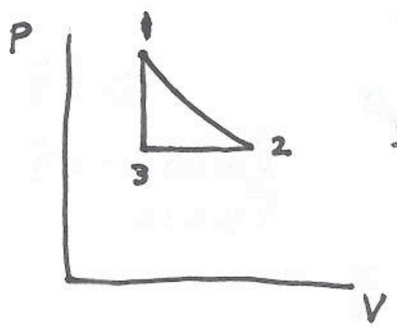
SMU Physics 1308 : Spring 2009

Exam 1

Problem 1 : One mole of a monatomic ($C_V = 3/2$ R) ideal gas at temperature $T_1 = 300\,K$ occupies $V_1 = 1\,\mathrm{L} = 10^{-3}\,\mathrm{m}^3$. It then expands adiabatically to volume $V_2 = 2V_1$. It is then compressed at constant pressure until it returns to the volume $V_3 = V_1$. Finally, the gas is heated at constant volume until it returns to the original state. Draw a diagram of the process, find $(p_1, p_2 = p_3)$, (T_2, T_3), (W_{12}, W_{23}, W_{31}), and (Q_{12}, Q_{23}, Q_{31}). Find the efficiency ε of the system by dividing the total work W done by the heat Q_+ exchanged during those parts of the cycle where heat is entering the system. How does this compare to the Carnot efficiency $\varepsilon_C = 1 - T_3/T_1$?



n, V_1, T_1 Known $\quad \underline{P_1 = nRT_1/V_1}$

$C_p = C_v + R$

$\underline{C_v = \tfrac{3}{2}R}$

$\underline{1\to2}$ $\quad P_1V_1^{\gamma} = P_2V_2^{\gamma} \qquad \gamma = C_p/C_v = 5/3$

$V_2/V_1 = 2 \qquad P_2 = P_1\left(V_1/V_2\right)^{\gamma} = \underline{P_1\, 2^{-5/3}}$

$T_2 = \dfrac{P_2V_2}{nR} = \dfrac{P_1V_1\,2^{1-5/3}}{nR} = \underline{T_1\, 2^{-2/3}}$

$\underline{Q_{12} = 0} \qquad \Delta E_{12} = -W_{12} = nC_v(T_2 - T_1)$

$\underline{W_{12} = nC_vT_1(1 - 2^{-2/3})}$

$V_3 = V_1 \quad P_3 = P_2$

$\underline{2\to3}$

$\dfrac{nRT_3}{V_3} = \dfrac{nRT_2}{V_2}$

$T_3 = T_2\dfrac{V_3}{V_2} = T_2/2 = \underline{T_1 2^{-5/3}}$

$W_{23} = P_2(V_3 - V_2)$

$= -P_2V_1 = \underline{-P_1V_1\,2^{-5/3}}$

$Q_{23} = \Delta E_{23} + W_{23}$

$= nC_v(T_3 - T_2) - P_1V_1\,2^{-5/3}$

$= -nC_vT_2/2 - P_1V_1\,2^{-5/3}$

$= -nT_1\left(C_v 2^{-5/3} + R\,2^{-5/3}\right) = \underline{-nC_pT_1\,2^{-5/3}}$

$\underline{3\to1}$ $\quad \underline{W_{31} = 0} \qquad \Delta E_{31} = nC_v(T_1 - T_3)$

$Q_{31} = \Delta E_{31} = \underline{nC_vT_1(1 - 2^{-5/3})}$

$\varepsilon = 1 + \left(\dfrac{Q_{31}}{Q_{23}}\right)^{-1} = 1 - \left(\dfrac{C_v}{C_p}\,\dfrac{(1-2^{-5/3})}{2^{-5/3}}\right)^{-1}$

$= 1 - \left(\tfrac{3}{5}(2^{5/3} - 1)\right)^{-1} = \underline{0.234}$

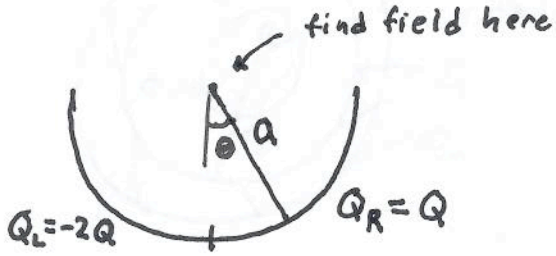
$\varepsilon_c = 1 - T_3/T_1 = 1 - 2^{-5/3} = \underline{0.685}$

