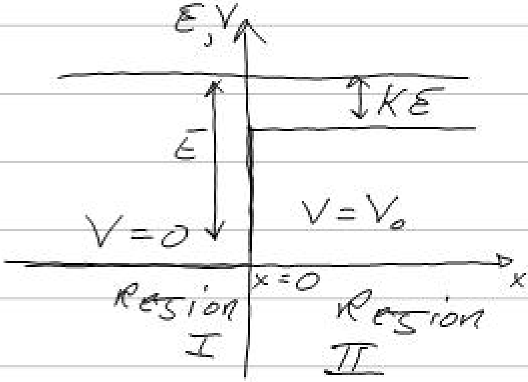
Potential Step



Consider a free particle passing thru a sudden impulse

→ KE ↓

→ Pot. E ↑

We need to understand wave-function, ψ, when $E > eV_0$

→ with potential well, used $\sin kx$, $\cos kx$ → had standing waves

→ for STEP, need to <u>direction of wave motion</u>

$$\boxed{\Psi(x,t) = A e^{\pm ikx - i\omega t}}$$

General solution

We have two regions

$$V(x) = \begin{cases} 0 & x < 0 & RI \\ V_0 & x \geq 0 & RII \end{cases}$$

For region I, Schrödinger Eq.
(time-independent)

$$\frac{\partial^2 \psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi(x)$$

→ as in potential well before

$$\rightarrow \boxed{\psi_I(x) = Ae^{ikx} + Be^{-ikx}}$$

right & left moving waves

where $k = \sqrt{2mE}/\hbar$

right-moving: "incident" wave, A
left-moving: "reflected" wave, B

Classically, if $E > V_0$ there is no reflected particle

In region II

$$\frac{d^2\psi}{dx^2} = \frac{-2m(E-V_0)}{\hbar^2}\psi(x) \left(= -k'^2\psi\right)$$

So, same functional form of $\psi(x)$ as in region I, but kinematics modified:

$$\psi_{II} = Ce^{ik'x} + De^{-ik'x}$$

$\to 0$ since no wave from right

"transmitted" wave (i.e. thru $x=0$ step)

If $E \gg V_0$: almost total transmission

$$R = \text{reflection} = \frac{|\psi_I(\text{refl.})|^2}{|\psi_I(\text{incid.})|^2} = \frac{B^*B}{A^*A} = \frac{(k-k')^2}{(k+k')^2}$$

$$\boxed{R = (\sqrt{E} - \sqrt{E-V_0})^2 / (\sqrt{E} + \sqrt{E-V_0})^2}$$

$$T = \text{transmission} = \frac{|\psi_{II}(\text{trans})|^2}{|\psi_I(\text{incid.})|^2} = \frac{C^*C}{A^*A}$$

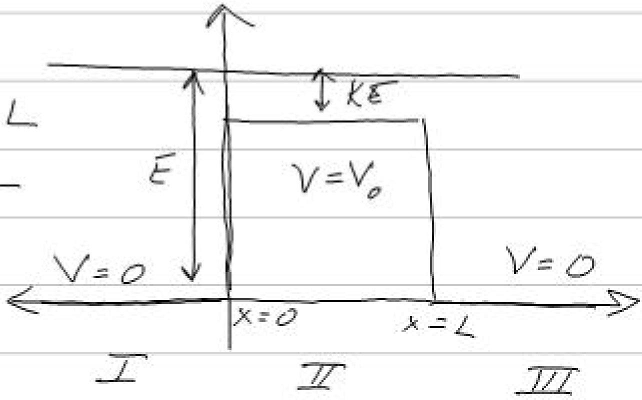
$$= \frac{4kk'}{(k+k')^2} = \boxed{\frac{4\sqrt{E(E-V_0)}}{(\sqrt{E}+\sqrt{E-V_0})^2}}$$

If $E \sim V_0$: almost total reflection

Potential Barrier

$V = 0 \quad x < 0, x > L$

$V = V_0 \quad 0 < x < L$



Consider $E > V_0$ case

classically, particle has enough E to pass 'over' barrier
→ no reflection : just slow down
($KE_{II} < KE_{I}$)

As before, need to consider solutions' direction of propagation.

