SMU Physics 1307 : Fall 2008

Notes on Rotational Motion

Fixed Axis

Consider a rigid object rotating about a fixed axis. There is an angular frequency $\omega = d\theta/dt$ which we can think of as a vector $\vec{\omega} = \omega\hat{z}$ pointing along the axis of rotation. Positive ω is counterclockwise rotation and negative ω is clockwise rotation. Correspondingly, if the fingers of the right hand curl in the direction of rotation, the thumb points in the direction of $\vec{\omega}$. Any point in the object with displacement vector from the origin $\vec{r} = r\hat{r}$, with $r = |\vec{r}|$, has purely tangential velocity $\vec{v} = v\hat{\theta}$ given by

$$\vec{v} = \vec{\omega} \times \vec{r} = \omega r \, \hat{z} \times \hat{r} = \omega r \hat{\theta}$$

Here $\hat{r}$ and $\hat{\theta}$ are the radial and tangential unit vectors we introduced in class

$$\hat{r} = \cos\theta \, \hat{x} + \sin\theta \, \hat{y} \qquad\qquad \hat{\theta} = \cos\theta \, \hat{y} - \sin\theta \, \hat{x}$$

Since $\hat{x} \times \hat{y} = \hat{z}$, it may be seen that $\hat{r} \times \hat{\theta} = \hat{z}$. Thus we may write $v = r\omega$ but it must be understood that v is the tangential component of $\vec{v}$, not the (positive) magnitude $|\vec{v}|$ of $\vec{v}$; however, $|\vec{v}| = |v| = r|\omega|$. The acceleration of this point is given by

$$\vec{a} = \frac{d}{dt}\,\vec{v} = (\frac{d}{dt}\vec{\omega}) \times \vec{r} + \vec{\omega} \times (\frac{d}{dt}\vec{r})$$

Now,

$$(\frac{d}{dt}\vec{\omega}) \times \vec{r} = r\frac{d\omega}{dt}\,\hat{z} \times \hat{r} = r\alpha\hat{\theta}$$

where we have defined the angular acceleration $\alpha = d\omega/dt$. Also,

$$\vec{\omega} \times (\frac{d}{dt}\vec{r}) = \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{\omega} \times (\omega r\hat{\theta}) = -r\omega^2\hat{r}$$

Thus $\vec{a}$ breaks up into tangential and centripetal parts

$$\vec{a} = r\alpha\hat{\theta} - r\omega^2\hat{r}$$

where $r\omega^2 = v^2/r$.

The total angular momentum about the axis of rotation is given by the sum of all the angular momenta of the particles making up the rigid object :

$$\vec{L} = \sum_j m_j\vec{r}_j \times \vec{v}_j = \sum_j \omega\, m_j|\vec{r}_j|^2\,\hat{r} \times \hat{\theta} = I\omega\hat{z}$$

Here I is the moment of inertia about the axis of rotation

$$I = \sum_j m_j|\vec{r}_j|^2$$

Taking $\vec{L} = L\hat{z}$ leads to the important relation

$$\boxed{L = I\omega}$$

Taking the time derivative of $\vec{L}$ we find

$$\frac{d}{dt}\vec{L} = \sum_j m_j\,(\vec{v}_j \times \vec{v}_j + \vec{r}_j \times \vec{a}_j)$$

Since $\vec{v}_j \times \vec{v}_j = 0$ and $m_j\vec{a}_j = \vec{F}_j$,

$$\frac{d}{dt}\vec{L} = \sum_j \vec{r}_j \times \vec{F}_j$$

The net force on m_j may be written as $\vec{F}_j = \vec{F}_j^{\,\mathrm{ext}} + \vec{F}_j^{\,\mathrm{int}}$. Here $\vec{F}_j^{\,\mathrm{int}}$ is the net force on m_j coming from other particles within the object, and $\vec{F}_j^{\,\mathrm{ext}}$ is the net force on m_j coming from external sources. As discussed in class, if all internal forces between pairs of particles are directed along the line separating the particles, an assumption we will make, the internal forces make no net contribution to the sum. Thus,

$$\frac{d}{dt}\vec{L} = \vec{\tau}^{\,\mathrm{ext}} = \sum_j \vec{r}_j \times \vec{F}_j^{\,\mathrm{ext}}$$

