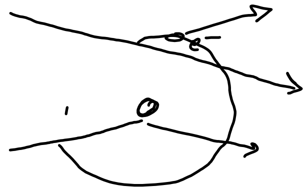
Electromagnetic Interactions of Charged Particles w/Matter

- inelastic collisions with atomic electrons
- elastic scattering from nuclei

These processes give



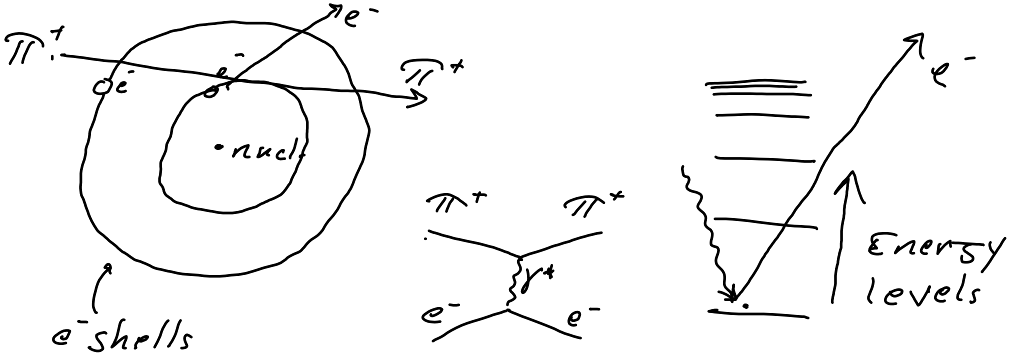
- loss of incident particle energy
- change in particle trajectory

Additional reactions

- Cerenkov emission
- transition radiation
- bremsstrahlung
- scintillation

Aside from high energy electrons and positrons
- inelastic collisions dominate electromagnetic interactions

Ionization and Inelastic Collisions



charged particle interacts with atomic electrons

ionization or excitation result

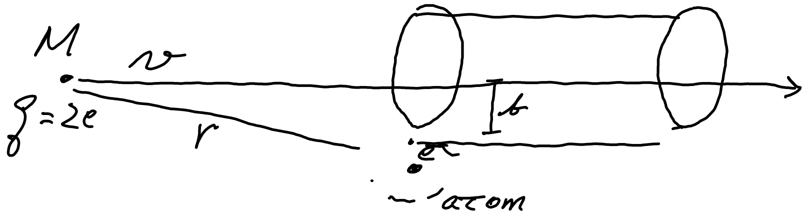
Only few eV ~~eue~~ lost: but many interactions in material

Remember σ_{atom}!

Example: 10 MeV proton incident on copper plate
→ total energy loss in $\frac{1}{4}$ mm

Some collisions provide sufficient energy to ionization electrons
→ cause ionization themselves
"δ rays"

Classical Calculation of Energy Loss



Coulomb interaction with e^-, nucleus

Assumptions

- $M \gg m_e$
- electron free and initially at rest
- electron only moves slightly during interaction
  - E-field remains as in initial position
- incident particle undeviated

Momentum gained by electron

$$\Delta p = \int F\,dt = e\int E_\perp\,dt = e\int E_\perp \frac{dt}{dx}\,dx$$

$$= \frac{e}{v}\int E_\perp\,dx$$

(only care about $E_\perp$ due to symmetry.)

Momentum Transfer

Use Gauss's Law for the integral

$$\int \vec{\mathcal{E}}_\perp \cdot d\vec{A} = 4\pi z e$$
$$= \int \mathcal{E}_\perp \, 2\pi b \, dx$$

$$\int \mathcal{E}_\perp dx = 2ze/b$$

This gives

$$\Delta p = \frac{e}{v}\int \mathcal{E}_\perp dx = \frac{2ze^2}{bv}$$

$$\Delta E = \frac{\Delta p^2}{2m_e} = \frac{2z^2e^4}{m_e b^2 v^2} \propto \frac{1}{b^2 v^2}$$

$\therefore$ most of energy transfer due to close collisions

Where is energy transfer happening?

atomic electrons: $m = m_e$; $e^2 \to e^2$

nuclei: $m = Am_p$; $e^2 \to Z^2e^2$

$$\frac{\Delta E(e)}{\Delta E(\text{nucl.})} = \frac{2z^2e^4}{m_e b^2 v^2} \cdot \frac{b^2 v^2 A m_p}{2z^2 Z^2 e^4} \sim \frac{2Z m_p}{Z^2 m_e}$$
$$= 2m_p/Z m_e \gg 1$$

