Parton Distribution Functions, Part 1

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A. Introduction
B. Properties of the PDFs
C. Results of CT10-NNLO Global Analysis
D. Uncertainties of the PDFs
E. Applications to LHC Physics

A. Introduction. QCD and High-Energy Physics

QCD is an elegant theory of the strong interactions – the gauge theory of color transformations. It has a simple Lagrangian

$$L = \overline{\psi}\,[\, i\not{\partial} - g\not{A} - m \,]\,\psi - \tfrac{1}{4} F_a^{\mu\nu} F_{a\mu\nu}$$

$$F_a^{\mu\nu} = \partial^{\mu} A_a^{\nu} - \partial^{\nu} A_a^{\mu} - g f_{abc} A_b^{\mu} A_c^{\nu}$$

$$\not{A} = \gamma_{\mu} A_a^{\mu} T_a$$

(sums over flavor and color are implied)

Parameters: g; m_1, m_2, m_3, …, m_6

$$L = \overline{\psi}[\, i\not{\partial} - g\not{A} - m \,]\psi - \tfrac{1}{4} F_a^{\mu\nu} F_{a\mu\nu}$$

However, the calculation of experimental observables is quite difficult, for 2 reasons:

(i) there are divergent renormalizations; the theory requires

- regularization
- perturbation theory
- a singular limit
  - $\Lambda \to \infty$ (GeV)
  - or, $a \to 0$ (fm)
  - or, $n \to 4$

(ii) quark confinement;
the asymptotic states are color singlets, whereas the fundamental fields are color triplets.

Nevertheless, certain cross sections can be calculated reliably ---
--- inclusive processes with large momentum transfer (i.e., short-distance interactions)

The reasons that QCD can provide accurate predictions for short-distance interactions are

- asymptotic freedom

$$\alpha_s(Q^2) \sim \textit{const.}/\ln(Q^2/\Lambda^2) \quad \text{as} \quad Q^2 \to \infty$$

- the factorization theorem

$$d\sigma_{hadron} \sim \text{PDFs} \otimes C$$

where C is calculable in perturbation theory.

The PDFs provide a connection between quarks and gluons (*the partons*) and the nucleon (*a bound state*).

Global Analysis of QCD and Parton Distribution Functions

$$d\sigma_{\text{hadron}} \sim \text{PDFs} \otimes C$$ (sum over flavors implied)

The symbol $\sim$ means “asymptotically equal as $Q \to \infty$”; the error is $O(m^2/Q^2)$ where Q is an appropriate (high) momentum scale.

The C’s are calculable in perturbation theory.

The PDFs are not calculable today, given our lack of understanding of the nonperturbative aspect of QCD (binding and confinement). But we can determine the PDFs from Global Analysis, with some accuracy.

The symbol $\otimes$ means "convolution"; $f \otimes g(x) = \int_x^1 f(\xi)\, g(x/\xi)\, d\xi$

next …

B. Properties of the PDFs -- Definitions

First, what are the Parton Distribution Functions? (PDFs)

The PDFs are a set of 11 functions,

$f_i(x,Q^2)$ where $0 \leq x \leq 1$ longitudinal momentum fraction

$Q >\sim 2\ GeV$ momentum scale

$i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ parton index

$f_0 = g(x,Q^2)$ the gluon PDF

$f_1 = u(x,Q^2)$ the up-quark PDF

$f_{-1} = \bar{u}(x,Q^2)$ the up-antiquark PDF

$f_2 = d$ and $f_{-2} = \bar{d}$

$f_3 = s$ and $f_{-3} = \bar{s}$

etc.

Exercise:
What about the top quark?

Second, what is the *meaning* of a PDF?

We tend to think and speak in terms of

"Proton Structure"

$u(x,Q^2)\ dx$ = the mean number of up quarks with longitudinal momentum fraction from x to $x + dx$, appropriate to a scattering experiment with momentum transfer Q.

$u(x,Q^2)$ = the up-quark density in momentum fraction

This heuristic interpretation makes sense from the LO parton model. More precisely, taking account of QCD interactions, $d\sigma_{\text{proton}}$ = PDFs $\otimes$ C .

$f_i(x,Q^2)$ = the *density* of parton type i
w.r.t. longitudinal momentum fraction x

Momentum Sum Rule

$$\int_0^1 x\, f_i(x,Q^2) =$$

longitudinal momentum fraction, carried by parton type i

$$\sum_{i=-5}^{5} \int_0^1 x\, f_i(x,Q^2) = 1$$

Flavor Number Sum Rules

$$\int_0^1 [\, u(x,Q^2) - \bar{u}(x,Q^2)\,]\, dx = 2$$

valence up quark integral,

$$\int_0^1 [\, d(x,Q^2) - \bar{d}(x,Q^2)\,]\, dx = 1$$

valence down quark integral,

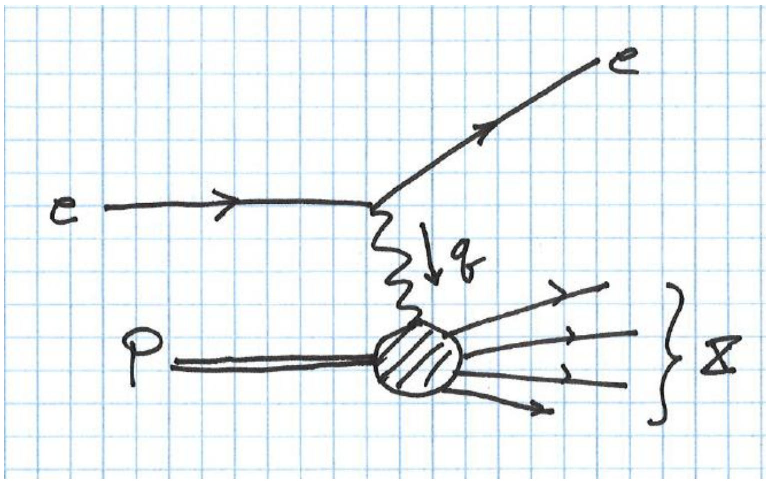
$$\int_0^1 [\, s(x,Q^2) - \bar{s}(x,Q^2)\,]\, dx = 0$$

valence strange quark integral

Example.