Taking $\vec{\tau}^{\,\mathrm{ext}} = \tau^{\,\mathrm{ext}}\hat{z}$ and using $dL/dt = I\,d\omega/dt = I\alpha$ we find the second important relation

$$\boxed{\frac{dL}{dt} = \tau^{\,\mathrm{ext}} = I\alpha}$$

Lets examine $\vec{\tau}^{\,\mathrm{ext}}$ more carefully. The summation given above is in terms of forces acting on very small elements m_j of the object. It is in essence a microscopic description. The external forces we would like to consider are either gravity, which acts on all particles at once, or are direct contact forces, like a string tension or frictional force pushing or pulling on a particular point of the object. So let us write $\vec{F}_j^{\,\mathrm{ext}}$ as a sum over these forces

$$\vec{F}_j^{\,\mathrm{ext}} \;=\; m_j\,\vec{g} + \sum_q \vec{F}_{jq}$$

Here q indexes the non-gravitational contact forces and j indexes the (microscopic) masses m_j . Thus,

$$\vec{\tau}^{\,\mathrm{ext}} = \sum_j m_j\,\vec{r}_j \times \vec{g} \,+\, \sum_j \sum_q \vec{r}_j \times \vec{F}_{jq}$$

The center of mass $\vec{R}$ is defined by

$$M\vec{R} = \sum_j m_j\,\vec{r}_j$$

Where $M = \sum_j m_j$. Thus,

$$\sum_j m_j\,\vec{r}_j \times \vec{g} = \vec{R} \times (\,M\vec{g}\,)$$

Now, it can be assumed that $\vec{F}_{jq}$ is non-zero only for masses m_j which are in the immediate vicinity of $\vec{r}_q$, the point application of the qth force. Thus we may take $\vec{r}_j = \vec{r}_q$, for the masses subject to the qth force. This leads to

$$\sum_j \sum_q \vec{r}_j \times \vec{F}_{jq} \;=\; \sum_q \vec{r}_q \times \sum_j \vec{F}_{jq} \;=\; \sum_q \vec{r}_q \times \vec{F}_q$$

Here $\vec{F}_q = \sum_j \vec{F}_{jq}$ is the total qth force, the sum of the forces exerted by the qth force on each m_j . Thus, finally,

$$\boxed{\vec{\tau}^{\,\rm ext} = \vec{R} \times (\,M\vec{g}\,) \,+\, \sum_q \vec{r}_q \times \vec{F}_q}$$

It is very important to realize that $\sum_q \vec{r}_q \times \vec{F}_q$ looks very similar to $\sum_j \vec{r}_j \times \vec{F}_j^{\rm ext}$ but the meaning is very different. The sum over j extends over all masses m_j , with respective positions $\vec{r}_j$, with $\vec{F}_j^{\rm ext}$ being the total external force on m_j . The sum over q extends over all external contact forces, acting at points $\vec{r}_q$ on the object, with $\vec{F}_q$ being the qth force. In solving physical problems it is the expression in the box above which will be used.

The kinetic energy of a rigid object rotating around a fixed center of rotation is given by

$$K = \sum_j \tfrac{1}{2} m_j \, \vec{v}_j \cdot \vec{v}_j = \sum_j \tfrac{1}{2} m_j \, \omega^2 \, |\vec{r}_j|^2 \, \hat{\theta} \cdot \hat{\theta}$$

Using the definition of I and recognizing that $\hat{\theta} \cdot \hat{\theta} = 1$ we find

$$\boxed{K = \tfrac{1}{2} I \omega^2}$$

Given $dK/dt = I\alpha\,\omega$, the change in kinetic energy between t_1 and t_2 is given by

$$\Delta K \;=\; \int_{t_1}^{t_2} dt \, I\alpha\,\omega \;=\; \int_{t_1}^{t_2} dt \, \omega \, \tau^{\,\rm ext}$$

