CONSTRUCT $\mathcal{L}$: (cf BJORKEN & DRELL II)

INGREDIENTS: $\varphi(x)$: FOR 1 PART (ASSUME REAL)

∂_μ

DIMENSIONS: CLASSICAL $S = \int dt\, L(t) = \int dt\, [T - V]$

$(= \int d^4x\, \mathcal{L}(x))$ $\quad [\![M^{-1}]\!]\,[\![M]\!]$

$\Rightarrow$ DIM-LESS (i.e. DIM OF ANG MOM IN UNITS OF $\hbar$) (will use M to indicate dim)

$[\![\ ]\!] \equiv$ DIM. OF

POSSIBLE TERMS: φ^n (OK - SCALAR) $n = 1, 2, \ldots$

$\partial_\mu \partial^\mu \varphi^n = \partial^2 \varphi^n$, $\quad \partial^{2n} \varphi^m$ i.e. $(\partial_\mu \partial^\mu)^n$

$[(\partial_\mu \varphi)(\partial^\mu \varphi)]^n$

PRODUCTS OF THESE: ex $\varphi^n \partial_\mu \partial^\mu \varphi^{n'} \ldots$

EVEN w/ LOR. INVAR, A MESS

LIST TERMS:

(a) $\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi = \frac{1}{2}\left[(\partial_t \varphi)^2 - \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi\right]$

- SCALAR
- GIVES INTERESTING DYNAMICS (FROM BOTH ∂_0 & $\vec{\nabla}$) $\Rightarrow$ WAVES
- NOT SIMPLEST, BUT MENTION 1ST BEC.

CONVENTION: DIM-LESS CONST. $\frac{1}{2}$ IN FRONT
(can mult $\mathcal{L}$ by overall const w/ no effect, so get to pick one)

$\Rightarrow \underbrace{\int d^4x}_{M^{-4}}\, \mathcal{L} \Rightarrow [\![\mathcal{L}]\!] = [\![M^4]\!] \qquad [\![\partial_\mu]\!] = [\![M^{-1}]\!]$

$\Rightarrow [\![\varphi]\!] \doteq [\![M]\!]$

(b) $\varphi \partial^2 \varphi$

- SAME AFTER INT. BY PARTS ⇒ NOTHING NEW
(ALWAYS ASSUME $\varphi \to 0$ AT $|x| \to \infty$)

ALSO, JUST $\partial^2 \varphi = \partial_\mu (\partial^\mu \varphi)$ = 4-D SURFACE INT.
= 0 IF $\varphi \to 0$

(c) $c_1 \varphi$

- OK (SCALAR)
- c_1 : $[\![M^3]\!]$
- CAN DISCARD : IF HAVE φ^2 (OR MORE)
CAN CHG VARIABLES $\varphi \equiv c + \varphi'$
(i.e. re-define field ⇒ advantage of Lag. mech.)
COMPLETE SQ & REMOVE ⇒ NO PHYSICS

(d) $c_2 \varphi^2$
- c_2 : $[\![M^2]\!]$
- CONVENTION : $c_2 \equiv -\frac{1}{2} m^2$

(AFTER QUANTIZE, m IS MASS OF FREE PARTICLE)

(e) $c_3 \varphi^3$ c_3 : $[\![M^1]\!]$

(f) $c_4 \varphi^4$ c_4 : $[\![M^0]\!]$ CONVENTION : $c_4 \equiv -\frac{1}{4!} \lambda$
(DIM-LESS)

(g) $c_5 \varphi^5$ ETC : c_5 : $[\![M^{-1}]\!]$

⇒ NON-RENORMALIZABLE

⇒ FUNDAMENTALLY INCONSISTENT ⇒ DISCARD
(as mentioned, can relax this; expect
it's suppressed by $\sim \frac{1}{\Lambda^n_{\text{new physics}}}$)

(h) $c_5' \varphi^2 \partial^2 \varphi$,

$c_6 \varphi^2 \partial_\mu \varphi \partial^\mu \varphi$, $c_8 \partial_\mu \varphi \partial^\mu \varphi \partial_\nu \varphi \partial^\nu \varphi$, ...

