Physics 7361
Homework 10 30 Oct 06
Due Nov 13

1. Read AH II Ch 19 (Higgs), Appx M (Gp Theory)

2. Consider the action of permuting three objects.
   (a) Show that this forms a group.
   (b) Give a matrix version of the defining representation, listing all elements.
   (c) Show that this representation is reducible (Hint: find a vector which is an eigenvector of all the matrices.) Give the matrices in the reduced representation.

3. Apart from the trivial (scalar) representation, the simplest representations of the rotation group $SU(2)$ include vectors that are two-component objects called spinors. The generators are the $2 \times 2$ Pauli matrices (divided by two). Construct the matrix that rotates a spinor by a finite angle $\theta$ about the $y$ axis using the Pauli matrix $(1/2)\sigma_x$. What is the effect of a rotation by $2\pi$?

4. Show that the combinations $\hat{x} \pm i\hat{y}$ rotate in the same was as states $|j, m\rangle$ with $j = 1$ and $m = \pm 1$.

5. The Lagrangian for a free, massive, vector boson is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} M^2 A^\mu A_\mu \,.$$

   (Note the positive sign for the mass term, which is correct, unlike my lecture notes.)
   (a) Derive the Euler-Lagrange equations discussed in lecture.
   (b) Find the three independent solutions in momentum space for a free particle, $\epsilon^\mu(\mathbf{k}, \lambda) \exp(-ik \cdot x)$, in a frame in which $\mathbf{k}$ is along the $z$ axis. Choose the $\epsilon^\mu$ vectors to be orthonormal,

$$\epsilon(\mathbf{k}, \lambda)^\mu \, \epsilon(\mathbf{k}, \lambda')^*_\mu = \delta_{\lambda,\lambda'}$$

   and with definite helicity.
   (c) Give the form of these solutions in the particle's rest frame, and in the limit where the magnitude of $\mathbf{k}$ is large.

For a massless field ($M = 0$), this becomes a gauge field. The equation

$$\partial_\mu A^\mu = 0$$

is no longer an equation of motion, but we may impose it as a gauge condition ("Lorentz gauge"). We may also impose $A^0 = 0$ (cf AH section 7.3). There are now only two independent solutions.
(d) Give these in a frame in which $\mathbf{k}$ is along the $z$ axis (again, with definite helicity).