Wave Properties of Matter Particles

De Broglie Waves:

If light has wave + particle properties

=> What about matter particles?

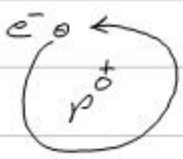
de Broglie wavelength

$$\lambda = h/p$$

λ: wave property; p: particle property

this generalization combines insights from Relativity and Planck

Consider an atom:

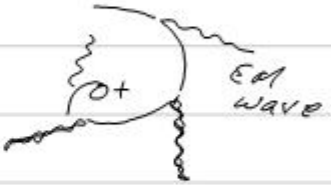


H atom

Classical idea that e^- revolve around positive nucleus: planetary model

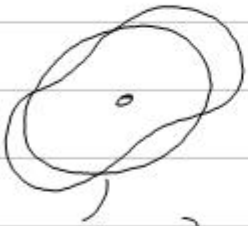
But accelerating e' emits EM waves:

Collapse of atom!



Using de Broglie idea:

- have a standing wave in orbital around nucleus (only integer # waves in orbit)



$2\pi r = n\lambda$

$$2\pi r = n\lambda = n\frac{h}{p}$$

$$pr = n\frac{h}{2\pi}$$

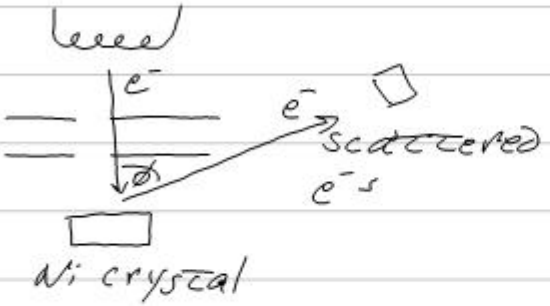
$$\boxed{L = n\hbar}$$

condition for stable atoms!

→ known experimentally to be case (Bohr model)

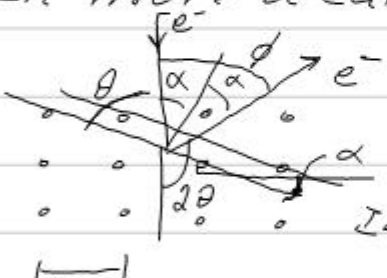
So how do we establish wave-like nature of e^-?

Electron Scattering



Look @ intensity vs. scattering angle

In more detail:



$d = D \sin\alpha$

Condition for constructive interf.

$$n\lambda = 2d\sin\theta = 2d\cos\alpha$$
$$= 2D\sin\alpha\cos\alpha$$
$$= D\sin 2\alpha = \boxed{D\sin\phi}$$

Turning this into a measure of λ:

$$\lambda = (d \sin\phi)/n$$

$n = 1$ if 1st max. @ $50°$

$\phi =$

$$\lambda_e = (0.215\,nm)\sin 50° = 0.165\,nm$$

→ uses directly wave-like picture describing classical EM-waves

How can we verify de Broglie?

- Experiment sets $\phi = 50°$ + varies K.E. of e^-'s

I_{max} @ $\phi = 50°$ when K.E. $= 54\,eV$

Calculate momentum

$V_0 = 54V$ to get to K.E. $= 54\,eV$

$$K.E. = eV_0 = \frac{p^2}{2m}$$

$$p = \sqrt{2meV_0}$$

We can now calculate the de Broglie wavelength

$$\lambda = \frac{h}{p} = \frac{hc}{pc} = \frac{hc}{\sqrt{2(mc^2)eV_0}}$$

mc^2 ↓ $\mathcal{E}_0$ for $e^- = 0.511\,MeV$

$$= \frac{1240\ eV \cdot nm}{\sqrt{2(0.511 \times 10^6\ eV)(eV_0)}}$$

$$= \frac{1.226\ nm\ V^{1/2}}{V_0}$$

$$\lambda_e = \underline{0.167\ nm!!!}$$

The agreement of the de Broglie derived result with the classical wave optics result confirms the wave-particle nature of e^-s.

The geometry of the crystals is of same order to λ_e and it causes interference of e^- consistent with Bragg's law.