In general $\Delta K \,=\, W_{net} \,=\, W_c + W_n$, where W_c is the work done by conservative torques and W_n is the work done by non-conservative torques. We also write $\tau^{\,\rm ext} \,=\, \tau_c^{\rm ext} + \tau_n^{\rm ext}$. The conservative torques $\tau_c^{\rm ext}(\theta)$ are purely a function of the angle. Thus we may define the potential energy as $W_c = -\Delta U$. Since $d\theta = \omega dt$,

$$W_c \;=\; \int_{t_1}^{t_2} dt \, \omega \, \tau_c^{\rm ext}(\theta) \;=\; \int_{\theta_1}^{\theta_2} d\theta \, \tau_c^{\rm ext}(\theta) \;=\; -(U(\theta_2) - U(\theta_1))$$

Given some reference angle θ_0 the potential energy is defined as

$$U(\theta) = U(\theta_0) - \int_{\theta_0}^{\theta} d\theta \, \tau_c^{\rm ext}(\theta)$$

Note that $\tau_c^{\rm ext}(\theta) \;=\; -dU/d\theta$ and that $U(\theta_0)$ is an arbitrary constant, often set to zero. Thus we can define a mechanical energy

$$E \;=\; \tfrac{1}{2} I\omega^2 + U(\theta)$$

which leads to the very important relation

$$\boxed{\Delta E = W_n}$$

In the case that we are considering a segment of the motion with $|\omega| > 0$ (motion is monotonic) for which we know $\tau_n^{\rm ext}$ at each θ then we may also write

$$W_n \;=\; \int_{\theta_1}^{\theta_2} d\theta \, \tau_n^{\rm ext}(\theta)$$

In practice $\tau_n^{\text{ext}}(\theta)$ will often be a constant (independent of time, and thus θ). In this case

$$W_n \;=\; \tau_n^{\text{ext}}\,(\theta_2 - \theta_1)$$

In most problems assigned in this class gravity will provide the only conservative torque. In this case the position of the center of mass is

$$\vec{R} \;=\; R\,\hat{r} = R(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})$$

and thus, from the expression above for $\vec{\tau}^{\text{ext}}$, and given $\vec{g} = -g\hat{y}$,

$$\vec{\tau}_g^{\text{ext}} \;=\; \tau_g^{\text{ext}}\,\hat{z} \;=\; \vec{R}\times(\,M\vec{g}\,) \;=\; -MgR\,(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})\times\hat{y} \;=\; -MgR\,\cos\theta\,\hat{z}$$

So, choosing $\theta_0 = 0$ and $U_g(\theta_0) = 0$,

$$U_g(\theta) \;=\; \int_0^\theta d\theta\,MgR\,\cos\theta \;=\; MgR\,\sin\theta$$

But $R\sin\theta \;=\; Y$, the y coordinate of the center of mass. Thus, in the case that gravity provides the only conservative torque we have

$$\boxed{E \;=\; \tfrac{1}{2}I\omega^2 + Mg\,Y}$$

Rolling

For rolling motion we will only consider a circular object, with radius denoted by r, whose center of mass lies at the center of the circle. This will lead to the important property that the velocity of the center of mass $\vec{V} = V\hat{t}$, where $\hat{t}$ is the unit vector tangent to the surface, is related to the angular velocity ω by

$$\boxed{V \;=\; -\omega\,r}$$

The negative sign is because rightward motion (positive V) produces clockwise rotation (negative ω). For a rolling object the angular momentum will be highly dependent on where we choose the origin of coordinates. Choosing the center of mass as the origin introduces complications, since this is an accelerated reference frame in general. Thus the angular momentum and its derivative with respect to time is not a very helpful variable to use in solving problems. One property that holds in all cases, including the fixed axis situation considered above, is that the motion of the center of mass depends only on the external forces

$$\boxed{M\vec{A} = \sum_j F_j^{\text{ext}} = \sum_q \vec{F}_q}$$

where we have defined $\vec{A} = d\vec{V}/dt$. Another equation that may be used is

$$\boxed{\tau_{cm}^{\text{ext}} \;=\; I_c\,\alpha}$$

Note that this is not expressed in terms of the rate of change of angular momentum with respect to a fixed origin. It also differs from the analogous expression in the fixed axis case because I_c and τ_{cm}^{ext} are computed with respect to the center of mass, rather than a particular fixed rotation axis. Thus, given any origin with vector $\vec{R}$ to the center of mass,

$$I_{cm} = \sum_j m_j|\vec{r}_j - \vec{R}|^2 = \sum_j m_j\left(|\vec{r}_j|^2 - 2\vec{r}_j\cdot\vec{R} + |\vec{R}|^2\right)$$

