SMU Physics 3344 : Fall 2010

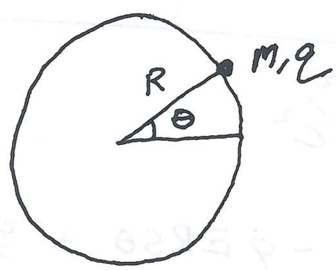
Final

Problem 1 : The figure below shows a circular track of radius R around which a particle of charge q and mass m is constrained to move. There is a constant electric field $\vec{E} = -E\hat{y}$ pointing downward in the figure. There is also a constant magnetic field $\vec{B} = B\hat{z}$ with $\hat{z}$ coming out of the page. The corresponding potential and vector potential are $\phi = Ey$ and $\vec{A} = B(x\hat{y} - y\hat{x})/2$. Write down the Lagrangian L in terms of the polar coordinates r and θ and their time derivatives, converting from x and y if necessary, and implementing the constraint via a Lagrange multiplier N (normal force). Compute the Euler-Lagrange (EL) equations for the system and then implement the constraint, producing a set of two equations which describe the constrained system. Now compute the free Lagrangian L_F by implementing the constraint. From this compute H_F, and verify that its conservation is equivalent to one of the EL equations found above(that is take its time derivative and compare). The free Hamiltonian H_F, which in this case is equivalent to the Hamiltonian H after the constraint is applied, permits an expression for $\dot{\theta}$ in terms of θ and initial conditions. Find the normal force N purely in terms of $\dot{\theta}$ and θ by eliminating $\ddot{\theta}$ from the EL equations.

$x = r c\theta$

$y = r s\theta$

$\vec{E} \downarrow$



$\odot \vec{B}$

$\hat{y}$, $\hat{x}$, $\odot\hat{z}$

$$L = -mc\sqrt{c^2 - \dot{r}^2 - r^2\dot{\theta}^2} - qEy + \frac{qB}{2c}(x\dot{y} - y\dot{x}) + N(r-R)$$

$$x\dot{y} - y\dot{x} = r c\theta(\dot{r} s\theta + r c\theta\dot{\theta}) - r s\theta(\dot{r} c\theta - r s\theta\dot{\theta}) = r^2\dot{\theta}$$

$$L = -\gamma^{-1}mc^2 - qEr s\theta + \frac{qB}{2c}\dot{\theta} r^2 + N(r-R)$$

$$\frac{\partial}{\partial\beta^2}(-\gamma^{-1}) = \tfrac{1}{2}\gamma \qquad \frac{\partial L}{\partial\dot{r}} = \tfrac{1}{2}mc^2\gamma\frac{\partial\beta^2}{\partial\dot{r}} = \gamma m\dot{r}$$

$$\frac{\partial L}{\partial\dot{\theta}} = \tfrac{1}{2}mc^2\gamma\frac{\partial\beta^2}{\partial\dot{\theta}} + \frac{qB}{2c}r^2 = \gamma m r^2\dot{\theta} + \frac{qB}{2c}r^2$$

$$c^2\frac{\partial\beta^2}{\partial r} = 2r\dot{\theta}^2$$

$$\frac{\partial L}{\partial r} = \gamma m r\dot{\theta}^2 - qE s\theta + \frac{qB}{c}\dot{\theta} r + N$$

$$\frac{\partial L}{\partial\theta} = -qEr c\theta$$

(cover)

EL before applying constraints:

$$\frac{d}{dt}(\gamma m \dot{r}) = \gamma m r \dot{\theta}^2 - qE\,s\theta + \frac{qB}{c}\dot{\theta} r + N$$

$$\frac{d}{dt}\left(\gamma m r^2 \dot{\theta} + \frac{qB}{2c} r^2\right) = -qE r c\theta$$

set $r = R$

① $$0 = \gamma m R \dot{\theta}^2 - qE s\theta + \frac{qB}{c} R \dot{\theta} + N$$