Problem

what is λ_e when $\mathcal{E} = 50\ GeV$?

$$\lambda_e = h/p = h/(\mathcal{E}/c) \qquad (p \sim \mathcal{E}/c \text{ @ high energy})$$

$$= \frac{4.14 \times 10^{-15}\ eV\,s}{50 \times 10^{9}\ eV / 3 \times 10^{8}\ m/s}$$

$$= \boxed{2.5 \times 10^{-17}\ m}$$

this is 1.2% of proton's diameter

Problem

thermal neutrons $K.E. = 0.025\,eV$

interatomic spacing $\Delta = 0.45\,nm$

Where is the 1st order ($n=1$) Bragg peak?

$$\lambda_n = \frac{h}{p} = \frac{h}{\sqrt{2m(K.E.)}} = \frac{4.14\times10^{-15}\,eV\cdot s}{\sqrt{2\,(939\times10^{6}\,eV)/(c^2)\,(0.025\,eV)}}$$

De Broglie

$$= 1.8\times10^{-10}\,m = 0.18\,nm$$

$$\lambda_n = \frac{2d\sin\theta}{n} = \Delta\sin\phi$$

Bragg

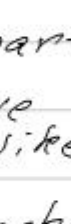
$$\phi = \sin^{-1}\left(\frac{\lambda}{\Delta}\right) = \sin^{-1}\left(\frac{0.18\,nm}{0.45\,nm}\right)$$

$$\boxed{\phi = 23.6^\circ}$$

Wave-Particle Duality

So we have something which behaves as both a wave & a particle - in fact, exhibits facets of one or the other in different situations

- when $d \gg \lambda$: particle
- when $d \sim \lambda$: wave-like



depends on measurement making!

Consider "slowing down" ⇒ very low intensity

Individual 'hits' still "build-up" to interference Intensity distribution ⇒ particles interfere w/self!

the wave-like behavior: probability to get γ with $E = h\nu$

⇒ interference of Probability waves ⇒ Intensity distribution

$$N_\gamma \propto \text{Prob}_\gamma \propto \langle E^2 \rangle$$

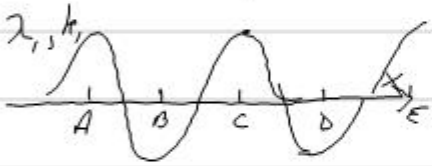
↳ E-field

Wave Packets

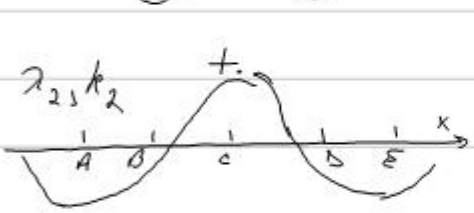
When multiple waves in same (x,t), they behave independently

principle of superposition — can add waves linearly to get net amplitude

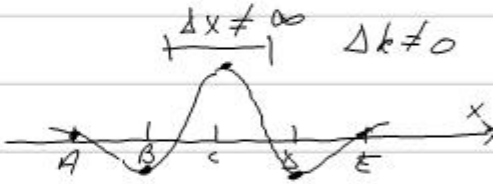
It is possible, by summing a large # of waves to get a localized (i.e. particle-like) amplitude.



$\Delta k = 0, \quad \Delta x = \infty$



+



a wave-packet is where a large amplitude exists

many wave #'s:

$\Delta k \neq 0$

$\Delta x \neq 0$

Wave Motion

Classical wave Equation

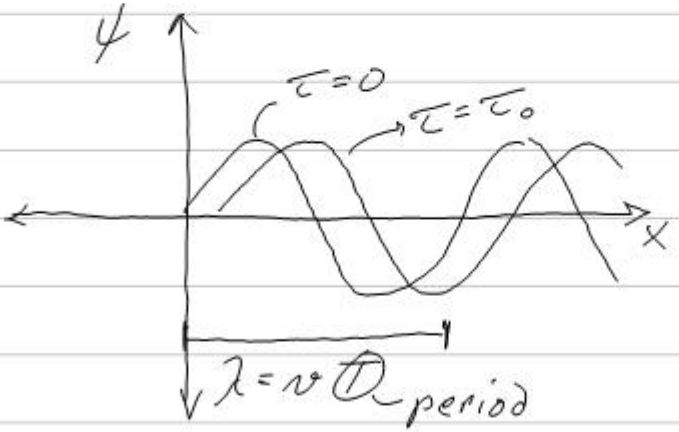
$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

v → phase velocity of wave

A general solution is

$$\psi(x,t) = A \sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

ψ → instantaneous amplitude (could be vertical displacement of string)



$v = \lambda/T = \frac{\omega}{k}$

$k = 2\pi/\lambda$ (wave #)

$\omega = 2\pi/T$ (angular freq.)

$$\psi(x,t) = A \sin(kx - \omega t + \phi)$$ (most general)

Quantum Wave Eq.