$c_5' : [\![M^{-1}]\!]$ $c_6 : [\![M^{-2}]\!]$ $c_8 : [\![M^{-4}]\!]$

DISCARD $\Rightarrow$ NON-REN.

SO FAR:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \tfrac{1}{2} m^2 \varphi^2 + c_3 \varphi^3 - \tfrac{1}{4!} \lambda \varphi^4$$

SYMMETRY:

SUPPOSE NEVER OBSERVE EVEN # $\to$ ODD # (OR V. V.)
IN SCATTERING (OR DECAYS if $>1\ \varphi$); SUSPECT SYMM: { DISCRETE SYMM: EVENNESS OR ODDNESS; (-1) FOR EACH PART.

$$\mathcal{L}(\varphi) \stackrel{?}{=} \mathcal{L}(-\varphi) \quad \text{ie} \quad c_3 = 0 \quad \text{(IDENTICALLY)}$$

will see: symm $\leftrightarrow$ cons. law
have $(-)$ for each field

- IF EXCLUDE, WILL NEVER GENERATE VIA INTS BY SYMM
(ie CAN'T CONSTRUCT IT VIA OTHER INTS)
$\Rightarrow$ CONSISTENT TO LEAVE OUT
(expt'l question)
(NOT TRUE FOR $m^2\varphi^2$)

$$\Rightarrow \boxed{\mathcal{L} = \tfrac{1}{2}(\partial_\mu \varphi)(\partial_\mu \varphi) - \tfrac{1}{2} m^2 \varphi^2 - \tfrac{1}{4!} \lambda \varphi^4}$$

- SIMPLEST INTERESTING FT
"$\lambda \varphi$ 4TH"

CLASSICAL EOM: EULER-LAGRANGE EQNS

$\mathcal{L}(\varphi, \partial_\mu \varphi)$ FNL OF φ & $\partial_\mu \varphi$

LEAST ACTION / HAM. PRINCIPLE: φ MINIMIZES

$$S = \int d^4x \, \mathcal{L}(\varphi, \partial_\mu \varphi)$$

THEN, ADD ARB. FN TO φ W/ SMALL PARAM ϵ:

$$\varphi_\epsilon(x) = \varphi(x) + \epsilon \eta(x) \qquad \partial_\mu \varphi_\epsilon = \partial_\mu \varphi + \epsilon \partial_\mu \eta$$

LEAST ACTION

$$\partial_\epsilon S(\epsilon) = \partial_\epsilon \int d^4x \, \mathcal{L}(\varphi_\epsilon, \partial_\mu \varphi_\epsilon) = 0 \qquad \text{AT } \epsilon = 0$$

(ie STA. UNDER SMALL VAR. IN φ)

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} \frac{\partial \varphi_\epsilon}{\partial \epsilon} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \frac{\partial(\partial_\mu \varphi)}{\partial \epsilon} \right]$$

THINK OF $\mathcal{L}$ AS FN OF φ, $\partial_\mu \varphi$ SEPARATELY; SOMETIMES SEE $\frac{\delta \mathcal{L}}{\delta \varphi}$ TO DIST. FROM $\partial_\mu \varphi$ (ie WRT FN, NOT x)

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} \eta + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu \eta \right] \quad (\text{note } \Sigma_\mu)$$

ASSUME η (& ALL FIELDS) $\to 0$ AT $|x| \to \infty$

BY PARTS:

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right] \eta(x) = 0$$

can now pull out η

$\Rightarrow$ TRUE FOR ANY $\eta(x)$. (FOR EX, $\eta(x) = \delta(x-y)$)

$\Rightarrow$ E-L EQNS: $$\boxed{\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = 0}$$

$\Rightarrow$ ANY TYPE OF φ (not just scalar).