Wave functions

Region I: $\psi_I(x) = Ae^{+ikx} + Be^{-ikx}$

Incident Reflected

Region II: $\psi_{II}(x) = Ce^{ik'x} + \underline{De^{-ik'x}}$

new reflected wave from end of barrier @x=

Region III: $\psi_{III}(x) = Fe^{ikx}$ → Note: same as in I

Transmitted

$$k = \sqrt{2mE}/\hbar \; ; \; k' = \sqrt{2m(E-V_0)}/\hbar$$

We want to know reflection + transmission probabilities:

$$R = \frac{|\psi_I(\text{refl.})|^2}{|\psi_I(\text{inc.})|^2} = \frac{B^*B}{A^*A}$$

$$T = \frac{|\psi_{III}(\text{trans.})|^2}{|\psi_I(\text{inc.})|^2} = \frac{F^*F}{A^*A}$$

→ we need to know B + F in terms of A, at least

Continuity Requirements

$\psi_{I}(0) = \psi_{II}(0)$: $Ae^{ik0} + Be^{-ik0} = Ce^{ik'0} + De^{-ik'0}$

$$\boxed{A + B = C + D}$$

$\left.\frac{d\psi_{I}}{dx}\right|_{x=0} = \left.\frac{d\psi_{II}}{dx}\right|_{x=0}$: $ikAe^{ik0} - ikBe^{-ik0}$
$= ik'Ce^{ik'0} - ik'De^{-ik'0}$

$$\boxed{k(A-B) = k'(C-D)}$$

$\psi_{II}(L) = \psi_{III}(L)$: $\boxed{Ce^{ik'L} + De^{-ik'L} = Fe^{ikL}}$

$\left.\frac{d\psi_{II}}{dx}\right|_{x=L} = \left.\frac{d\psi_{III}}{dx}\right|_{x=L}$: $ik'(Ce^{ik'L} - De^{-ik'L})$
$= ikFe^{ikL}$

$$\boxed{k'(Ce^{ik'L} - De^{-ik'L}) = kFe^{ikL}}$$

Have 4 equations in 5 unknowns
- can solve for B, F in terms of A

Reflection

$$R = \frac{B^*B}{A^*A} = \frac{\sin^2(k'L)}{\sin^2(k'L) + \frac{4k^2k'^2}{(k^2-k'^2)^2}}$$

$$= \boxed{\frac{\sin^2\left[\sqrt{2m(E-V_0)}\, L/\hbar\right]}{\sin^2\left[\sqrt{2m(E-V_0)}\, L/\hbar\right] + 4(E/V_0)\left[(E/V_0)-1\right]}}$$

Note: numerator has a sine func.

Reflection $= 0$ when

$$k'L = n\pi = \frac{\sqrt{2m(E-V_0)}}{\hbar} L$$

Solving for E: $\boxed{E = V_0 + \frac{n^2\pi^2\hbar^2}{2mL^2}}$

complete transmission only @ certain E's

Transmission

$$T = \frac{F^*F}{A^*A} = \left(\frac{4k^2k'^2}{(k^2-k'^2)^2}\right) \Big/ \left[\sin^2(k'L) + \frac{4k^2k'^2}{(k^2-k'^2)^2}\right]$$

$$= \boxed{1 \Big/ \left[1 + \frac{V_0^2 \sin^2\left[\sqrt{2m(E-V)}\, L/\hbar\right]}{4E(E-V)}\right]}$$

$E < V_0$ Case of Potential Barrier

Regions I & III same general form of wave-function

$$\psi_I(x) = \underbrace{Ae^{ikx}}_{\text{Incident}} + \underbrace{Be^{-ikx}}_{\text{Reflected}}$$