$\therefore$ atomic electrons mostly where happening

Energy Loss per Unit Length

If N_e is density of electrons

$$dE(b) = \Delta E\, N_e\, dV = \frac{2z^2e^4}{m_e b^2 v^2} N_e (2\pi b\, db\, dx)$$

$$= \frac{4\pi z^2 e^4}{m v^2} N_e \frac{db}{b} dx$$

energy lost to all electrons between b and $b+db$ in dx

Need to integrate over 'b' to get

$$\frac{dE}{dx} = \frac{4\pi z^2 e^4}{m_e v^2} N_e \int_{b_{min}}^{b_{max}} \frac{db}{b}$$

Problem: 1) a large b takes a long time: e^- moves

2) as $b \to 0$, $\Delta p \to \infty$

$$\therefore \frac{dE}{dx} = \frac{4\pi z^2 e^4}{m_e v^2} N_e \ln \frac{b_{max}}{b_{min}}$$

Minimum impact parameter:

→ maximum ΔE when a head-on collision

$$\Delta E_{max} = \frac{(2p)^2}{2m} = 2\gamma^2 m_e v^2$$

$$\text{also} = \frac{2z^2e^4}{m_e v^2 b^2_{min}}$$

$$\therefore \underline{b_{min} = ze^2/\gamma m_e v^2}$$

Maximum impact parameter

e^- has orbital frequency, ν

period $T = 1/\nu$

Interaction with passing particle short compared to T

$$\tau/\gamma = (b/v)/\gamma = b/v\gamma \le T = 1/\nu$$

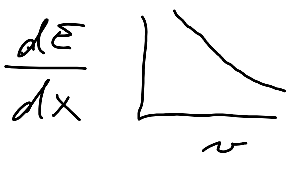
$$\therefore \underline{b_{max} = \gamma v/\nu}$$ → typical (or average) frequency

Substitution gives:

$$\boxed{\frac{dE}{dx} = \frac{4\pi z^2 e^4}{m_e v^2} N_e \ln\left(\frac{\gamma^2 m_e v^3}{z e^2 \nu}\right)}$$

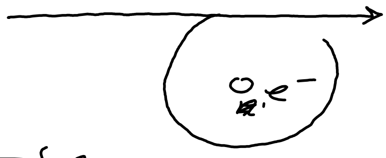
$\frac{1}{v^2}$ @ low energy

$\ln \gamma^2$ @ high energy



Classical Electromagnetic Cross Section

hen a particle travels within 'b' of atom



want the cross section as a function of Energy transfer W

Differential cross-section in area of ring

$$d\sigma(b) = \frac{d\sigma(b)}{db}\,db = 2\pi b\,db$$

Using $W \cancel{\Delta E} = 2z^2Z^2e^4/b^2v^2m$,

Then: $b = \sqrt{\frac{2z^2Z^2e^4}{W\cancel{\Delta E}\,v^2m}}$; $db = -\frac{1}{2}\sqrt{\frac{2z^2Z^2e^4}{v^2m}}\,\frac{1}{W^{3/2}}\,dW$

This gives us

$$d\sigma(b) = 2\pi b\,db = \frac{2\pi z^2Z^2e^4}{v^2m}\,\frac{dW}{W^2} = d\sigma(W)$$

$$\boxed{\frac{d\sigma}{dW} = \frac{2\pi z^2Z^2e^4}{v^2m}\,\frac{1}{W^2}}$$

cross section of particle with energy $E = E(v)$ to loose energy W in a collision with a free electron

Quantum treatment of Energy Loss

Must account for:

- particle energy transfers to atomic electrons only in discrete amounts
- particles have a wave nature i.e. for small 'b', point-like particle is not valid

We think in terms of small (large) momentum transfers, not large (small) impact parameters

→ also Dirac wave-functions with Coulomb potential coupling to γ field

To get the total energy loss, need to integrate over all excitation energies weighted by cross section

$$\frac{dE}{dx} = n_a \int E \, d\sigma$$

In close collisions

$$\left.\frac{dE}{dx}\right|_{W>\eta} = n_e \int_{\eta}^{W_{max}} W \frac{d\sigma}{dW} dW$$

η = an energy cutoff limiting energy transfers to > 50 keV

W_{max} = maximum energy transfer

For the differential cross section, $\frac{d\sigma}{dW}$

$$\frac{d\sigma}{dW} = \underbrace{2\pi \frac{e^4}{mc^2} \frac{1}{\beta^2 W^2}}_{\text{semi-classical}} \left(1 - \beta^2 \frac{W}{W_{max}}\right)$$

(ie. when $W \ll W_{max}$)