DIS of electrons by protons; e.g., HERA experiments

$$e + p \rightarrow e + X$$



Kinematic Variables

$$Q^2 = -q^2$$

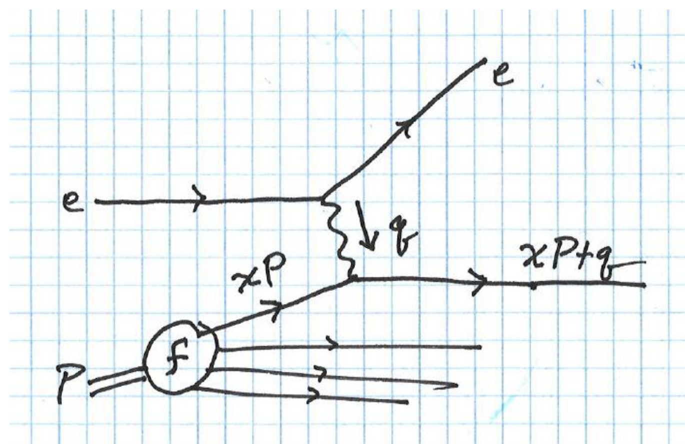
$$\nu = P \cdot q / M$$

$$x = \frac{Q^2}{2M\nu} = \frac{-q^2}{2P \cdot q}$$

$$y = \frac{P \cdot q}{P \cdot \ell}$$

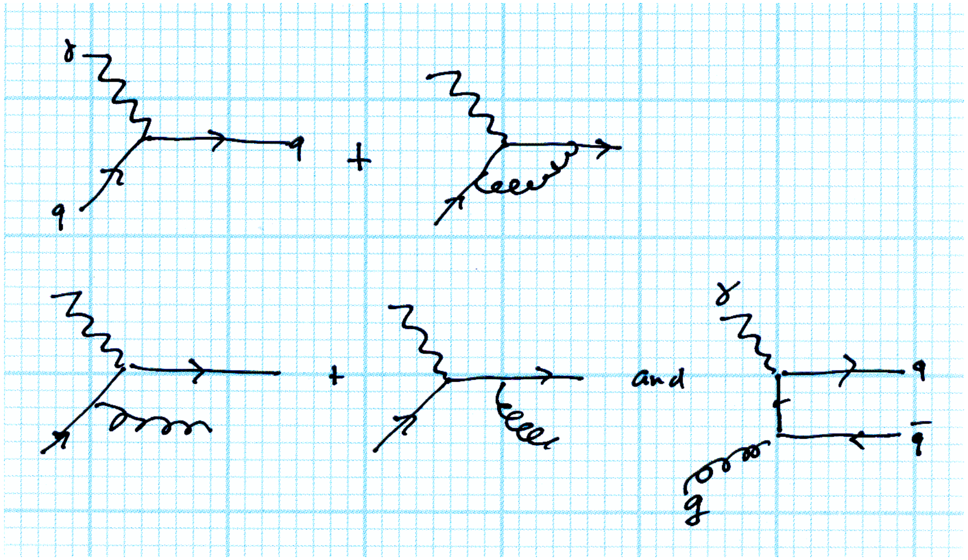
in lowest order of QCD

(summed over flavors!)



But QCD radiative corrections must be included to get a sufficiently accurate prediction.

The NLO approximation will involve these interactions …



From these *perturbative calculations*, we determine the coefficient functions $C_i^{(NLO)}$, and hence write

$$\sigma_{ep} \sim \sum_i (PDF)_i^{(NLO)} \otimes C_i^{(NLO)}$$

Approximations available today: LO, NLO, NNLO

The Factorization Theorem

For short-distance interactions,

$$\sigma_{ep} \sim \sum_{i} (\mathrm{PDF})_i \otimes C_i$$

$$\sigma_{pp} \sim \sum_{i,j} (\mathrm{PDF})_i \; (\mathrm{PDF})_j \otimes \otimes C_{ij}$$

and <u>the</u> <u>PDFs</u> <u>are</u> <u>universal</u> **!**

We can write a formal, field-theoretic expression,

Correlator function

$$f_i(x,\mu) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dy^- \; e^{iy^- p^+}$$

$$\times \langle p | \, \overline{\psi}_i(0, y^-, 0_T) \, \gamma^+ \, \psi_i(0,0,0_T) \, | p \rangle$$

although we can't evaluate it because we don't know the bound state |p>.

Exercise.

Suppose the parton densities for the proton are known,

$$f_i\,(x,Q^2) \quad \text{for} \quad \{\ i\ =\ 0, \pm 1, \pm 2, \ldots, \pm 5\ \}$$

(A) In terms of the $f_i(x,Q^2)$, write the 11 parton densities for the neutron, say, $g_i(x,Q^2)$.

(B) In terms of the $f_i(x,Q^2)$, write the 11 parton densities for the deuteron, say, $h_i(x,Q^2)$.

next …

B. Properties of the PDFs – Q^2 evolution

<u>Evolution in Q</u>

The PDFs are a set of 11 functions,

$f_i(x,Q^2)$ where $0 \leq x \leq 1$ longitudinal momentum fraction

$Q >\sim 2\ GeV$ momentum scale

$f_i(x,Q^2)$ = the density of partons of type *i,* carrying a fraction *x* of the longitudinal momentum of a proton, when resolved at a momentum scale Q.

The DGLAP Evolution Equations

- We know how the f_i vary with Q.
- That follows from the *renormalization group*.
- It's calculable in perturbation theory .

The DGLAP Evolution Equations

V.N. Gribov and L.N. Lipatov, Sov J Nucl Phys 15, 438 (1972);
G. Altarelli and G. Parisi, Nucl Phys B126, 298 (1977);
Yu.L. Dokshitzer, Sov Phys JETP 46, 641 (1977).

$$\frac{\partial q(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_S(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} P(z,\alpha_S(Q^2))\, q(x/z,Q^2)$$

for a NON-SINGLET distribution,

e.g., $q = q_i - q_j$

$$P(z,\alpha_S) = P^{(0)}(z) + \tfrac{\alpha_s}{2\pi} P^{(1)}(z) + (\tfrac{\alpha_s}{2\pi})^2 P^{(2)}(z)$$

Solve the 11 coupled equations numerically.
For example, you could download the program HOPPET.
G. P. Salam and J. Rojo, *A Higher Order Perturbative Parton Evolution Toolkit*; download from http://projects.hepforge.org/hoppet
… a library of programs written in Fortran 90.

A Higher Order Perturbative Parton Evolution Toolkit (HOPPET)

G. P. Salam and J. Rojo
LPTHE,
UPMC – Univ. Paris 6,
Université Paris Diderot – Paris 7,
CNRS UMR 7589,
75252 Paris cedex 05, France

Abstract

This document describes a Fortran 95 package for carrying out DGLAP evolution and other common manipulations of parton distribution functions (PDFs). The PDFs are represented on a grid in x–space so as to avoid limitations on the functional form of input distributions. Good speed and accuracy are obtained through the representation of splitting functions in terms of their convolution with a set of piecewise polynomial basis functions, and Runge-Kutta techniques are used for the evolution in Q. Unpolarised evolution is provided to NNLO, including heavy-quark thresholds in the $\overline{\text{MS}}$ scheme, and longitudinally polarised evolution to NLO. The code is structured so as to provide simple access to the objects representing splitting functions and PDFs, making it possible for a user to extend the facilities already provided. A streamlined interface is also available, facilitating use of the evolution part of the code from F77 and C/C++.

Some informative results obtained using HOPPET

Starting from a set of "benchmark input PDFs", let's use HOPPET to calculate the evolved PDFs at selected values of Q.

