 e^{-b^2x^2}\,dx\right]$$

Both are Gaussian integrals. $I_0(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}$, $\alpha > 0$

Thus:

$$\int_{-\infty}^{+\infty} e^{-b^2x^2}\,dx = \frac{\sqrt{\pi}}{b}, \quad \int_{-\infty}^{+\infty} x^2 e^{-b^2x^2}\,dx = \frac{\sqrt{\pi}}{2b^3}$$

Thus, $\overline{p^2} = -\hbar^2\left(\frac{b}{\sqrt{\pi}}\right)\left(-b^2\frac{\sqrt{\pi}}{b} + b^4\frac{\sqrt{\pi}}{2b^3}\right) = \frac{1}{2}\hbar^2 b^2$

$$= \frac{1}{2}\hbar^{\cancel{2}}\frac{\sqrt{mk}}{\cancel{\hbar}} \qquad \Delta p = \sqrt{\overline{p^2} - \bar{p}^2} = \sqrt{\frac{\hbar}{2}}\,(mk)^{1/4}$$

Exercise 6.e

$$\Delta x\,\Delta p = \frac{1}{\sqrt{2}}\left(\frac{\hbar^2}{mk}\right)^{1/4} \times \sqrt{\frac{\hbar}{2}}\,(mk)^{1/4} = \frac{1}{2}\hbar$$

The minimum possible product $\Delta x\,\Delta p$, because the function is Gaussian.

Exercise 78

We have $\psi(x) = \begin{cases} 2\sqrt{a^3}\,x\,e^{-ax} & x>0 \\ 0 & x<0 \end{cases}$

Verify that the normalization constant $2\sqrt{a^3}$ is correct.

i.e.

$$\int_{\text{all Space}} \psi^2 dx = \int_0^{\infty} \left(2\sqrt{a^3}\,x\,e^{-ax}\right)^2 dx = 4a^3 \int_0^{+\infty} x^2 e^{-2ax}\,dx$$

we have: $\displaystyle\int_0^{\infty} x^n e^{-bx}\,dx = \frac{n!}{b^{n+1}} \Rightarrow \int_0^{\infty} x^2 e^{-2ax}\,dx = \frac{2!}{(2a)^3}$

thus

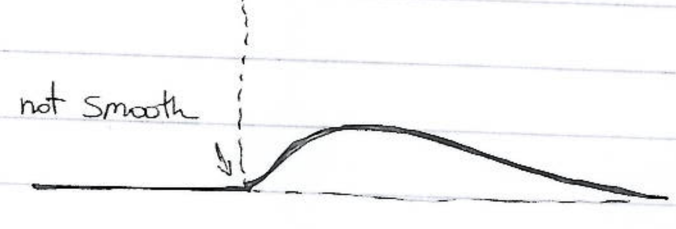
$$\int_{\text{all Space}} \psi^2 dx = 4a^3\,\frac{2!}{2^3 a^3} = \cancel{4}\cancel{a^3}\,\frac{\cancel{2}}{8\cancel{a^3}} = 1 \checkmark$$

Exercise 79:

It is smooth except at the origin.

It has a positive slope on the right, but zero on the left.

The derivative must be discontinuous because, as with both walls of the infinite well, the potential energy is infinite here



Exercise 80

The particle's most probable position

The probability density is $\psi^2(x) = 4a^3 x^2 e^{-2ax}$

$4a^3$ is a constant that has no effect on the location of a maximum of a function, thus.

$$\psi^2(x) \propto x^2 e^{-2ax}$$

proportional.

$$\frac{d x^2 e^{-2ax}}{dx} = \left(2x - 2ax^2\right) e^{-2ax} = 0 \quad \text{if } e^{-2ax} = 0 \Rightarrow x = \infty$$

if $\cancel{2}x - \cancel{2}ax^2 = 0 \Rightarrow x(1 - ax) = 0$

$x = 0$ or $1 - ax = 0 \quad x = \frac{1}{a}$

to see ~~if these~~ which one of these solutions is a maxima, we need to see for which solution

$$\frac{d^2}{dx^2}\left(x^2 e^{-2ax}\right) < 0$$

It is obvious that 0, and ∞ are minimas and $x = \frac{1}{a}$ is a maxima.