Using the definitions of I and $\vec{R}$ given above, this leads to the parallel axis theorem

$$\boxed{I = I_{cm} + M|\vec{R}|^2}$$

The torque about the center of mass is given by

$$\vec{\tau}_{cm}^{\,\text{ext}} = \tau_{cm}^{\text{ext}}\,\hat{z} \;=\; \sum_q (\vec{r}_q - \vec{R}) \times \vec{F}_q$$

It is often very useful in rolling motion problems to make use of energy concepts. In general, with $\vec{V}$ the velocity of the center of mass, for a rigid body moving in two dimensions

$$K = \tfrac{1}{2}\,M\,|\vec{V}|^2 + \tfrac{1}{2}\,I_{cm}\,\omega^2$$

For a rolling object, $V = -\omega r$, so K takes the form

$$\boxed{K = \tfrac{1}{2}\left(Mr^2 + I_{cm}\right)\omega^2 = \tfrac{1}{2}\left(M + I_{cm}/r^2\right)V^2}$$

Using the parallel axis theorem, this has the same form as the kinetic energy of an object rotating about a fixed axis a distance r from the center of mass. In fact this is the case; the point on the objects perimeter which is in contact with surface is momentarily at rest and the entire object is instantaneously rotating about this point.
To compute the work done by various forces, it is most simple to express an infinitesimal change in K as

$$dK = M\,\vec{V}\cdot d\vec{V} + I_{cm}\,\omega\,d\omega = M\,\vec{A}\cdot d\vec{R} + I_{cm}\,\alpha\,d\theta$$

Where we have used $\omega\,d\omega = \omega\,\alpha dt = \alpha\,d\theta$ and $\vec{V}\cdot d\vec{V} = \vec{V}\cdot\vec{A}\,dt = d\vec{R}\cdot\vec{A}$. Using the above force laws we may write this as

$$dK = \vec{F}^{\,\text{ext}}\cdot d\vec{R} + \tau_{cm}^{\text{ext}}\,d\theta$$

Note that the normal force cannot change the kinetic energy. First it is perpendicular to $d\vec{R}$, and thus cannot change the first term in dK . Secondly, the normal force points directly at the center of mass (we chose our object to have this property) thus does not contribute to τ_{cm}^{ext} . Consider the contribution of the frictional force dK_f to dK . It points tangent to the surface so we may write its contribution to $\vec{F}^{\,\text{ext}}$ as $\vec{f} = f\hat{t}$, where $\hat{t}$ is the unit tangent to the surface and the sign of f has been left undetermined. The torque it exerts may be written as $-r\hat{n}\times\vec{f} = rf\hat{z}$, where $\hat{n}$ is the unit normal to the surface. Thus we find that the frictional force does no net work when an object rolls

$$dK_f = f\hat{t}\cdot d\vec{R} + rfd\theta = 0$$

where we have used $\hat{t}\cdot d\vec{R} = V\,dt = -r\omega\,dt = -r\,d\theta$. Finally, gravity exerts no torque about the center of mass since

$$\sum_j (\vec{r}_j - \vec{R}) \times (m_j\vec{g}) = M\vec{g}\times\vec{R} - \vec{g}\times\sum_j m_j\,\vec{r}_j \;=\; 0$$

Thus

$$dK_g = M\vec{g}\cdot d\vec{R} = -Mg\,dY$$

Where, as above, Y is the y position of the center of mass. Thus we can ignore friction and the normal force and define an energy with $\Delta E = W_n$ as

$$\boxed{E = \tfrac{1}{2}\left(M + I_{cm}/r^2\right)V^2 \;+\; Mg\,Y}$$

It can also be shown that tensions between objects in a system may change the kinetic energy of individual objects but cannot change the entire kinetic energy of the system.