$\varepsilon < \varepsilon_c$

Problem 2 : The figure below shows a semicircle of radius $a = 0.05\,\mathrm{m}$. The right half of the semicircle has a uniform linear charge density with total charge $Q_R = Q = 0.2\,\mathrm{C}$. The left half of the semicircle has a uniform linear charge density with total charge $Q_L = -2Q = -0.4\,\mathrm{C}$. Find the field at the center of the semicircle. You will need the following integrals :

$$\int_{\theta_1}^{\theta_2} d\theta \, \sin\theta \;=\; -(\cos\theta_2 - \cos\theta_1) \qquad\qquad \int_{\theta_1}^{\theta_2} d\theta \, \cos\theta \;=\; (\sin\theta_2 - \sin\theta_1)$$

You may do this problem algebraically in terms of k, Q, and a. If you want to find numerical answers, you will need $k = 9.0\times10^{9}\,\mathrm{N\,m^2/C^2}$, where the units of the electric field are N/C.



$$dq = \lambda(\theta)\, a\, d\theta$$

$$\boxed{\lambda(\theta):\quad \begin{array}{ll} \dfrac{Q}{\pi a} & \theta > 0 \\[2ex] \dfrac{-2Q}{\pi a} & \theta < 0 \end{array}}$$

$\vec{r} = 0$

$\vec{r}\,' = -a\cos\theta\,\hat{y} + a\sin\theta\,\hat{x}$

$$\vec{E} = K\int dq\,\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} = K\int_{-\pi/2}^{\pi/2} d\theta\, a\,\frac{\lambda(\theta)}{a^3}(a\cos\theta\,\hat{y} - a\sin\theta\,\hat{x})$$

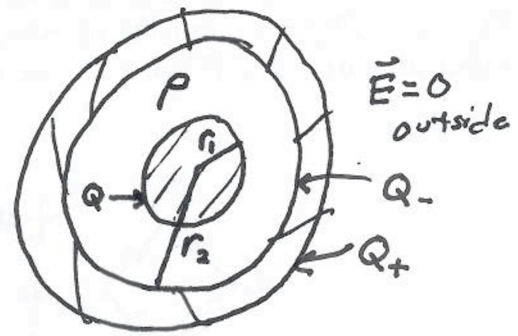
$$\vec{E} = \frac{KQ}{\pi a^2}\int_0^{\pi/2} d\theta\,(\cos\theta\,\hat{y} - \sin\theta\,\hat{x}) - 2\frac{KQ}{\pi a^2}\int_{-\pi/2}^{0} d\theta\,(\cos\theta\,\hat{y} - \sin\theta\,\hat{x})$$

$$= \frac{KQ}{\pi a^2}\,\hat{y}\,(\sin\tfrac{\pi}{2} - \sin 0) + \frac{KQ}{\pi a^2}\,\hat{x}\,(\cos\tfrac{\pi}{2} - \cos 0)$$

$$- \frac{2KQ}{\pi a^2}\,\hat{y}\,(\sin 0 - \sin(-\tfrac{\pi}{2})) - \frac{2KQ}{\pi a^2}\,\hat{x}\,(\cos 0 - \cos(-\tfrac{\pi}{2}))$$

$$= \frac{KQ}{\pi a^2}\,\hat{y} - \frac{KQ}{\pi a^2}\,\hat{x} - \frac{2KQ}{\pi a^2}\,\hat{y} - \frac{2KQ}{\pi a^2}\,\hat{x} = \underline{\underline{-\frac{KQ}{\pi a^2}(3\hat{x} + \hat{y})}}$$

Problem 3 : The figure below shows a conducting sphere of radius $r_1 = 0.05\,\text{m}$ with a charge $Q = 1\,\text{C}$ placed on it. Between this sphere and a conducting shell of inner radius $r_2 = 0.1\,\text{m}$ lies a constant volume charge density ρ. If the conducting shell has no net charge and the electric field outside the conducting shell is zero, what is ρ? Also find the electric field $\vec{E}(r)$ between the conducting surfaces. This problem may be done algebraically.