② $$m R^2 \frac{d}{dt}(\gamma \dot{\theta}) = -qERc\theta$$

① is already in right form (no $\ddot{\theta}$)

below: $\gamma^{-2} = 1 - R^2\dot{\theta}^2/c^2$

$$L_F = -\gamma^{-1} mc^2 - qERs\theta + \frac{qB}{2c} R^2 \dot{\theta}$$

$$H_F = \dot{\theta}\frac{\partial L_F}{\partial \dot{\theta}} - L_F \qquad \frac{\partial L_F}{\partial \dot{\theta}} = \tfrac{1}{2}\gamma mc^2 \frac{\partial \beta^2}{\partial \dot{\theta}} + \frac{qB}{2c} R^2$$

$$\boxed{H_F = \gamma m R^2 \dot{\theta}^2 + \frac{qB}{2c} R^2 \dot{\theta} + \gamma^{-1} mc^2 + qERs\theta - \frac{qB}{2c} R^2 \dot{\theta}}$$

$$\frac{\partial L_F}{\partial \dot{\theta}} = \gamma m R^2 \dot{\theta} + \frac{qB}{2c} R^2$$

$$H_F = \gamma mc^2 + qERs\theta$$

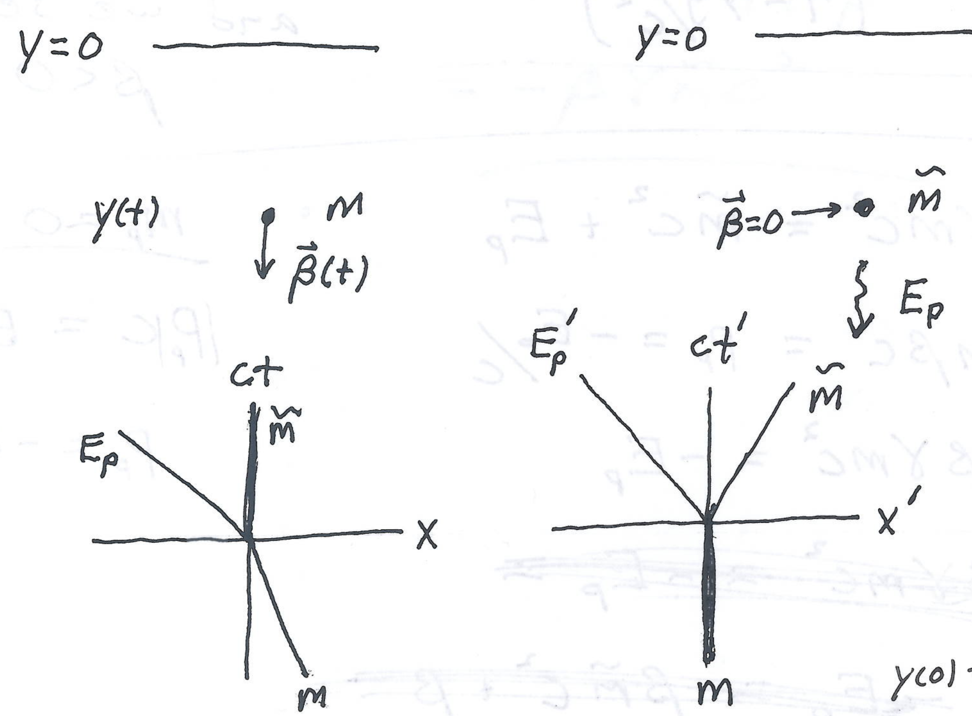
Thus, $\dot{\gamma} = -\frac{qER}{mc^2} c\theta \dot{\theta}$

so,
$$\frac{d}{dt}(\gamma\dot{\theta}) = \dot{\gamma}\dot{\theta} + \gamma\ddot{\theta} = \gamma^3 R^2 \dot{\theta}^2 \ddot{\theta}/c^2 + \gamma\ddot{\theta} = \gamma^3 \ddot{\theta} = -\frac{qE}{MR} c\theta$$
which is equiv to ②

