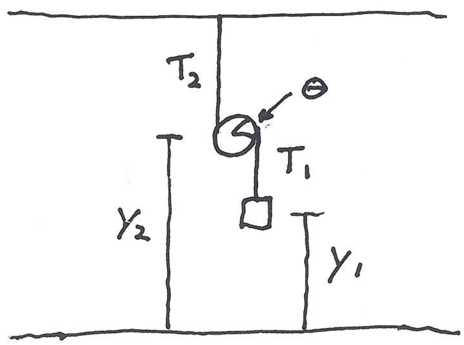
SMU Physics 3344 : Fall 2010

Exam 1

Problem 1 : The figure below shows a mass m_1 hanging from a string with tension T_1 which wraps as shown around a cylinder of mass m_2, moment of inertia about the center of mass I, and radius r. A separate string with tension T_2 also wraps as shown around the cylinder before attaching vertically to a fixed support. The motion of the objects is described as shown by $y_1(t)$, $y_2(t)$, and $\theta(t)$. Find the three equations of motion for these variable in terms of the forces and torques applied. Given that at $t = 0$ we take $\theta(0) = 0$, determine both $y_1(t)$ and $y_2(t)$ in terms of $y_1(0)$, $y_2(0)$, and $\theta(t)$; thus reducing the problem to one dimension. Now use these constraints to rewrite the equations of motion in terms of $\theta(t)$. Solve for T_1 and T_2 and produce an equation of motion for $\theta(t)$ alone. Write the energy E of the system in terms of $\theta(t)$ and determine that $\dot{E} = 0$ produces the equation of motion for $\theta(t)$.



$$m_1 \ddot{y}_1 = T_1 - m_1 g$$
$$m_2 \ddot{y}_2 = T_2 - T_1 - m_2 g$$
$$I \ddot{\theta} = -r(T_1 + T_2)$$
$$2 m_1 r \ddot{\theta} = T_1 - m_1 g$$
$$m_2 r \ddot{\theta} = T_2 - T_1 - m_2 g$$
$$E = \tfrac{1}{2} m_1 \dot{y}_1^2 + \tfrac{1}{2} m_2 \dot{y}_2^2 + \tfrac{1}{2} I \dot{\theta}^2 + m_1 g y_1 + m_2 g y_2$$
$$\dot{E} = \dot{y}_1 (T_1 - m_1 g) + \dot{y}_2 (T_2 - T_1 - m_2 g) + \dot{\theta}(-r(T_1 + T_2)) + m_1 g \dot{y}_1 + m_2 g \dot{y}_2$$
$$= T_1(\dot{y}_1 - \dot{y}_2 - r\dot{\theta}) + T_2(\dot{y}_2 - r\dot{\theta}) = 0$$

so, $\dot{y}_2 = r\dot{\theta}$ $\quad \dot{y}_1 = 2r\dot{\theta}$ (as above)

$$y_2 - y_{20} = r\theta = y_1 - y_{10} - r\theta$$
$$y_1 - y_{10} = 2r\theta$$

plug ③ into ①&②

$$T_1 = m_1 g + 2 m_1 r \ddot{\theta} \quad ①$$
$$T_2 = (m_1 + m_2) g + (2m_1 + m_2) r \ddot{\theta} \quad ②$$
$$-I\ddot{\theta} = (2m_1 + m_2) g r + (4m_1 + m_2) r^2 \ddot{\theta}$$
$$(I + (4m_1 + m_2) r^2) \ddot{\theta} = -(2m_1 + m_2) g r \quad ③$$
$$E = \tfrac{1}{2}(I + 4m_1 r^2 + m_2 r^2) \dot{\theta}^2 + m_1 g y_1 + m_2 g y_2$$
$$\dot{E} = \dot{\theta}\left((I + (4m_1 + m_2) r^2)\ddot{\theta} + (2m_1 + m_2) g r\right) = 0$$

which is ③

Problem 2 : The figure below shows a massless rod of length a with one end free to pivot around a fixed point and the other end free to pivot about a point a distance b from the center of mass of an object of mass m and moment of inertia about the center of mass I. The motion of the system is described by the angles $\theta(t)$ and $\varphi(t)$ as shown in the figure. We will not produce equations of motion for the massless rod; it will simply be treated like a string which only produces a tension T directed away from the object along the rod, with negative T corresponding to compression of the rod. Write the position of the center of mass of the object as a function of θ and φ, and write down the equations of motion for the object in terms of the forces and torques applied. Express T in terms of θ, $\dot\theta$, φ, and $\dot\varphi$. Eliminate T from the equations of motion for θ and φ.