For the ***input*** (not realistic but used here to study the evolution qualitatively):

$$xu_v(x) = 5.107200\, x^{0.8}\, (1-x)^3$$
$$xd_v(x) = 3.064320\, x^{0.8}\, (1-x)^4$$
$$x\bar{d}(x) = 0.1939875\, x^{-0.1}\, (1-x)^6$$
$$x\bar{u}(x) = x\,\bar{d}(x)(1-x)$$
$$xs(x) = x\bar{s}(x) = 0.2(x\bar{d}(x) + x\bar{u}(x))$$
$$xg(x) = 1.7\, x^{-0.1}\, (1-x)^5$$

Output tables

The expected output from the program is:

Evaluating PDFs at Q = 100.000 GeV

x	u-ubar	d-dbar	2(ubr+dbr)	c+cbar	gluon
1.0E-05	3.1907E-03	1.9532E-03	3.4732E+01	1.5875E+01	2.2012E+02
1.0E-04	1.4023E-02	8.2749E-03	1.5617E+01	6.7244E+00	8.8804E+01
1.0E-03	6.0019E-02	3.4519E-02	6.4173E+00	2.4494E+00	3.0404E+01
1.0E-02	2.3244E-01	1.3000E-01	2.2778E+00	6.6746E-01	7.7912E+00
1.0E-01	5.4993E-01	2.7035E-01	3.8526E-01	6.4466E-02	8.5266E-01
3.0E-01	3.4622E-01	1.2833E-01	3.4600E-02	4.0134E-03	7.8898E-02
5.0E-01	1.1868E-01	3.0811E-02	2.3198E-03	2.3752E-04	7.6398E-03
7.0E-01	1.9486E-02	2.9901E-03	5.2352E-05	5.6038E-06	3.7080E-04
9.0E-01	3.3522E-04	1.6933E-05	2.5735E-08	4.3368E-09	1.1721E-06

The Running Coupling of QCD

$\alpha_S(Q^2)$

$$\frac{d\alpha_S}{d\ln Q^2} = \beta(\alpha_S) = -b_0\alpha_S^2 - b_1\alpha_S^3 - b_2\alpha_S^4$$

$$b_0 = (33 - 2n_f)\,/\,12\pi$$

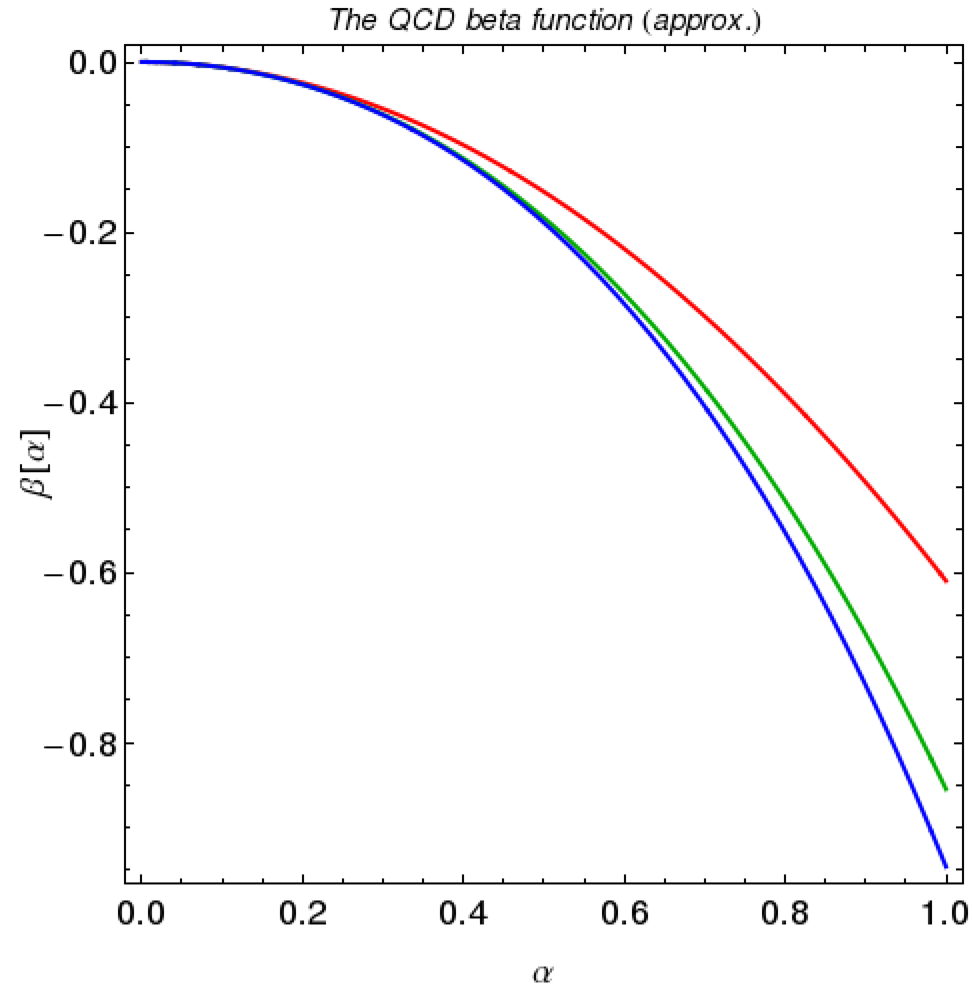
$$b_1 = (153 - 19n_f)\,/\,24\pi^2$$

$$b_2 = (77139 - 15099n_f + 325n_f^2)\,/\,3456\pi^3$$

The QCD Running Coupling Constant

$$\frac{d\alpha_S}{d\ln Q^2} = \beta(\alpha_S)$$

$$\ln\frac{Q^2}{Q_0^2} = \int_{\alpha_0}^{\alpha_s}\frac{d\alpha}{\beta(\alpha)}$$



For Global Analysis, we need an accurate $\alpha_S(Q^2)$.