also, $\dot{\gamma} = \gamma^3 R^2 \dot{\theta} \ddot{\theta}/c^2$

or,

Problem 2 : The figure at left below shows a particle of mass m dropping straight down in the negative y direction in a constant gravitational field. The Lagrangian for this system is the same as that for a constant electric field $\vec{E} = -E\hat{y}$ (also $\vec{B} = 0$) with the replacement $qE \to mg$. From the Euler-Lagrange equation, find $\beta(t)$ for the particle with $\beta(0) = 0$. From the (conserved) Hamiltonian, find $\beta(y)$ and from this $y(t)$, with $y(0) = 0$. As shown in the figure at right, at some time in the free-fall the particle emits a photon straight down in such a way that the particle left behind comes to a stop in the inertial frame of the ground. Find the mass $\tilde{m}$ of the new particle, and find the energy E_p of the photon in the ground frame. These should both be expressed in terms of m and the parameter β of the mass m just before the emmission. Also find the the energy $\tilde{E}_p$ of the photon in the instantaneous rest frame of the mass m just before the emmission. Now express this in terms of $\tilde{m}$. This emmission process is shown in the two space-time diagrams below.



for $t>0$
$y(t)<0$
$\beta(t)<0$

$y(0)=0 \quad \beta(0)=0 \quad \gamma(0)=1$

$$\frac{d}{dt}(\gamma\dot{y}) = -g$$

$$\dot{y}(0)=0 \Rightarrow \gamma\dot{y}/c = \gamma\beta = -gt/c$$

$$\gamma^2\beta^2 = \gamma^2 - 1$$

$$\gamma^2 = 1 + (gt/c)^2$$

$$\beta = \frac{-(gt/c)}{(1+(gt/c)^2)^{1/2}}$$

$$H = \gamma mc^2 + mgy = mc^2$$

$$y(t) = -(\gamma-1)c^2/g$$

$$y(t) = -c^2/g\left(\sqrt{1+(gt/c)^2} - 1\right)$$

$$\gamma = 1 - yg/c^2$$

$$1-\beta^2 = (1-yg/c^2)^{-2}$$

$$\beta^2 = -(1-yg/c^2)^{-2} + 1$$

$$\beta^2 = \left((1-\gamma g/c^2)^2 - 1\right) / (1-\gamma g/c^2)^2$$

$$\beta^2 = \frac{-2\gamma g/c^2 + (\gamma g/c^2)^2}{(1-\gamma g/c^2)^2}$$

$$\cancel{\beta = -(\gamma g/c^2)\left(1 - \frac{2c^2}{\gamma g}\right)^{1/2}}$$

$$\beta = -\frac{\left((\gamma g/c^2)^2 - 2\gamma g/c^2\right)^{1/2}}{(1-\gamma g/c^2)}$$

note $\gamma < 0$

and we set $\beta < 0$

$$\gamma m c^2 = \tilde{m} c^2 + E_p$$

$m_p = 0$

$|P_p| c = E_p$

$P_p = -E_p/c$

$\beta < 0$

γ, β Known

$$\gamma m \beta c = P_p = -E_p/c$$

$$\beta \gamma m c^2 = -E_p$$

$$\cancel{\beta \gamma m c^2 = -E_p}$$

$$\cancel{-E_p = \beta \tilde{m} c^2 + \beta}$$

$$E_p = -\beta \gamma m c^2$$

$$\gamma m c^2 = \tilde{m} c^2 - \beta \gamma m c^2$$

$$\gamma(1+\beta) m = \tilde{m}$$

$$\tilde{m} = m\sqrt{\frac{1+\beta}{1-\beta}} \le m$$

$$E_p = \frac{-\beta m c^2}{\sqrt{1-\beta^2}}$$

For $\tilde{E}_p$ use doppler formula or lorentz:

$$\begin{pmatrix} \tilde{E}_p \\ -\tilde{E}_p \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E_p \\ -E_p \end{pmatrix}$$

$$\tilde{E}_p = \gamma(1+\beta) E_p = \sqrt{\frac{1+\beta}{1-\beta}}\, E_p$$

(redshift)

$$= -\beta\gamma m c^2 \sqrt{\frac{1+\beta}{1-\beta}}$$

$$= -\beta\tilde{\gamma}\tilde{m}c^2$$