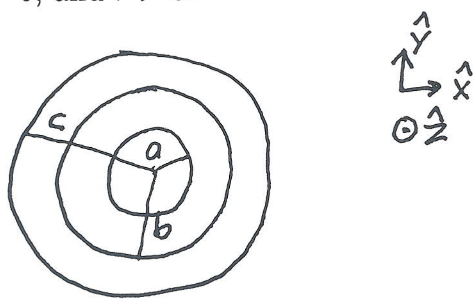
SMU Physics 1304 : Spring 2011

Final Exam

Problem 1 : The figure below shows the circular cross section of a wire of radius a which carries a current I. Outside of this is a capacitor plate of inner radius b and outer radius c has a field $\vec{E} = -\alpha t \hat{z}$. Between the capacitor and wire ($b > r > a$) there is empty space. Find the function $B(r)$ in terms of α and I, where the magnetic field is given by $\vec{B} = B(r)\hat{\theta}$, for $r < a$, $b > r > a$, $c > r > b$, and $r > c$.



$r<a$

$$2\pi r B = \mu_0 I r^2/a^2 \qquad \vec{B} = \frac{\mu_0 I r}{2\pi a^2}\hat{\theta}$$

$b>r>a$

$$2\pi r B = \mu_0 I \qquad \vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\theta}$$

$c>r>b$

$$2\pi r B = \mu_0 I + \mu_0 \varepsilon_0 \frac{d}{dt}\left((-\alpha t)\pi(r^2-b^2)\right)$$

$$= \mu_0 I - \mu_0 \varepsilon_0 \alpha \pi(r^2-b^2)$$

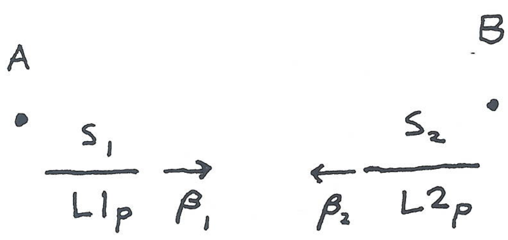
$$\vec{B} = \frac{\mu_0}{2\pi r}\left(I - \varepsilon_0 \alpha \pi (r^2-b^2)\right)\hat{\theta}$$

$r>c$

$$2\pi r B = \mu_0 I - \mu_0 \varepsilon_0 \alpha \pi (c^2-b^2)$$

$$\vec{B} = \frac{\mu_0}{2\pi r}\left(I - \varepsilon_0 \alpha \pi (c^2-b^2)\right)\hat{\theta}$$

Problem 2 : The figure below shows the rest frame of two planets A and B. A ship S_1 moves from A to B with unknown velocity $\beta_1 > 0$, while another ship S_2 moves from B to A with unknown velocity $\beta_2 < 0$. In the frame of the planets the length of ship S_1 is $L1_p = 12\,\mathrm{m}$, and length of ship S_2 is $L2_p = 18\,\mathrm{m}$. In the frame of ship S_1 the length of ship S_1 is $L1_1 = 16\,\mathrm{m}$. In the frame of ship S_2 the length of ship S_2 is $L2_2 = 24\,\mathrm{m}$. Find β_1 and β_2. In the frame of ship S_1, find the velocity β_2' and length $L2_1$ of ship S_2. In the frame of ship S_2, find the velocity β_1' and length $L1_2$ of ship S_1.



$$\gamma_1 = \frac{1}{\sqrt{1-\beta_1^2}}$$

$$\gamma_2 = \frac{1}{\sqrt{1-\beta_2^2}}$$

$$L1_p = L1_1/\gamma_1 \qquad L2_p = L2_2/\gamma_2$$

$$1-\beta_1^2 = \left(\frac{L1_p}{L1_1}\right)^2 \qquad 1-\beta_2^2 = \left(\frac{L2_p}{L2_2}\right)^2$$

$$\beta_1^2 = 1-\left(\frac{12}{16}\right)^2 \qquad \beta_2^2 = 1-\left(\frac{18}{24}\right)^2$$