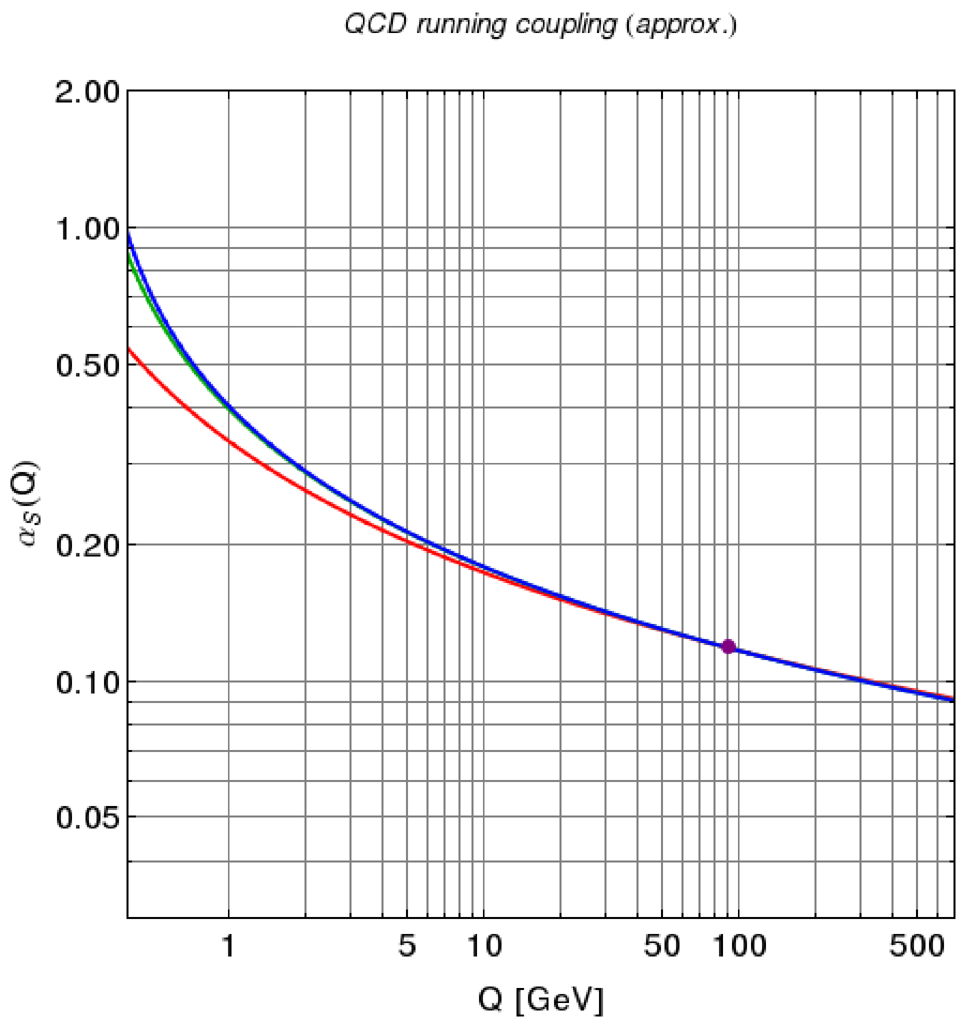
(approx. : N_F = 5 massless)

The QCD Running Coupling Constant

Evolution of α_S as a function of Q, using

- the 1-loop beta function (red)
- the 2-loop beta function (green)
- the 3-loop beta function (blue)

For Global Analysis, we need an accurate $\alpha_S(Q^2)$.

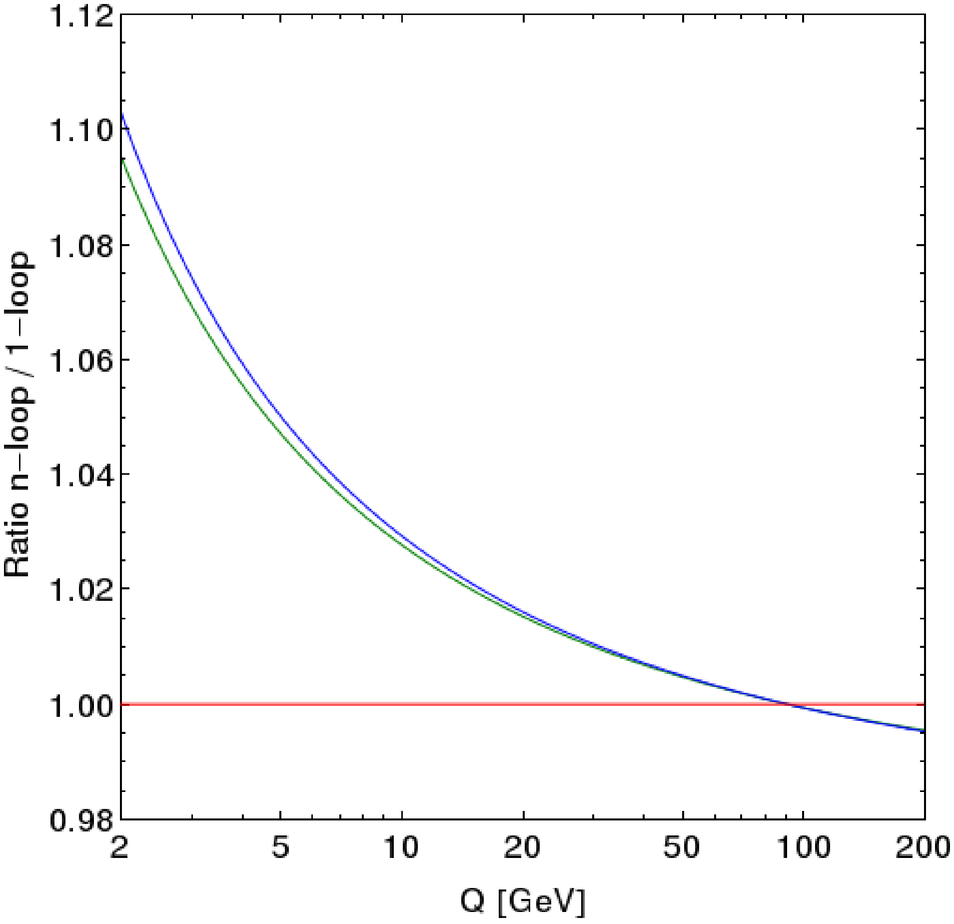


$\alpha_S(M_Z) = 0.1184 \pm 0.0007$

S. Bethke, G. Dissertori and G.P. Salam; Particle Data Group 2012

The QCD Running Coupling Constant

How large are the 2-loop and 3-loop corrections for $\alpha_S(Q^2)$?



Comments on $\alpha_S(Q^2)$

- An important improvement is to include the quark masses.

- The central fit has $\alpha_S(M_Z) = 0.118$. *(Bethke)*

- CTEQ also provides alternative PDFs with a range of value of $\alpha_S(M_Z)$; these are called the α_S-*series.*

- Another approach is to use the Global Analysis to "determine" the value of $\alpha_S(M_Z)$.