→ technically, this is expression for spin 0 particles

→ when $W \ll$ incident energy, spin dependence goes away

For massive particle

$$\boxed{\frac{dE}{dx} = \frac{4\pi n_e}{mv^2} z^2 e^4 \left(\ln\left(\frac{2mv^2\gamma^2}{I} \right) - \beta^2 \right)}$$

Bethe - Bloch

- Depends on
  - velocity of incident particle
  - material electron density, n_e
  - I = mean ionization potential of material (or, excitation)

I is main parameter

→ the average orbital frequency, ν times Planck constant

$$I = h\nu \sim 16\, Z^{0.9}\, eV \quad (Z>1)$$

→ the value of ν is a weighted average of frequencies for atomic levels
- weighted by "osillator strengths" (mainly from empirical measurement)

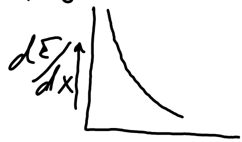
→ some varition as move thru periodic table

Behavior of energy loss

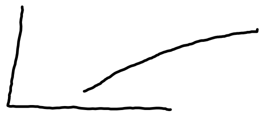
$$\frac{dE}{dx} \propto \frac{1}{\beta^2} \ln \underbrace{A}_{\text{const.}} \beta^2 \gamma^2$$

For low velocities ($\beta\gamma < 3$)

$$\frac{dE}{dx} \propto 1/\beta^2$$



When $\beta\gamma > 4$, $dE/dx \propto \ln \beta^2 \gamma^2$



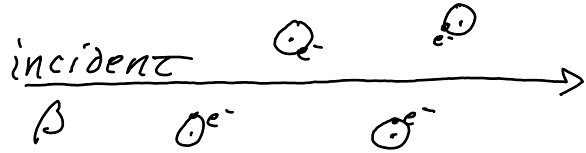
"relativistic rise"

Minimum ionizing
- → condition when $\beta\gamma \sim 3\text{-}4$
- → all incident particles roughly same $\boxed{dE/dx \sim 2\,\text{MeV}/g\,cm^2}$

(note: the overall structure here has similarity to semi-classical picture.)

Density and Fermi Plateau

We've considered target atoms as if they are each isolated
— for dense materials



Relativistic incident particle polarizes atoms along path

∴ atomic electrons furthest from path see reduced $\vec{E}$ field (~~prog~~ beam is "screened")

— they contribute less ionization than expected

As incident particle γ rises:

$b_{max} \approx \gamma v/\nu \rightarrow$ larger volume of integration

(in quantum case, this relates to small momentum transfer)

∴ density effect larger

$$\frac{dE}{dx} \propto \ln \gamma^2 \rightarrow \ln \gamma \rightarrow \underline{\text{constant}}$$

<u>Fermi plateau</u>

Bethe-Bloch modified to

$$\boxed{\frac{dE}{dx} = \frac{4\pi n_e z^2 e^4}{mv^2}\left[\ln\frac{2mv^2\gamma^2}{I} - \beta^2 - \frac{\delta(\gamma)}{2}\right]}$$

$\delta(\gamma)$ = density parameter

- in terms of atomic properties

Fluctuations in Energy Loss

$\frac{d\sigma}{dw} \propto \frac{1}{w^2}$ ∴ small energy transfers most likely

→ the $\frac{dE}{dx}$ calculated (Bethe-Bloch)

→ ~~most~~ average value

→ most probable energy transfers lower E

→ high tail from δ-rays

Number δ-rays produced with energy $> E_1$ in thickness x

$$N(E \geq E_1) = \frac{2\pi n_e z^2 e^4}{m v} \times \left(\frac{1}{E_1} - \frac{1}{E_{max}} \right)$$

maximum possible energy transfer

Electrons + Positrons

- Ionization not primary energy loss mechanism for $e^\pm$ (in most cases)

→ somewhat different than for other particles

$$\frac{d\sigma}{dW}(E,W) = \frac{2\pi e^4}{mv^2}\left[\frac{1}{W^2} - \frac{1}{W(E-W)} \times \frac{mc^2(2E+mc^2)}{(E+mc^2)^2} + \frac{1}{(E-W)^2} + \frac{1}{(E+mc^2)^2}\right]$$