$$\beta_1 = -\beta_2 = \sqrt{7}/4 = \underline{0.661}$$

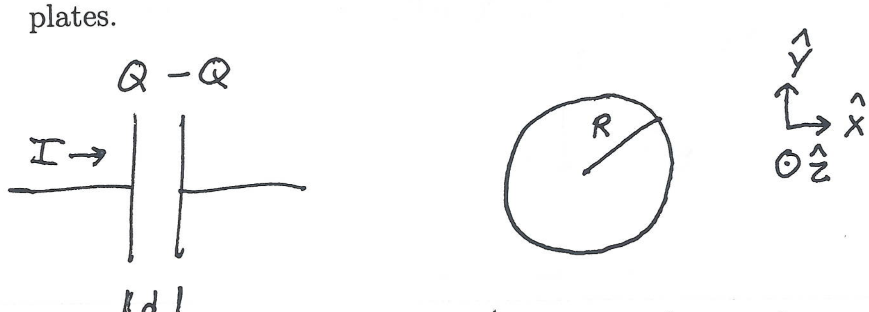
$$\beta_2' = \frac{\beta_2-\beta_1}{1-\beta_1\beta_2} = \frac{-2\beta_1}{1+\beta_1^2} = \underline{-0.920}$$

$$\beta_1' = \frac{\beta_1-\beta_2}{1-\beta_1\beta_2} = -\beta_2' = \underline{0.920}$$

$$\gamma_1' = \frac{1}{\sqrt{1-\beta_1'^2}} = \underline{2.56} \qquad L2_1 = L2_2/\gamma_2' = \underline{9.39}$$

$$\gamma_2' = \frac{1}{\sqrt{1-\beta_2'^2}} = \gamma_1' \qquad L1_2 = L1_1/\gamma_1' = \underline{6.26}$$

Problem 3 : The figure below shows a circular capacitor of radius $R = 0.4\,\mathrm{m}$ onto which a current of $I = 1\,\mathrm{A}$ is running such that $Q = It$. Find the electric field $\vec{E}$ between the plates, in terms of axes shown below. Calculate the magnetic field $\vec{B}$ for $r < R$ and $r > R$, indicating the direction in terms of $\hat{r}$ and/or $\hat{\theta}$ in the $\hat{x}$ and $\hat{y}$ plane. Also find the Poynting vector $\vec{S}$ as a function of r and t. Assuming the plates are separated by $d = 0.1\,\mathrm{m}$, find the flux of $\vec{S}$ out of the volume between the plates. Extra Credit (2 points) : Show that this flux is equal to the time rate of change of the total electromagnetic field energy between the plates.



$r<R$

$$\vec{E} = \frac{Q\hat{z}}{\pi R^2 \varepsilon_0} = \frac{It\hat{z}}{\pi R^2 \varepsilon_0}$$

$r>R$ $\quad \vec{E} = 0$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

$r>R$ $\quad \vec{S} = 0$

$r<R$ $\quad \hat{z}\times\hat{\theta} = -\hat{r}$

$$\vec{S} = \frac{1}{\mu_0}\frac{It\hat{z}}{\pi R^2\varepsilon_0}\times\frac{\mu_0 I}{2\pi r}\hat{\theta}$$

$$\vec{S} = \frac{-I^2 t\,\hat{r}}{\pi R^2 \varepsilon_0\, 2\pi r}$$

inside

$$\frac{dU}{dt} = -\int dA\,\hat{n}\cdot\vec{S}$$

$\hat{n} = \hat{r}$ $\quad$ sides of cylinder at $r=R$

$$\frac{dU}{dt} = \left(\frac{I^2 t}{\pi R^2 \varepsilon_0}\frac{1}{2\pi R}\right) d\, 2\pi R = \frac{I^2 d t}{\pi R^2 \varepsilon_0}$$

$$\int d\vec{r}\cdot\vec{B} = \mu_0\varepsilon_0 \frac{d}{dt}\int dA\,\hat{n}\cdot\vec{E}$$