(a) MORE THAN ONE φ^i: 1 EQN EACH (consider varying separately)

(b) φ^i COULD BE A^μ (4 FIELDS) $\Rightarrow$ MAX. EQNS

OUR CASE: (1 REAL SCALAR)

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -m^2\varphi - \frac{\lambda}{3!}\varphi^3$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \frac{\partial}{\partial(\partial_\mu\varphi)}\left[\tfrac{1}{2}\partial_\nu\varphi\, g^{\nu\rho}\,\partial_\rho\varphi\right]$$

$$= \frac{1}{2}\left[\frac{\partial(\partial_\nu\varphi)}{\partial(\partial_\mu\varphi)}\, g^{\nu\rho}\,\partial_\rho\varphi + \partial_\nu\varphi\, g^{\nu\rho}\,\frac{\partial(\partial_\rho\varphi)}{\partial(\partial_\mu\varphi)}\right]$$

{note: can't do $\frac{\partial}{\partial(\partial_\mu\varphi)}\left[\tfrac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi\right]$

(1) 2^{ND} set is summed over ⇒ takes all values, but asking about deriv wrt partic $\partial_\mu\varphi$

(2) $\partial^\mu\varphi$ is diff kind of object than $\partial_\mu\varphi$; clearer to write in terms of $\partial_\mu\varphi$

(but can show: $\frac{\partial(\partial^\rho\varphi)}{\partial(\partial_\mu\varphi)} = \frac{\partial(g^{\rho\sigma}\partial_\sigma\varphi)}{\partial(\partial_\mu\varphi)}$

$$= g^{\rho\sigma}\delta^\mu{}_\sigma = g^{\rho\mu} \text{)}$$

$$= \frac{1}{2}\left[\underbrace{\delta^\mu{}_\nu}_{= g^\mu{}_\nu} g^{\nu\rho}\partial_\rho\varphi + \partial_\nu\varphi\, g^{\nu\rho}\delta^\mu{}_\rho\right]$$

$$= \frac{1}{2}\left[g^{\mu\rho}\partial_\rho\varphi + g^{\nu\mu}\partial_\nu\varphi\right] = g^{\mu\rho}\partial_\rho\varphi = \partial^\mu\varphi$$

$$\Rightarrow \frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = 0 = +m^2\varphi + \frac{\lambda}{3!}\varphi^3 - \partial_\mu\partial^\mu\varphi$$

OR

$$\boxed{(\partial^2 + m^2)\varphi - \frac{\lambda}{3!}\varphi^3 = 0}$$

w/ $\lambda = 0$: KLEIN-GORDON EQN

⇒ SIMPLEST RELATIVISTIC WV EQN

(NOTE: ALL TERMS ARE SCALARS)

{ 1^{ST} attempt at relativistic qm; problem: ∂_t^2 ⇒ solutions w/ so travel of $-E$ as well as $+E$; non-sense ⇒ not stable; in FT, these become antiparticles }

Dirac attempted to fix this via rel. eqn w/ 1 time deriv.; still failed, but fermions ⇒ comp filled Dirac sea etc

COMMENTS:

1. E-L EQNS FOR FIELDS SIMILAR TO PARTICLE

   (ie $S = \int dt\, L(q(t), \dot{q}(t))$

   AND VARY $q + \epsilon\eta(t)$)

2. IF THINK OF $\bar{x}$ AS PARAM:

   $\varphi(t, \bar{x}) \equiv \varphi_{\bar{x}}(t) \approx q^i(t)$

   i LABELS ∞-# OF COORDS (OR PUT ON LATTICE)

   ⇒ EXACTLY SAME (WILL USE THIS)