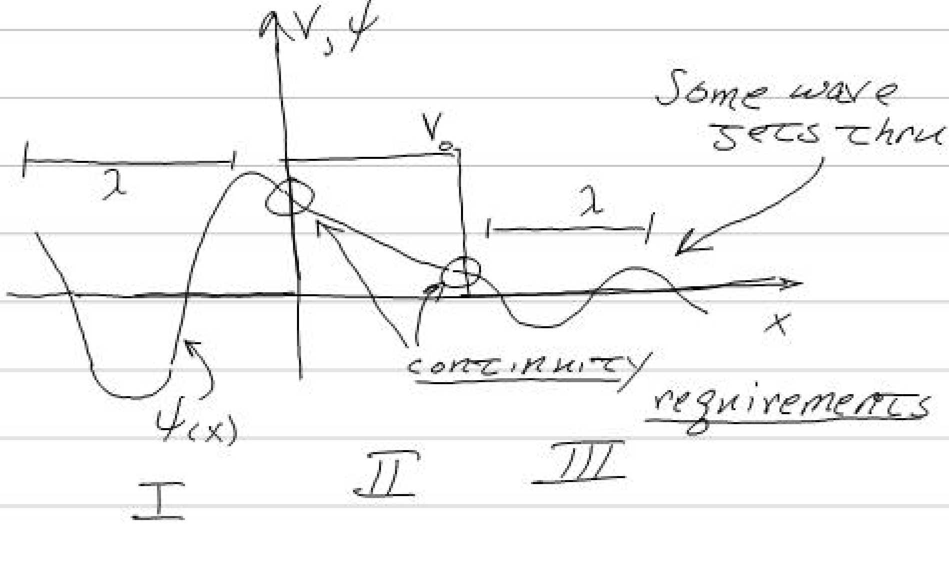
$$\psi_{III}(x) = \underbrace{Fe^{ikx}}_{\text{Transmitted}}$$

Region II different:

$$\frac{d^2\psi_{II}}{dx^2} = \underbrace{\frac{2m(V_0-E)}{\hbar^2}}_{\text{"}\alpha\text{"}}\psi_{II}(x) = \alpha^2\psi_{II}$$

$$\boxed{\psi_{II}(x) = Ce^{\alpha x} + De^{-\alpha x}}$$

0 for positive x



Tunneling

Apply same steps to account for continuity as $E > V_0$ case:

$$R = \frac{\sinh^2\left[\sqrt{2m(V_0 - E)}\, L/\hbar\right]}{\sinh^2\left[\sqrt{2m(V_0 - E)}\, L/\hbar\right] + 4(E/V_0)\left[1 - E/V_0\right]}$$

$$T = \left[1 + \frac{V_0^2 \sinh^2\left(\sqrt{2m(V_0 - E)}\, L/\hbar\right)}{4E(V_0 - E)}\right]^{-1} \quad (= 1 - R)$$

$\sinh^2$ term gives amplitude on other side of wall @ $x = L$

If $\alpha L = \frac{\sqrt{2m(V_0 - E)}}{\hbar} L \gg 1$,

Wide Barrier

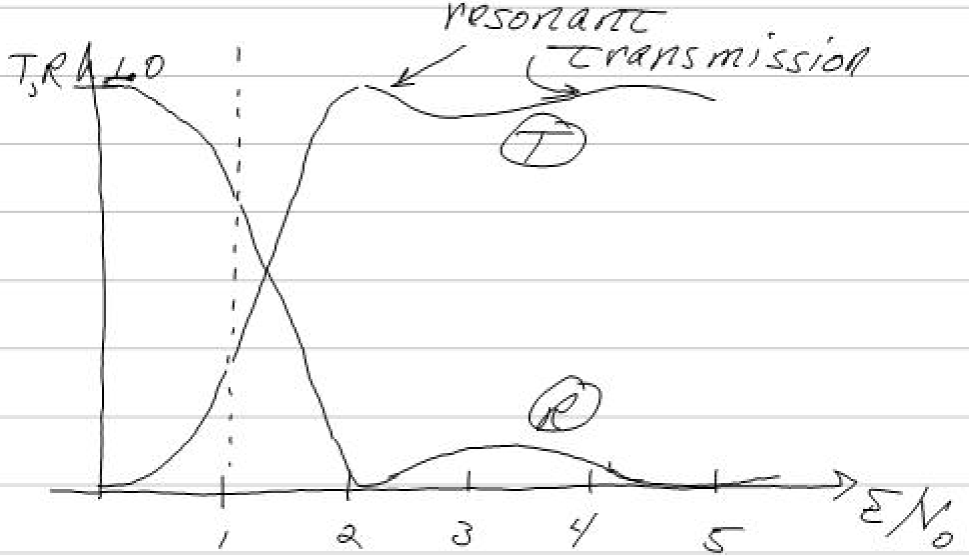
$$\boxed{T = 16 \frac{E}{V_0}\left(1 - \frac{E}{V_0}\right) e^{-2\alpha x}}$$

As $L \to \infty$, $T \to 0$

→ So we don't see tunneling except @ atomic + smaller distances

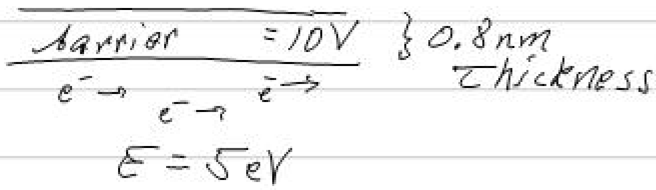
Final term $(e^{-2\alpha x})$ means T has enormous dependence on particle energy.