$r<R$

$$2\pi r B = \mu_0\varepsilon_0 \frac{d}{dt}\left(\pi r^2 \frac{It}{\pi R^2\varepsilon_0}\right)$$

$$= \mu_0 I\, r^2/R^2$$

$$\vec{B} = \frac{\mu_0 I r}{2\pi R^2}\hat{\theta}$$

$r>R$

$$2\pi r B = \mu_0\varepsilon_0 \frac{d}{dt}\left(\pi R^2 \frac{It}{\pi R^2\varepsilon_0}\right)$$

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\theta}$$

Extra credit:

$$U = \tfrac{1}{2}\varepsilon_0 \vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^2$$

$$\frac{dU}{dt} = \int dV \frac{du}{dt}$$

over volume of cylinder

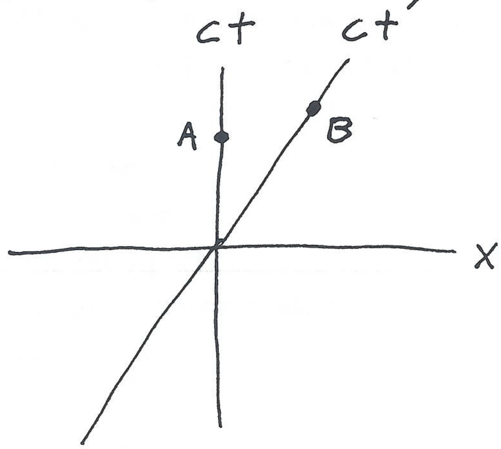
$$\frac{du}{dt} = \tfrac{1}{2}\varepsilon_0 \frac{d}{dt}\left(\frac{It}{\pi R^2 \varepsilon_0}\right)^2$$

$$= \frac{I^2 t}{(\pi R^2)^2 \varepsilon_0}$$

$$\frac{dU}{dt} = d(\pi R^2)\frac{du}{dt}$$

$$= \frac{I^2 d t}{\pi R^2 \varepsilon_0}$$

Problem 4 : The figure below shows two events as seen by an observer O with coordinate system (x, ct). The first event A is at $x_A = 0$ and $ct_A = 6 \times 10^5\, c \cdot s$. The second event B is seen by an observer O' with coordinate system (x', ct') to be at $x'_B = 0$ and $ct'_B = 5 \times 10^5\, c \cdot s$. If O' is moving with $\beta = 0.8$ with respect to O, find x'_A, ct'_A, x_B, and ct_B. Compare the time difference between the events as seen by O and O', in particular which event happens first to the respective observers. Find the velocity β_3 with which a third observer O_3 would have to be moving with respect to O in order that the events A and B are seen as simultaneous by O_3. Find the corresponding β'_3 of this third observer O_3 with respect to O'.



$$ct' = \gamma(ct - \beta x)$$
$$x' = \gamma(x - \beta ct)$$
$$ct = \gamma(ct' + \beta x')$$
$$x = \gamma(x' + \beta ct')$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

A/ $x_A = 0 \quad ct'_A = \gamma ct_A$

$x'_A = -\gamma\beta ct_A = -\beta ct'_A$

$ct'_A = 10^6\, c\cdot s$

$x'_A = -8 \times 10^5\, c\cdot s$

B/ $x'_B = 0 \quad ct_B = \gamma ct'_B$

$x_B = \gamma\beta ct'_B = \beta ct_B$

$ct_B = 6.67 \times 10^5\, c\cdot s$

$x_B = 5.33 \times 10^5\, c\cdot s$

O/ $ct_B - ct_A = 6.67 \times 10^4\, c\cdot s$

O'/ $ct'_B - ct'_A = -6.0 \times 10^5\, c\cdot s$

O₃/ $ct''_A = \gamma_3(ct_A - \beta_3 x_A)$

$ct''_B = \gamma_3(ct_B - \beta_3 x_B)$

$ct''_A = ct''_B$

$\gamma_3(ct_A - \beta_3 x_A) = \gamma_3(ct_B - \beta_3 x_B)$

$$\beta_3 = \frac{ct_B - ct_A}{x_B - x_A} = 0.125$$

$$\beta'_3 = \frac{\beta_3 - \beta}{1 - \beta_3\beta} = -0.75$$