$$\vec R = \vec a + \vec b = (a\hat a + b\hat b)$$

$$\hat a = (s\varphi\,\hat x - c\varphi\,\hat y) \qquad \dot{\hat a} = \dot\varphi\,\hat\varphi$$

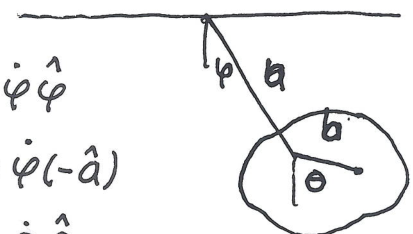
$$\hat\varphi = (c\varphi\,\hat x + s\varphi\,\hat y) \qquad \dot{\hat\varphi} = \dot\varphi(-\hat a)$$

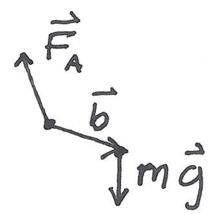
$$\hat b = (s\theta\,\hat x - c\theta\,\hat y) \qquad \dot{\hat b} = \dot\theta\,\hat\theta$$

$$\hat\theta = (c\theta\,\hat x + s\theta\,\hat y) \qquad \dot{\hat\theta} = \dot\theta(-\hat b)$$

$$\ddot{\hat a} = \ddot\varphi\,\hat\varphi - \dot\varphi^2\,\hat a$$

$$\ddot{\hat b} = \ddot\theta\,\hat\theta - \dot\theta^2\,\hat b$$





$$\vec F_A = -T\hat a$$

$$m\ddot{\vec R} = m(a\ddot{\hat a} + b\ddot{\hat b}) = -T\hat a - mg\hat y$$

resolve into $\hat a$ and $\hat\varphi$ directions.

$$\hat a\cdot m\ddot{\vec R} = m(a(-\dot\varphi^2) + b(\ddot\theta\,\hat a\cdot\hat\theta - \dot\theta^2\,\hat a\cdot\hat b)) = -T - mg\,\hat a\cdot\hat y$$

$$\hat\varphi\cdot m\ddot{\vec R} = m(a\ddot\varphi + b(\ddot\theta\,\hat\varphi\cdot\hat\theta - \dot\theta^2\,\hat\varphi\cdot\hat b)) = -mg\,\hat\varphi\cdot\hat y$$

$$\hat a\cdot\hat\theta = (s\varphi c\theta - c\varphi s\theta) = A$$

$$\hat a\cdot\hat b = (s\varphi s\theta + c\varphi c\theta) = B$$

$$\hat\varphi\cdot\hat\theta = B \qquad \hat\varphi\cdot\hat b = -A$$

$$\hat a\cdot\hat y = -c\varphi \qquad \hat\varphi\cdot\hat y = s\varphi$$

$$-ma\dot\varphi^2 + mAb\ddot\theta - mBb\dot\theta^2 = -T + mgc\varphi$$

$$ma\ddot\varphi + mBb\ddot\theta + mAb\dot\theta^2 = -mgs\varphi$$

divide by m:

$$a\ddot\varphi + Bb\ddot\theta + Ab\dot\theta^2 = -gs\varphi \quad]\,①$$

$$Ab\ddot\theta - a\dot\varphi^2 - Bb\dot\theta^2 = -T/m + gc\varphi \quad]\,②$$

$$I\ddot\theta\,\hat z = (-b\hat b)\times(-T\hat a) = bT\,\hat b\times\hat a = +bTA\,\hat z$$

$$I\ddot\theta = bTA \quad]\,③$$

plug into ② to eliminate T

$$\frac{b^2TA^2}{I} - a\dot\varphi^2 - Bb\dot\theta^2 = -T/m + gc\varphi$$

(solve for T)