Transmission & Particle Energy



Example

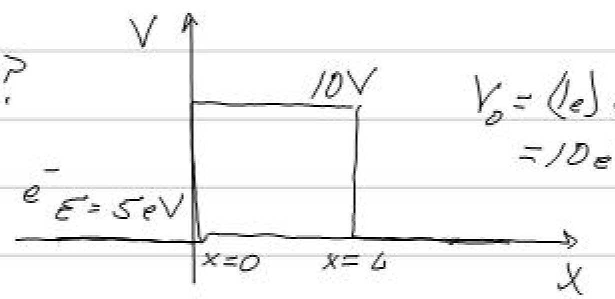
Consider a semiconductor:



e⁻ have kinetic energy in a "conduction band"

What is tunneling probability to get thru barrier?

Is this a 'wide' barrier?



$V_0 = (1e)(10V)$
$= 10eV$

$$\alpha L = \frac{\sqrt{2m(V_0 - E)}}{\hbar} L$$

$$= \frac{\sqrt{2(5.11\times10^5 eV/c^2)(5eV)}}{6.6\times10^{-16} eV\cdot s}(8\times10^{-10} m)$$

$$= \frac{2.2\times10^3 eV/c}{6.6\times10^{-16} eV\cdot s}(8\times10^{-10} m) = 1.2\times10^{10} m^{-1}(8\times10^{-10} m)$$

$$= 9.5 \gg 1 \Rightarrow \text{yes}$$

therefore

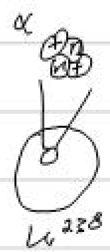
$$T \approx 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2\alpha L}$$

$$= 4e^{-19} \sim \boxed{3\times10^{-8}}$$

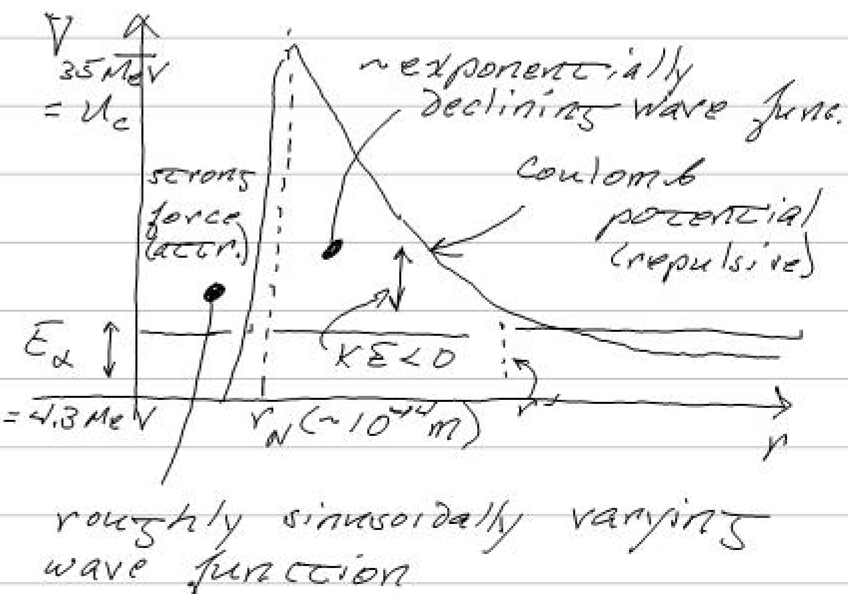
α Decay and Tunneling

Heavy atomic nuclei 'decay' spontaneously
→ α (alpha) decay

$$U^{238} \Rightarrow Th^{234} + \alpha$$
$$(92p + 146n) \qquad (90p + 144n) \quad (2p + 2n)$$