Exercise 81

$$P(0 < x < 1/a) = \int_0^{1/a} \psi^2 dx = \int_0^{1/a} \left(2\sqrt{a^3}\, x e^{-ax}\right)^2 dx$$

$$= 4a^3 \int_0^{1/a} x^2 e^{-2ax}\, dx$$

Here, To do the integral we need Integration by Part

Let $U' = e^{-2ax} \longrightarrow U = -\frac{1}{2a} e^{-2ax}$

$V = x^2 \longrightarrow V' = 2x$

$$\int U'V = [UV]_0^{1/a} - \int_0^{1/a} \overbrace{\frac{-2x}{2a} e^{-2ax}}^{UV'} dx$$

$$[uv]_0^{1/a} = \left[\frac{-x^2}{2a} e^{-2ax}\right]_0^{1/a} = \frac{-1/a^2}{2a} e^{-2} = \frac{-e^{-2}}{2a^3}$$

for $\int_0^{1/a} \frac{-x}{a} e^{-2ax}\,dx$ we need another integration by part.

let $u' = e^{-2ax} \longrightarrow u = \frac{-1}{2a} e^{-2ax}$

$v = x \longrightarrow v' = 1$

$$-\frac{1}{a}\int_0^{1/a} x e^{-2ax}\,dx = -\frac{1}{a}\left(\left[\frac{-x}{2a} e^{-2ax}\right]_0^{1/a} - \int \frac{-1}{2a} e^{-2ax}\,dx\right)$$

$$= -\frac{1}{a}\left(\frac{-1}{2a^2} e^{-2} + \frac{1}{2a}\int e^{-2ax}\,dx\right)$$

$$= -\frac{1}{a}\left(\frac{-1}{2a^2} e^{-2} + \frac{1}{2a}\left[\frac{-1}{2a} e^{-2ax}\right]_0^{1/a}\right)$$

$$= -\frac{1}{a}\left(\frac{-1}{2a^2} e^{-2} + \frac{1}{2a}\left[-\frac{1}{2a} e^{-2} + \frac{1}{2a}\right]\right)$$

$$= \frac{e^{-2}}{2a^3} - \frac{1}{2a^2}\left[\frac{-1}{2a^2} e^{-2} + \frac{1}{2a}\right]$$

$$= \frac{e^{-2}}{2a^3} + \frac{e^{-2}}{4a^3} - \frac{1}{4a^3}$$

Thus:

$$P(0<x<1/a) = 4a^3\left(\frac{-e^{-2}}{\cancel{2}a^3} - \frac{e^{-2}}{\cancel{2}a^3} - \frac{e^{-2}}{4a^3} + \frac{1}{4a^3}\right)$$

$$= \cancel{4a^3}\left(\frac{-5e^{-2}+1}{\cancel{4a^3}}\right)$$

$$\boxed{P(0<x<1/a) = 0.32}$$

Exercise 5.82:

The expectation value of the position of the particle:

$$\bar{X} = \int_{\substack{\text{all}\\\text{space}}} x\,\psi^2(x)\,dx = \int_0^\infty x\left(2\sqrt{a^3}\,x\,e^{-ax}\right)^2 dx$$

$$= 4a^3\int_0^\infty x^3 e^{-2ax}\,dx$$

$$= 4a^3\,\frac{3!}{(2a)^{3+1}} = \cancel{4}\cancel{a^3}\,\frac{3\times\cancel{2}}{\cancel{4}\times\cancel{4}_2\,a^{\cancel{4}}}$$

$$\boxed{\bar{X} = \frac{3}{2a} = \frac{1.5}{a}}$$