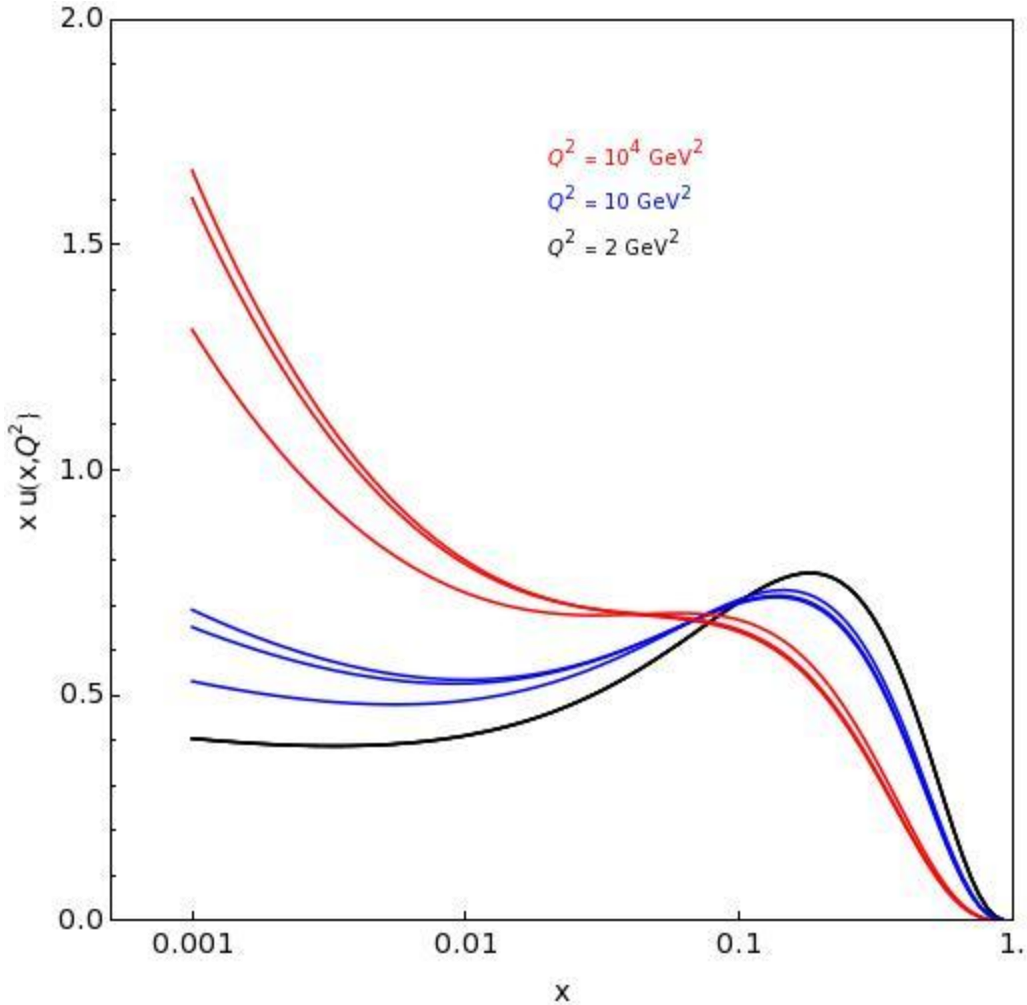
- *Asymptotic freedom*

How does the u-quark PDF evolve in Q?

Examples from HOPPET

U-quark PDF evolution :

Black : $Q = Q_0 = 1.414$ GeV
Blue : Q = 3.16 GeV
(1-loop, 2-loop, 3-loop)
Red : Q = 100.0 GeV
(1-loop, 2-loop, 3-loop)

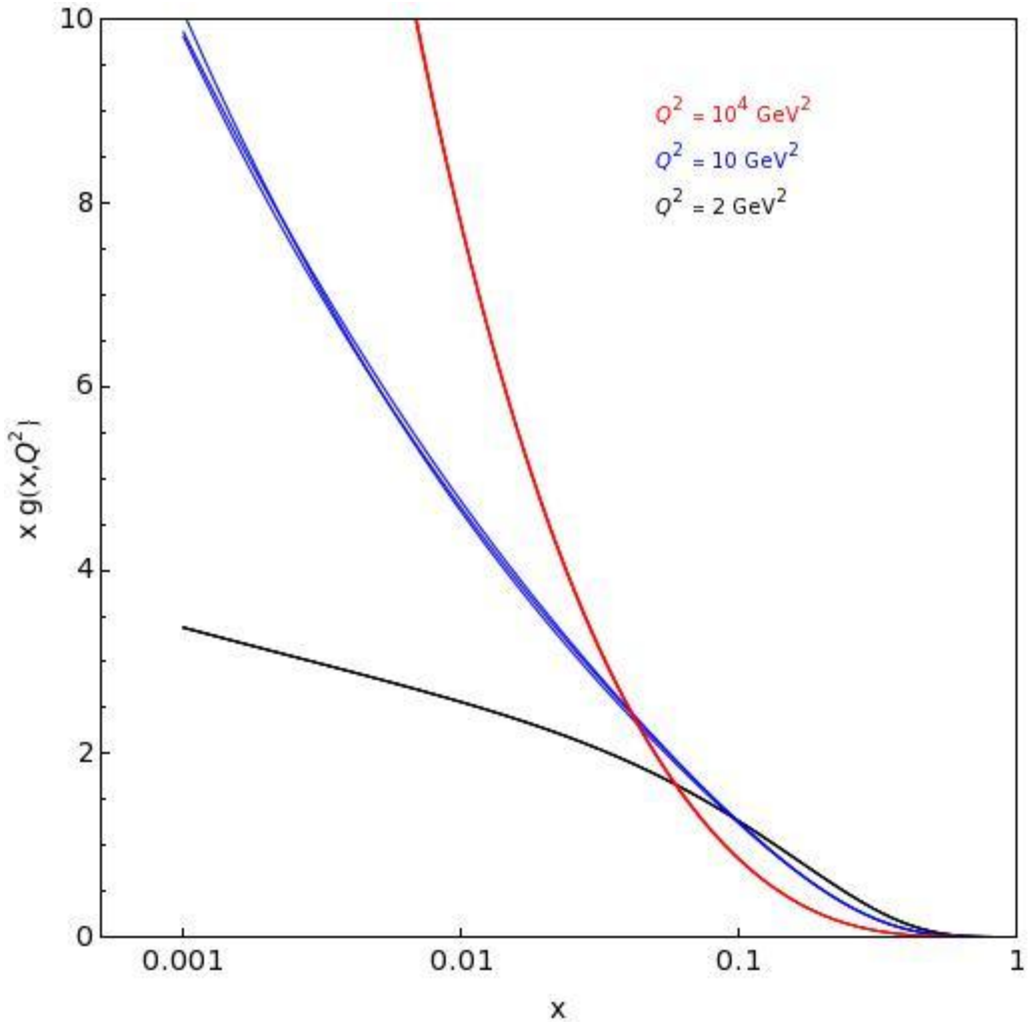


How does the gluon PDF evolve in Q?