3. THERE'S A COMPACT NOTATION THAT EMPHASIZES THIS (CF NASH) w/

   FUNCTIONAL DERIVS $\dfrac{\delta}{\delta\varphi(x)}$ { complements functional = fn of fns → deriv wrt fn (not param)

   ie DERIV WRT φ AT x

   BASIC RULE $\dfrac{\delta}{\delta\varphi(x)}\varphi(y) = \delta^4(x-y)$ (DIFF x^μ, y^μ ⇒ DIFF VAR)

   THEN LEAST ACTION:

   $$\frac{\delta S}{\delta\varphi(x)} = 0$$ (TRUE FOR φ AT ANY x)

   ⇒ TRY IT

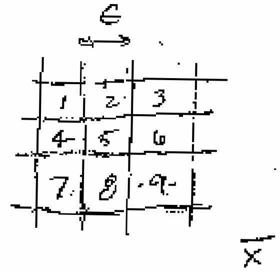
HAMILTONIAN: (cf B;D II)

FOLLOW USUAL METHOD: $(\dot q \equiv \partial_t q)$

(1) NEED CONJ. $p = \frac{\partial L}{\partial \dot q}$;

(a) MAKE # DOF FINITE:

(in 3-space) BREAK $\bar x$ INTO CELLS VOL $\Delta V = \epsilon^3$
LABEL w/ i



DEF $\varphi_i(t) \equiv \frac{1}{\Delta V}\int_{(i)} d^3x\, \varphi(t,\bar x)$

$\equiv$ φ AVE'D OVER CELL i (i.e. $\varphi_{\bar x}(t)$)

(b) $\mathcal{L}(\varphi, \partial_\mu \varphi) \equiv \mathcal{L}(\varphi, \dot\varphi, \bar\nabla \varphi)$

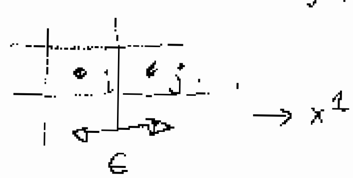
$\mathcal{L}$ IS LOCAL: BREAKS UP INTO PIECES $\mathcal{L}_i$ WHICH DEP. ON $\varphi_i, \dot\varphi_i$ & φ_{n_i}

n_i RUNS OVER NEXT SPATIAL NEIGHBORS OF i
$\Rightarrow$ NEED FOR DERIVS $\bar\nabla$

OUR CASE
$$\mathcal{L}(t,\bar x) = (\partial_t \varphi(t,\bar x))^2 - \bar\nabla\varphi \cdot \bar\nabla\varphi - m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

$$\nabla_1 \varphi \sim \frac{1}{\epsilon}(\varphi_i - \varphi_j) \quad \text{etc}$$

skip {



$$\Rightarrow \quad \mathcal{L}_i = \dot\varphi_i^2 - \frac{1}{\epsilon^2}\sum_{n_i}(\varphi_{n_i} - \varphi_i)^2 - m^2\varphi_i^2 - \frac{\lambda}{4!}\varphi_i^4$$

$$L = \int d^3x\, \mathcal{L} \sim \sum_i \Delta V\, \mathcal{L}_i(\varphi_i, \varphi_{n_i}, \dot\varphi_i)$$

(c) CONJ $p_i = \frac{\partial L}{\partial \dot\varphi_i(t)} = \Delta V \underbrace{\frac{\partial \mathcal{L}_i}{\partial \dot\varphi_i(t)}}$

$\equiv \pi_i(t)$ CONJ. MOM. DENSITY

→

(d) $H = \sum_i p_i \dot{\varphi}_i - L$

$$= \sum_i \Delta V \left(\pi_i \dot{\varphi}_i - \mathcal{L}_i \right)$$

(e) CONTINUUM: $\Delta V \to 0 \qquad i \to \bar{x}$

$$\pi(t,\bar{x}) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(t,\bar{x})}$$

CAN USE TO SOLVE FOR $\dot{\varphi}$ VS π

$$H = \int d^3x \underbrace{\left(\pi(t,\bar{x})\, \dot{\varphi}(t,\bar{x}) - \mathcal{L} \right)}$$