E = K.E. of incident $e^\pm$

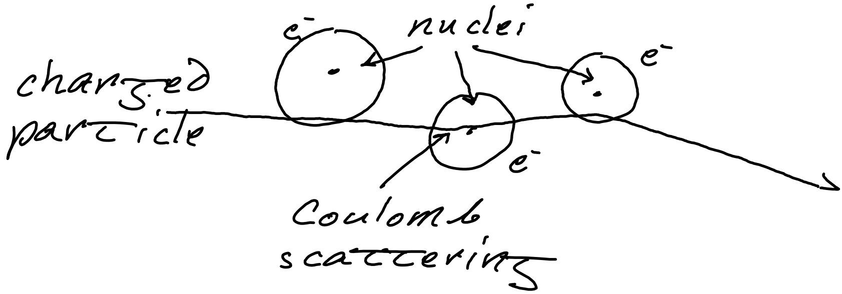
Moller cross-section (Bhabha for e^+)

$$\frac{dE}{dx} = \frac{2\pi n_e e^4}{mc^2}\left(2\ln\frac{2mc^2}{I} + \underbrace{3}_{=4 \text{ for } e^+}\ln\gamma - \underbrace{1.95}_{=2 \text{ for } e^+}\right.$$

for electrons

- relativistic rise smaller than for other particles (i.e. $\propto \ln\gamma$)

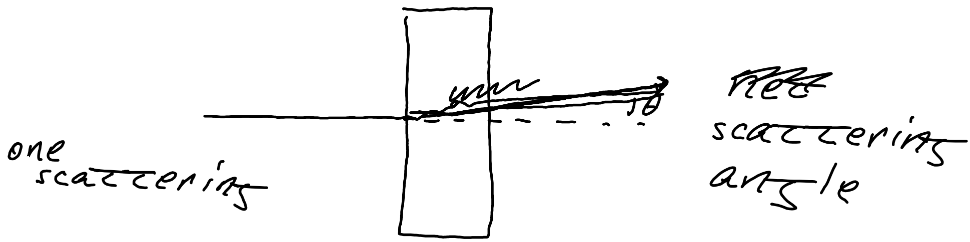
Elastic Scattering



Deflections of particle trajectory
- nuclei + e⁻ ~~are~~ involved
- nuclei dominate this process

Make some assumptions:
- no energy loss
- only a change charged particle trajectory results

Consider a particle
→ charge z, mass m
- incident on a block of material



Charged particles in medium
receive $\perp$ change in momentum
→ incident particle gets opposite

(one scattering) $$p_\perp = \frac{2zZe^2}{bv}$$ (from earlier)

This means

$$\theta = \frac{p_\perp}{p} = \frac{2zZe^2}{bvp}$$

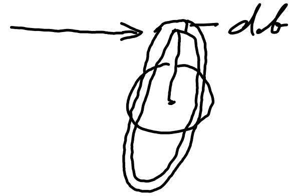
Or

$$\underline{b = 2zZe^2 / bv\theta}$$

Differential Cross Section

Want to know cross section for scattering between θ and $\theta + d\theta$

$$d\sigma = 2\pi b \, db$$



Substituting for b and db, we have

$$d\sigma = 2\pi \left(\frac{2zZe^2}{pv}\right)^2 \frac{d\theta}{\theta^3}$$

For small angles, $d\Omega \simeq 2\pi\theta \, d\theta$

$$\therefore \boxed{\frac{d\sigma}{d\Omega} = z^2 Z^2 r_e^2 \left(\frac{mc}{\beta p}\right)^2 \frac{1}{(\theta/2)^4}}$$

Full Rutherford form:

spin $\emptyset$

$$\theta/2 \to \sin(\theta/2)$$

(spin 1/2: multiply by $(1 - \beta^2 \sin^2 \theta/2)$)

Total Elastic Cross Section

Need to integrate

→ Rutherford breaks down when too far from atom

- nuclear charge screened by atomic electrons

$\therefore \theta^2/4 \rightarrow (\theta^2 + \theta_{min}^2)/4$

↳ cross section levels at very small angles

Estimating θ_{min}:

By atomic radius ~ b: $\theta'_{min} \simeq 2z Z^{4/3} \frac{mc^2}{pv} \left(\frac{r_e}{a_0}\right)$