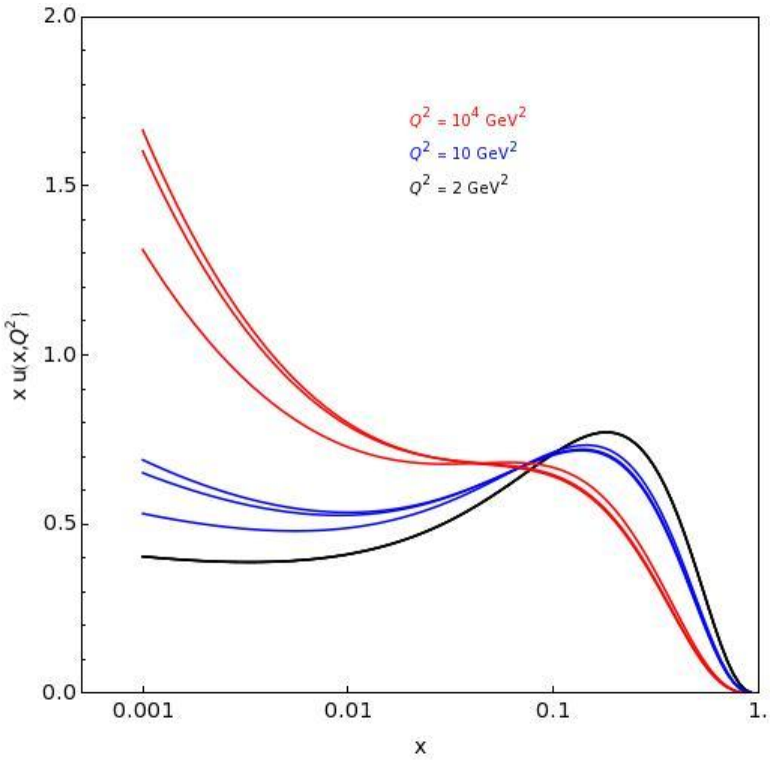
Examples from HOPPET

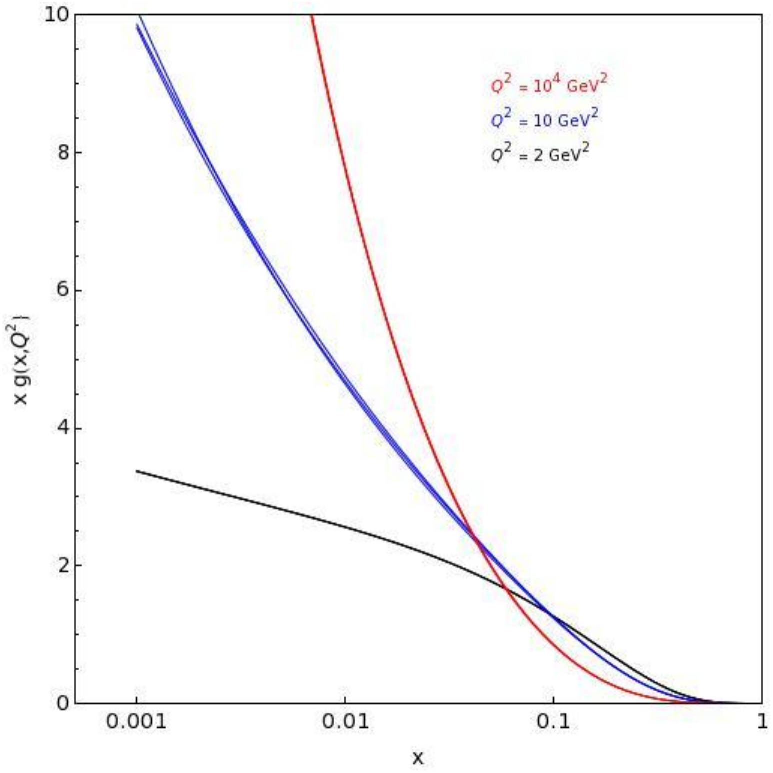
Gluon PDF evolution :

Black : Q = Q0 = 1.414 GeV
Blue : Q = 3.16 GeV
(1-loop, 2-loop, 3-loop)
Red : Q = 100.0 GeV
(1-loop, 2-loop, 3-loop)



DGLAP evolution of PDFs





- ❑ The "structure of the proton" depends on the resolving power of the scattering process. As Q increases …

  PDFs decrease at large x

  PDFs increase at small x

  as we resolve the gluon radiation and quark pair production.
- ❑ The momentum sum rule and the flavor sum rules hold for all Q.
- ❑ These graphs show the DGLAP evolution for LO, NLO, NNLO Global Analysis.

How large are the NLO and NNLO corrections?

… a good exercise using HOPPET

next …

Structure of the Proton

CTEQ Parton Distribution Functions

CTEQ6.6
P. M. Nadolsky, H.-L. Lai, Q.-H. Cao, J. Huston, J. Pumplin, D. Stump, W.-K. Tung, C.-P. Yuan, *Implications of CTEQ global analysis for collider observables,* Phys.Rev.D78:013004 (2008) ; arXiv:0802.0007 [hep-ph]

CT10
Hung-Liang Lai, Marco Guzzi, Joey Huston, Zhao Li, Pavel M. Nadolsky, Jon Pumplin, C.-P. Yuan, *New parton distributions for collider physics,* Phys.Rev.D82:074024 (2010) ; arXiv:1007.2241 [hep-ph]

CT10-NNLO
(2012)

The results shown below are for CT10-NNLO.

Structure of the Proton

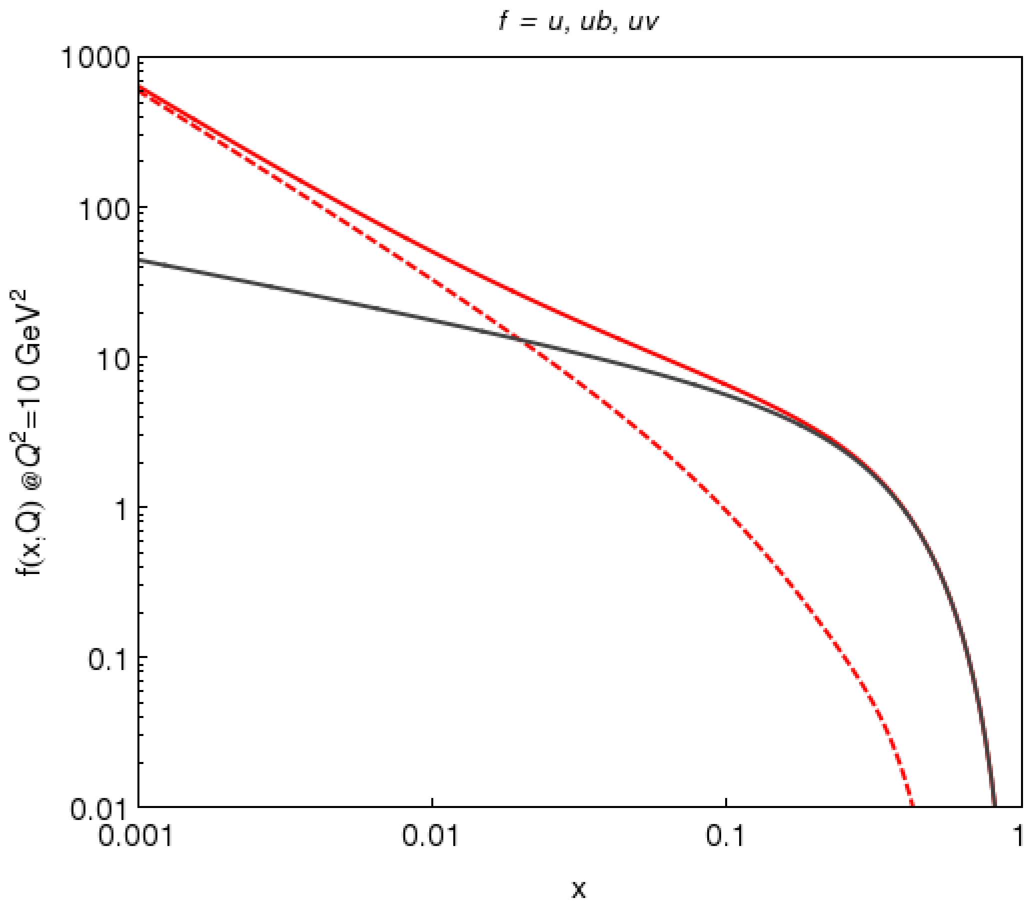
The U Quark

(Q^2 = 10 GeV^2; Q = 3.16 GeV)