$\equiv$ HAM DENSITY: $\mathcal{H}(\pi, \varphi)$

OUR CASE:

$$\mathcal{L} = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\nabla\varphi\cdot\nabla\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}$$

$$\mathcal{H} = \pi\dot{\varphi} - \mathcal{L}$$

} more than 1 $\varphi \Rightarrow$ sum

$$= \pi^2 - \left(\frac{1}{2}\pi^2 - (\nabla\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4 \right)$$

$$\boxed{\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{1}{4!}\lambda\varphi^4}$$

NOTE: – $\mathcal{H} \geq 0$ (φ & π are real)

– PROCEDURE TREATS $t, \bar{x}$ ON DIFF FOOTING
WILL SEE MORE COVARIANT TREATMENT LATER.

QUANTIZE:

USUALLY WORK IN HEIS. PICT:

- OPS. DEP. ON t, STATES ARE t-IND.

WHY? KEEPS t, $\bar{x}$ TOGETHER:

$\Rightarrow$ OPS ARE $\varphi_i(t)$, $\pi_i(t)$

$$\equiv \varphi(t,\bar{x}), \quad \pi(t,\bar{x})$$

QUANTIZE $\equiv$ (a) IMPOSE CAN. COMM. RELNS

(b) FIND REP. WHICH SATISFIES THESE

__EX__ 1-DOF q; $p \equiv \frac{\partial L}{\partial \dot{q}}$:

(a) $[p(t), q(t)] = -i\hbar = -i$

(__HEIS__: EQUAL-t __SCHR__: $[p(0), q(0)]$)

(b) $p \equiv -i\hbar\,\partial_q = -i\,\partial_q$

(THEN PUT INTO H, LOOK FOR E-STATES, ETC)

HERE:

$$[\varphi_i(t), \varphi_j(t)] = [p_i(t), p_j(t)] = 0$$

$$[p_i(t), \varphi_j(t)] = -i\,\delta_{ij}$$

CONTIN:

$$[\pi_i(t), \varphi_j(t)] = -i\,\frac{\delta_{ij}}{\Delta V} \xrightarrow[\Delta V \to 0]{} \delta^3(\bar{x}_i - \bar{x}_j)$$

ie $$\boxed{[\pi(t,\bar{x}), \varphi(t,\bar{x}')] = -i\,\delta^3(\bar{x}-\bar{x}')}$$

(IF SEVERAL FIELDS $\varphi_s(t, \bar{x})$, ONLY NON-ZERO RELN IS

$$[\pi_r(t, \bar{x}), \varphi^s(t, \bar{x}')] = -i\, \delta_r{}^s\, \delta^3(\bar{x} - \bar{x}') \quad)$$

NOTE: AT DIFF t's

$$[\varphi(t, \bar{x}), \varphi(t', \bar{x}')] \neq 0$$

$$\equiv [\varphi(x), \varphi(x')]$$

{ will see ⇒ interesting object; related to propagator from x' to x

TIME EVOLUTION:

HEIS. EQNS OF MOTION:

$$\boxed{\begin{aligned} \partial_t \varphi(t, \bar{x}) &= i[H, \varphi(t, \bar{x})] \\ \partial_t \pi(t, \bar{x}) &= i[H, \pi(t, \bar{x})] \end{aligned}}$$

H EVOLVES OPS (ALL OPS) TO LATER t's

CAN CHECK: (cf HW)

COMM RELNS BETW. π & φ GUARANTEES THESE REPRODUCE E-L (OR HAM.) EOM.

NOTE: PRESCRIPTION SEPARATES t & $\bar{x}$

FROM NON-REL QM; IS IT RIGHT FOR REL. THY?

⇒ PUT INTO BIGGER CONTEXT;

SHOW $H = P^0 =$ 0TH COMP OF 4-VECT