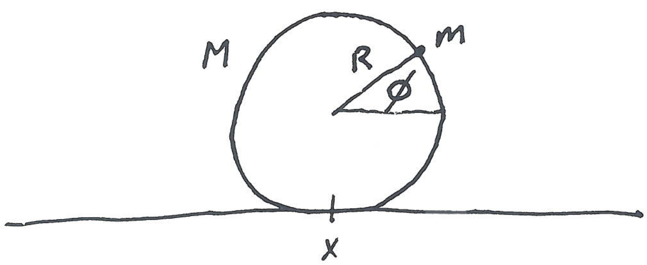
SMU Physics 3344 : Fall 2010

Exam 2

Problem 1 : The figure below shows a hoop of mass M, radius R, and moment of inertia $I = MR^2$ rolling without slipping on a horizontal surface. A mass m slides freely around the hoop. Take x to be the horizontal location of the center of the hoop and ϕ to be the angular position of m as shown in the figure. Ignoring all constraint forces (that is, by imposing the equations of constraint) in the problem, write the Lagrangian L in terms of the x and ϕ variables. Compute the Euler-Lagrange equation for x and ϕ. Find the conserved momentum in this system and, assuming that $\dot{x} = 0$ and $\dot{\phi} = 0$ at $t = 0$, use this to relate $\dot{x}$ and $\dot{\phi}$ at all times. Compute the Hamiltonian H for this system and express it entirely in terms of ϕ. If at $t = 0$ we choose $x = 0$ and $\phi = \pi/2$, use energy conservation to find $\dot{x}$ when $\phi = 0$.



$$x + R\theta = 0$$
$$\alpha = x + Rc\phi$$
$$\beta = Rs\phi$$

$$\tilde{M} = M + m + I/R^2$$

$$L = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}I\dot{\theta}^2 + \tfrac{1}{2}m(\dot{\alpha}^2 + \dot{\beta}^2) - mg\beta$$

$$L = \tfrac{1}{2}(M + I/R^2)\dot{x}^2 + \tfrac{1}{2}m\left((\dot{x} - Rs\phi\dot{\phi})^2 + R^2c^2\phi\dot{\phi}^2\right) - mgRs\phi$$

$$L = \tfrac{1}{2}\tilde{M}\dot{x}^2 + \tfrac{1}{2}mR^2\dot{\phi}^2 - mRs\phi\,\dot{x}\dot{\phi} - mgRs\phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left(\tilde{M}\dot{x} - mRs\phi\dot{\phi}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{d}{dt}\left(mR^2\dot{\phi} - mRs\phi\dot{x}\right) = -mRc\phi\dot{x}\dot{\phi} - mgRc\phi$$

$$p_x = \tilde{M}\dot{x} - mRs\phi\dot{\phi} = 0 \text{ (initial conditions)} \qquad \dot{x} = \frac{m}{\tilde{M}}Rs\phi\dot{\phi}$$

$$mR^2\ddot{\phi} - \cancel{mRc\phi\dot{\phi}\dot{x}} - mRs\phi\ddot{x} = -\cancel{mRc\phi\dot{x}\dot{\phi}} - mgRc\phi$$

$$R\ddot{\phi} - s\phi\ddot{x} = -gc\phi$$

$$H = \tfrac{1}{2}\tilde{M}\dot{x}^2 + \tfrac{1}{2}mR^2\dot{\phi}^2 - mRS\phi\,\dot{x}\dot{\phi} + mgRS\phi$$

$$\tilde{M}\dot{x} = mRS\phi\,\dot{\phi} \qquad (P_x = 0)$$

$$H = \frac{(mRS\phi)^2}{2\tilde{M}}\dot{\phi}^2 + \tfrac{1}{2}mR^2\dot{\phi}^2 - \frac{(mRS\phi)^2}{\tilde{M}}\dot{\phi}^2 + mgRS\phi$$

$$H = \tfrac{1}{2}mR^2\dot{\phi}^2\left(1 - S^2\phi\frac{m}{\tilde{M}}\right) + mgRS\phi \qquad \dot{H} = 0$$

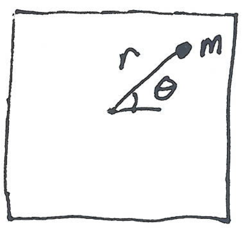
$t=0$ $(x=0,\ \phi=\pi/2)$
$(\dot{x}=0,\ \dot{\phi}=0)$

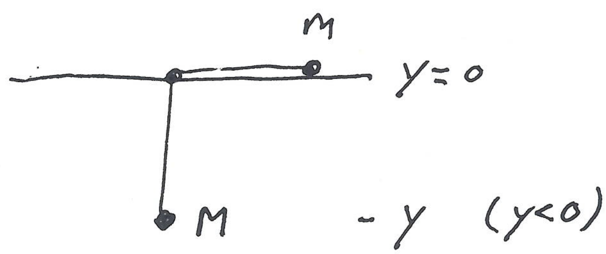
$$E = mgR$$

$$\phi = 0 \Rightarrow E = \tfrac{1}{2}mR^2\dot{\phi}^2 \qquad (S\phi = 0)$$

$$\dot{\phi}^2 = 2g/R \qquad \underline{\dot{x} = 0}$$

Problem 2 : The figure at left below shows the top view of a mass m on a horizontal table with coordinates r and θ. The mass is attached to a string which goes through the table and attaches to another mass M which is constrained to move vertically with coordinate y as shown from the side view in the figure at right. Write down the Lagrangian L for the system with the constraint that the string has fixed length ℓ being implemented through a lagrange multiplier given by λ. Compute the Euler-Lagrange equation for the y direction, thus deriving an expression for λ. Now impose the constraint and write down the free Lagrangian L_F in terms of r and θ. Compute the corresponding Euler-Lagrange equations for r and θ, and write down the Hamiltonian H_F associated with L_F. Find the (conserved) angular momentum p_θ for L_F, and write H_F in terms of p_θ, r, and $\dot{r}$. Now, using the various Euler-Lagrange equations, express λ in terms of p_θ and r.





$$L = \tfrac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \tfrac{1}{2} M\dot{y}^2 - Mgy + \lambda(r - y - \ell)$$

$$M\ddot{y} = -Mg - \lambda \qquad (\lambda = -\text{Tension})$$

$$L_F = \tfrac{1}{2}(m+M)\dot{r}^2 + \tfrac{1}{2} m r^2\dot{\theta}^2 - Mg(r-\ell)$$

$$① \quad (m+M)\ddot{r} = mr\dot{\theta}^2 - Mg$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = \frac{d}{dt}p_\theta = 0$$

$$H_F = \tfrac{1}{2}(m+M)\dot{r}^2 + \tfrac{1}{2} m r^2\dot{\theta}^2 + Mg(r-\ell)$$

$$H_F = \tfrac{1}{2}(m+M)\dot{r}^2 + p_\theta^2/(2mr^2) + Mg(r-\ell)$$

$$y = r - \ell$$

$$M\ddot{y} = M\ddot{r} = -Mg - \lambda$$

$$\lambda = -Mg - \frac{M}{(M+m)}\left(\frac{p_\theta^2}{mr^3} - Mg\right)$$

using ①

$$M(mr\dot{\theta}^2 - Mg) = -(M+m)(Mg + \lambda)$$