Red: $u(x,Q^2)$

Dashed Red: $\bar{u}(x,Q^2)$

Gray: $u_{valence}(x,Q^2)$



This log-log plot shows …

… for x $\gtrsim$ 0.1 the valence structure dominates

… for x $\lesssim$ 0.01 $\bar{u}$ approaches u

Structure of the Proton

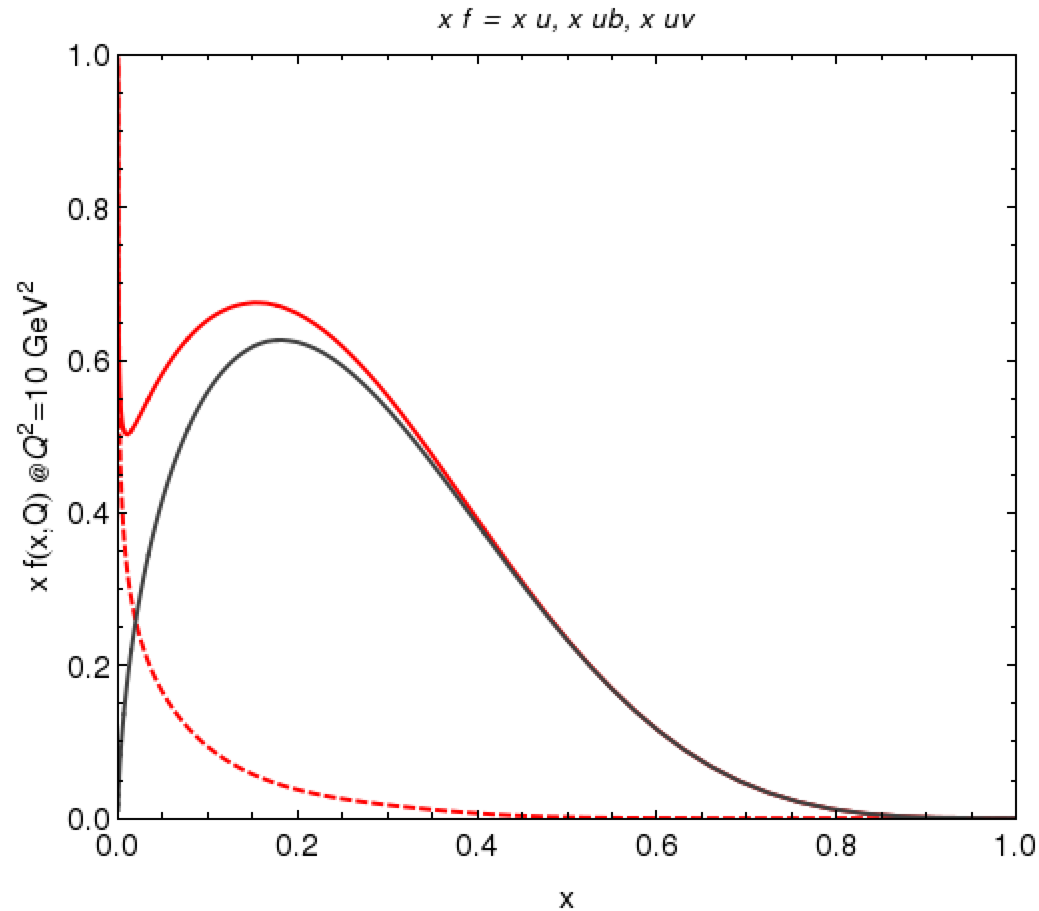
The U Quark

(Q^2 = 10 GeV^2; Q = 3.16 GeV)

Red: $x\ u(x,Q^2)$

Dashed Red: $x\ \bar{u}(x,Q^2)$

Gray: $x\ u_{valence}(x,Q^2)$



This linear plot shows the momentum fraction for Q = 3.16 GeV (= area under the curve)

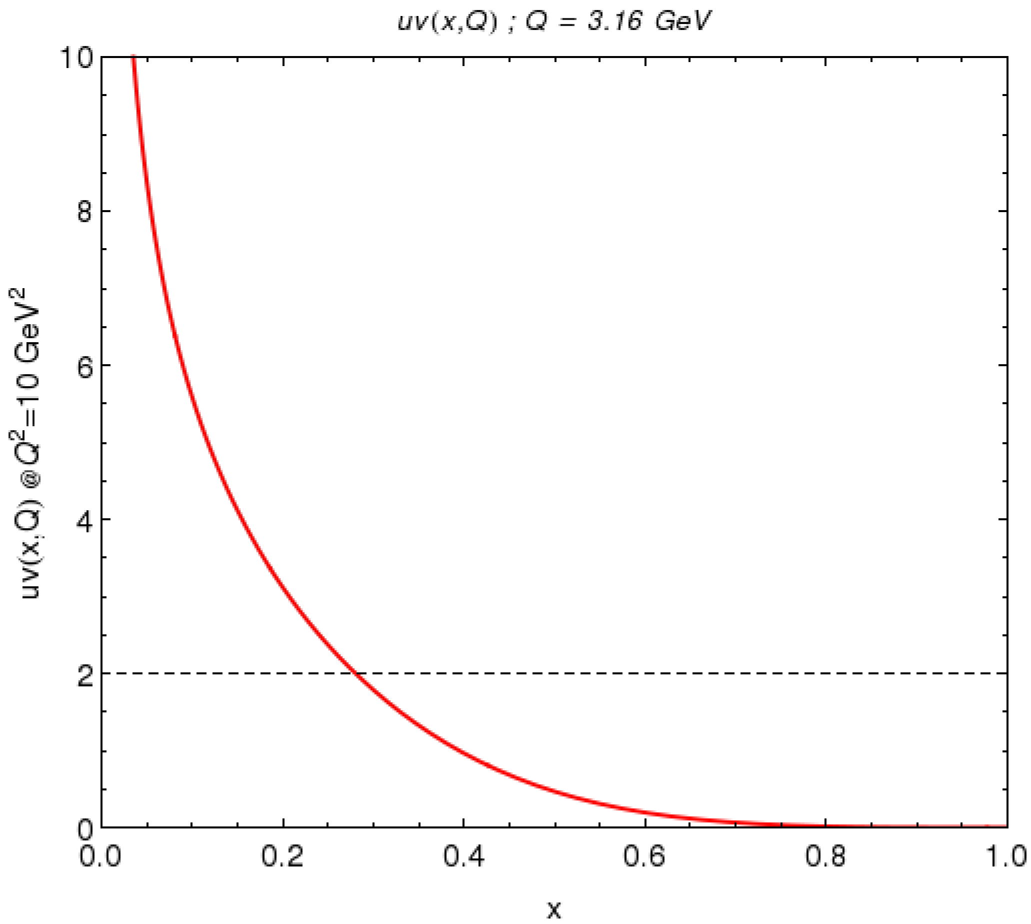
integral ≈ 0.32, 0.05, 0.27

Structure of the Proton

The U Quark

(Q^2 = 10 GeV^2; Q = 3.16 GeV)