An α is a fairly tightly bound nuclear element
- i.e. it exists in the confines of atomic nuclei



Inside $r = r_N$, α moves freely

Coulomb potential presents finite height to α
- Classically

$\underline{KE_\alpha > U_c}$ to 'escape'

$$U_c = \frac{q_1 q_2}{4\pi\epsilon_0 r_N} = \frac{(2\times 1.6\times 10^{-19} C)(90\times 1.6\times 10^{-19} C)}{4\pi\epsilon_0\ (7.4\times 10^{-15} m)}$$

$$= \underline{\underline{35\ MeV}}$$

But experimentally, $\underline{KE_\alpha = 4.3 MeV} \ll U_c$!

Finite width of barrier

- permits tunneling of α when $KE < U_c$
- potential more complicated than ⎍
  - difficult to solve
  - thickness, L, is very energy-dependent
    - r' is radius @ which $E = V$
    - $\boxed{L = r' - r_N}$

For a given KE_α, there is a characteristic depth, L to tunnel thru:

$$T = 16 \frac{KE_\alpha}{V_0}\left(1 - \frac{KE_\alpha}{V_0}\right) e^{-2\alpha L}$$

Since amplitude of transmitted wave $\propto e^{-2\alpha L}$ $\Rightarrow$ <u>small changes in L cause enormous changes in T.</u>

<u>Decay rate</u> $= \frac{\#\text{ decays}}{\text{time}}$

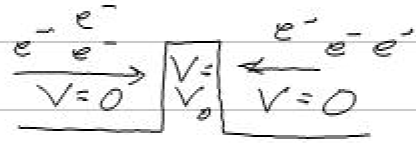
$$= \boxed{\frac{\nu T}{2 r_N}}$$

Probabilistic

$\Rightarrow$ <u>Q. Mech!</u>

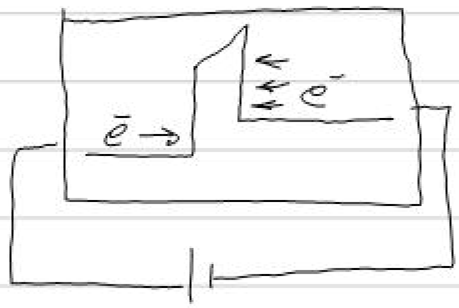
Tunneling Applications

Tunnel Diodes



A potential barrier
- equal e^- flow from both sides across barrier, $I=0$

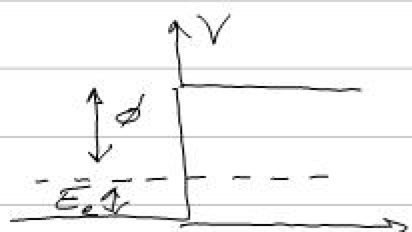
Apply a voltage:



$I \neq$ constant $* V$
(I only 1 way)
instantaneous change in I

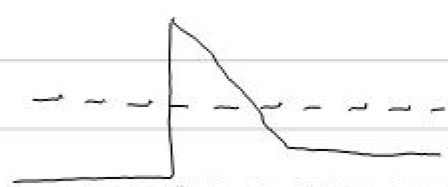
unequal e^- flow, $I > 0$

Field Emission



Electrons in a metal
- feel potential step to get out of metal
- e^- leave metal but eventually return

Apply voltage:



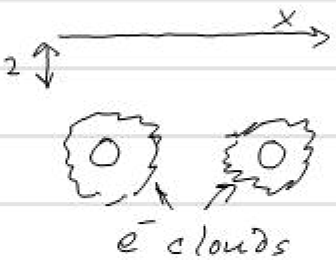
- now e^- tunnel thru barrier
- flow of e^- results from nearby application of E field
- less noise, power

Scanning Tunneling Microscope

Basic idea:

Use fact that tunneling very dependent on barrier width

$$\boxed{T = 16\,\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right) e^{-2\alpha L}}$$

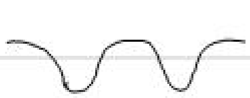


Have a metal tip move across a surface

- a potential barrier exists between e^- in surface atoms and tip

- when very close to atoms
  - width of barrier, L changes as translate in x-direction
  - moving in z also changes L

Dramatic changes in current



Describes 3-Dimensional "shape" of atoms' e^- clouds