Since particles are de Broglie waves:

- adhere to wave Equation
  1) a probabilistic interpretation
  2) depends on potential field a particle experiences
  3) obtain a wave function telling probable values of: x, E, p ("observables")

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V \psi(x,t)$$

V → potential

One Dimensional Time-dependent Schrödinger Equation

→ obtained because it describes observed results

Not same form as classical wave equation

→ so this is something new, physically

Uncertainty!

Gaussian wave packet:

It can be shown that such a wave satisfies

$$\Delta k \, \Delta x = 1/2$$

since $k = 2\pi/\lambda$, this means λ and x are uncertain as given in this relation

- Consider a de Broglie wave,

$$k = 2\pi/\lambda = 2\pi/(h/p) = p/\hbar$$

- then,

$$\Delta k \, \Delta x = \frac{\Delta p}{\hbar} \Delta x = 1/2$$

or,

$$\boxed{\Delta p \, \Delta x = \hbar/2}$$

this is smallest this product can be. Written

$$\boxed{\Delta p \, \Delta x \geq \hbar/2}$$

Heisenberg Uncertainty Principle

(1)

A similar relation for Gaussian wave packets,

$$\Delta\omega\,\Delta\tau = 1/2$$

gives us Energy-time uncertainty

(2) $\boxed{\Delta E\,\Delta\tau \geq \hbar/2}$

(1) + (2) are intrinsic uncertainties,

→ not measurement errors

→ reflect degree to which p_x, x, E & τ are definable for wave-particles

∴ provide fundamental limit to conservation of Energy & momentum

Note: if $\hbar \to 0$, live in completely classical universe

Problem

Consider π^0 decaying to 2γ's

$$\tau_{1/2} = 10^{-16}\,s$$

$$m_{\pi^0} = 136\ MeV/c^2$$

What is the "Energy width"?

This asks what is the range of masses for particle decaying this way.

Use Uncertainty Principle:

$$\Delta E \Delta \tau \geq \hbar/2$$

In this context, width sometimes labelled Γ (capital γ).

$$\frac{\Gamma}{2}\Delta\tau \geq \hbar/2$$

$$\Gamma = \hbar/\Delta\tau = \frac{6.6\times10^{-16}\ eV\cdot s}{10^{-16}\ s}$$

$$= 6.6\ eV$$

Bohr Model of Hydrogen Atom

Classical description plus some key assumptions

Stable atoms: "stationary states" exist where e^- don't radiate γ

Energy states:

- emission or absorption of EM radiation happens in transition between two "stationary states"

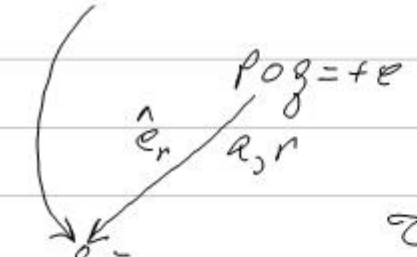
$$E = E_1 - E_2 = h\nu$$

- K.E. of e-nucleus system $= n\hbar\omega$ (ω — frequency of revolution)
  - ↳ equivalent to angular momentum

$$\boxed{L = n\hbar}$$ (de Broglie condition)

most difficult to accept

Classically,



$$\vec{F}_e = \frac{-1}{4\pi\epsilon_0}\frac{e^2}{r^2}\hat{e}_r$$

$$= -m\vec{a}$$

There is radial acceleration, $a_r = v^2/r$

$$\frac{e^2}{4\pi r^2 \epsilon_0} = m\left(\frac{v^2}{r}\right)$$

Which gives,

$$v = e/\sqrt{4\pi\epsilon_0 m_e r}$$

(m_e: mass of e^-; r: radius of atom)

$v \ll c$ $\quad$ $r \sim 5\times10^{-11}$ m

While from de Broglie, we have

$$L = |\vec{r}\times\vec{p}| = mvr = n\hbar$$

$$\hookrightarrow \boxed{v = n\hbar/mr}$$

corresponds to

$$K.E. = \frac{1}{2}mv^2 = \frac{e^2}{8\pi\epsilon_0 r^2}$$