since $\vec{E} = 0$ outside : $Q_+ = 0$

since $Q_+ + Q_- = 0$

$Q_- = 0$

(1) since $\vec{E} = 0$ inside conductor, no total charge, so $\rho V = -Q$

$$V = \tfrac{4}{3}\pi (r_2^2 - r_1^2)$$

$$\rho = \frac{-Q}{\tfrac{4}{3}\pi} (r_2^2 - r_1^2)^{-1}$$

V: Volume of constant volume charge density ρ

(2) Consider sphere of radius r within volume charge. Gauss' Law says:

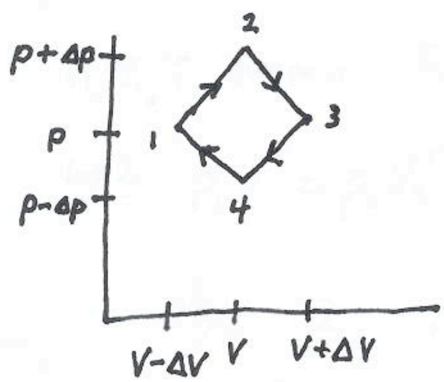
$$E\, 4\pi r^2 = (Q + \rho\, V(r))\, 4\pi K$$

$$V(r) = \tfrac{4}{3}\pi (r^2 - r_1^2)$$

$$E = \frac{KQ}{r^2} + K\rho\, \tfrac{4}{3}\pi \left(1 - r_1^2/r^2\right)$$

$$E = \frac{KQ}{r^2}\left(1 - \frac{(r^2 - r_1^2)}{(r_2^2 - r_1^2)}\right) = \frac{KQ}{r^2}\frac{(r_2^2 - r^2)}{(r_2^2 - r_1^2)}$$

Problem 4 : The figure below shows a closed process which traces out a square in the p-V plane, with all line segments making 45 degree angles with the horizontal. It may be shown that if $\frac{dp}{dV} > -\gamma p/V$, which we will assume, for all points of the process, then an adiabatic curve is steeper than all of the line segments, so that $Q_{12} > 0$, $Q_{23} > 0$, $Q_{34} < 0$, and $Q_{41} < 0$. By simple geometry, recalling that $W = \int p\,dV$, compute the work $W_+ = W_{12} + W_{23}$ and $W_- = W_{34} + W_{41}$ in terms of $p = 20\,\mathrm{N/m^3}$, $V = 10\,\mathrm{m^3}$, $\Delta p = 2\,\mathrm{N/m^3}$, and $\Delta V = 1\,\mathrm{m^3}$. Using $C_V = \frac{3}{2}R$, compute the change in energy $\Delta E_+ = \Delta E_{12} + \Delta E_{23}$ during the time that heat is added to the system. Use these results to compute $Q_+ = Q_{12} + Q_{23}$ and $Q_- = Q_{34} + Q_{41}$. Now compute the efficiency $\varepsilon = (W_+ + W_-)/Q_+$ of the system. Compare this to the Carnot efficiency $\varepsilon_C = 1 - T_4/T_2 = 1 - (p - \Delta p)/(p + \Delta p)$. This problem may be done algebraically.



$W_+ = 2p\,\Delta V + \Delta p\,\Delta V$ (Area)

$W_- = -2p\,\Delta V + \Delta p\,\Delta V$ (-Area)

$\Delta E_+ = n C_V (T_3 - T_1)$

$= \frac{C_V}{R}(P_3 V_3 - P_1 V_1)$

$= \frac{C_V}{R}\, 2p\Delta V = \underline{3p\Delta V}$

$= \underline{-\Delta E_-}$

$Q_+ = \Delta E_+ + W_+ = 5p\,\Delta V + \Delta p\,\Delta V$

$Q_- = \Delta E_- + W_- = -5p\,\Delta V + \Delta p\,\Delta V$

$\varepsilon = 1 + Q_-/Q_+ = 1 - \frac{(5p - \Delta p)}{(5p + \Delta p)} = 2/51$

$\varepsilon_c = 1 - T_4/T_2 = 1 - \frac{(p - \Delta p)}{(p + \Delta p)} = 2/11$