By uncertainty principle: $\theta_{min} = \alpha Z^{1/3} \frac{mc}{p}$

↳ use the smaller one

Also a θ_{max} when λ ~ nuclear radius

$$\theta_{max} \simeq \frac{2A^{-1/3}}{\alpha} \frac{mc}{p}$$

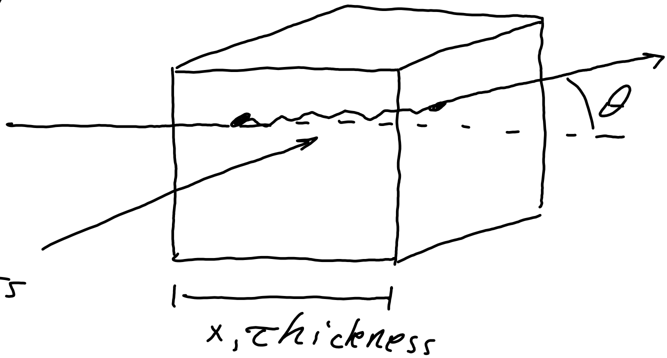
These give us

$$\sigma_{el.} = \int_{\theta_{min}}^{\theta_{max}} \frac{d\sigma}{d\Omega} \sin\theta \, d\theta \, d\phi = \boxed{\pi r_e^2 4 \Big/ \left(\frac{z Z^{2/3}}{\alpha \beta}\right)^2}$$

Multiple Scattering

particle passing thru block of material

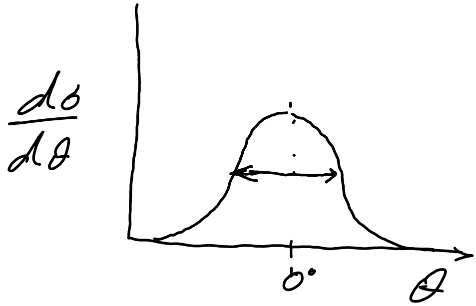
series of scatterings



→ average deflection of each scattering 0°

→ but a dispersion in outgoing particle net angle

What is typical value of $|\theta|$?



Calculate average θ^2

$$\theta_s^2 = x\, n_a \int_{\theta_{min}}^{\theta_{max}} \theta^2 \frac{d\sigma}{d\Omega}\, d\Omega$$

use Rutherford

statistical approach

$$\theta_s \propto \sqrt{n_a}$$

Note:
θ_s = root-mean squared θ
$= \sqrt{\langle \theta^2 \rangle}$

$$= 8\pi x\, n_a r_e^2 z^2 Z^2 \left(\frac{mc}{\beta p}\right)^2 \ln \frac{2}{\alpha^2 A^{1/3} Z^{2/3}}$$

this can be expressed (if $A = 2Z$)

$$\boxed{\theta_s = \frac{\sqrt{4\pi/\alpha}\; mc^2}{pv}\, z \sqrt{x/X_0}}$$

The 'radiation length', X_0 is distance over which a significant energy is lost due to bremsstrahlung

$$\boxed{X_0 = \frac{A}{4\alpha n_a Z^2 r_e^2 \ln(183/Z^{1/3})}}$$

Example: an electron ($z=1$) of 16 MeV energy passes thru 20 mg/cm^2 of iron ($X_0 = 13.8$ g/cm^2)

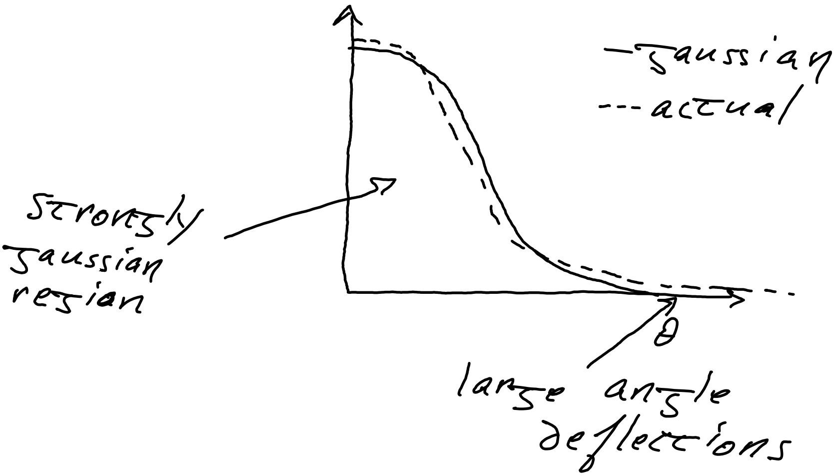
$$\theta_s \sim \frac{13.6\,\text{MeV}}{v p}\sqrt{\frac{x}{X_0}} \qquad (v = \beta c \sim 1)$$

$$= \frac{13.6\,\text{MeV}}{16\,\text{MeV}}\sqrt{\frac{20\,\text{mg/cm}^2}{13.8\,\text{g/cm}^2}}$$

$$= \underline{\underline{0.031\,\text{rad}}}$$

Gaussian Behavior

Scattering angle obeys a rough gaussian shape



large angle tail

- most non-gaussian
- generally from one large angle deflection in material
- these obey Rutherford formula