Red: $u_{valence}(x,Q^2)$



This linear plot demonstrates the flavor sum rule:

integral = 2

Structure of the Proton

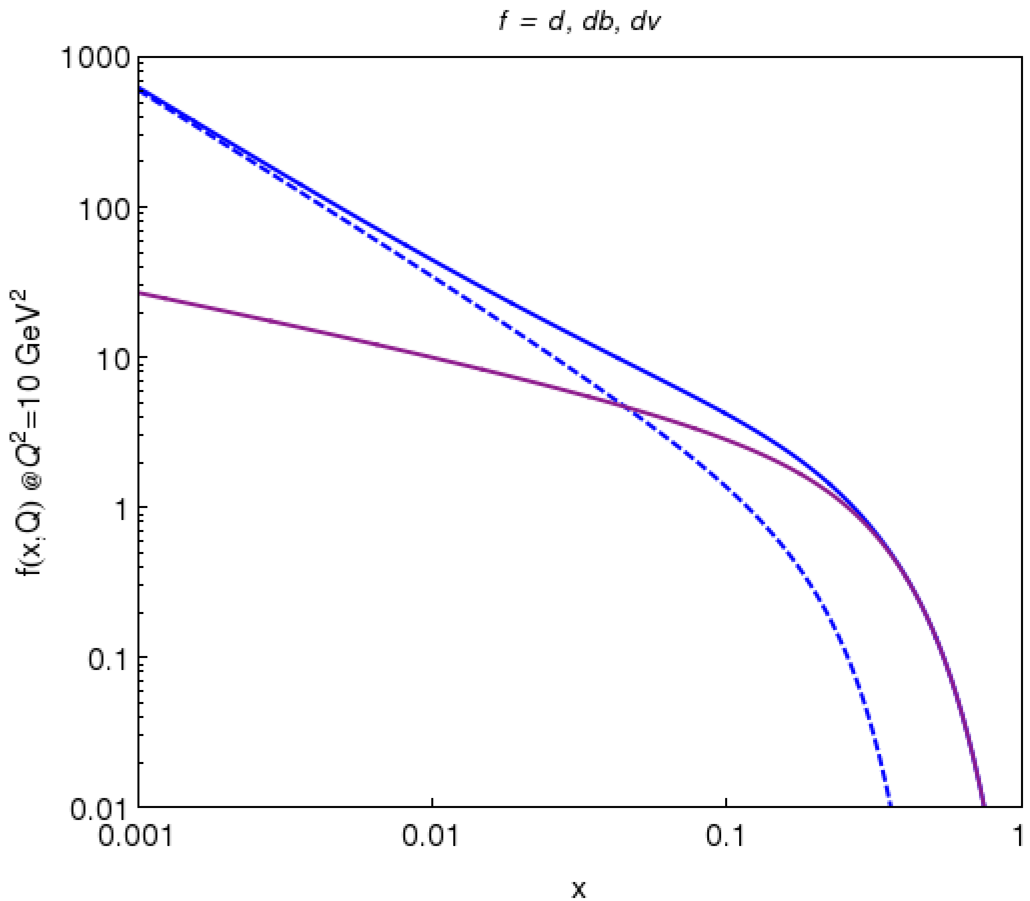
The D Quark

(Q^2 = 10 GeV^2; Q = 3.16 GeV)

Blue: $d(x,Q^2)$

Dashed Blue: $\bar{d}(x,Q^2)$

Gray: $d_{valence}(x,Q^2)$



This log-log plot shows …

… for $x \gtrsim 0.1$ the valence structure dominates

… for $x \lesssim 0.01$ $\bar{d}$ approaches d

Structure of the Proton

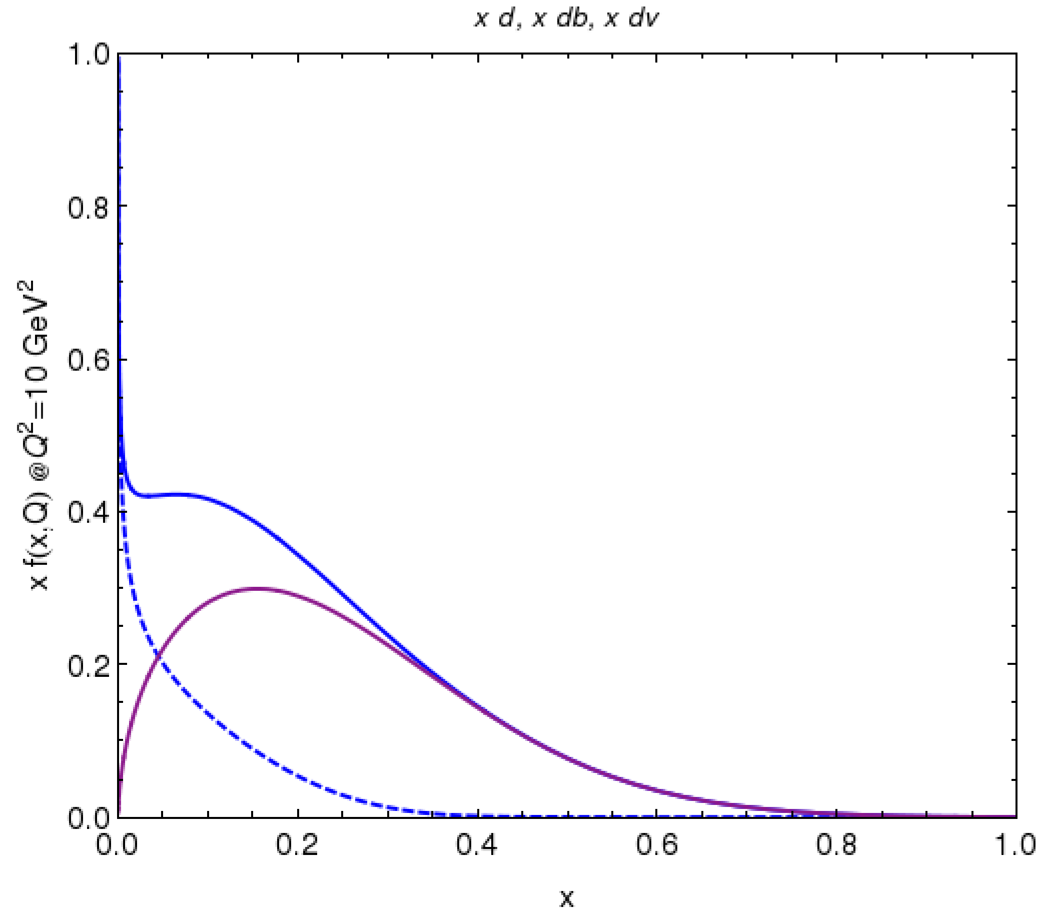
The D Quark

(Q^2 = 10 GeV^2; Q = 3.16 GeV)

Blue: $d(x,Q^2)$

Dashed Blue: $\bar{d}(x,Q^2)$

Gray: $d_{valence}(x,Q^2)$



This linear plot shows the momentum fraction for Q = 3.16 GeV (= area under the curve)

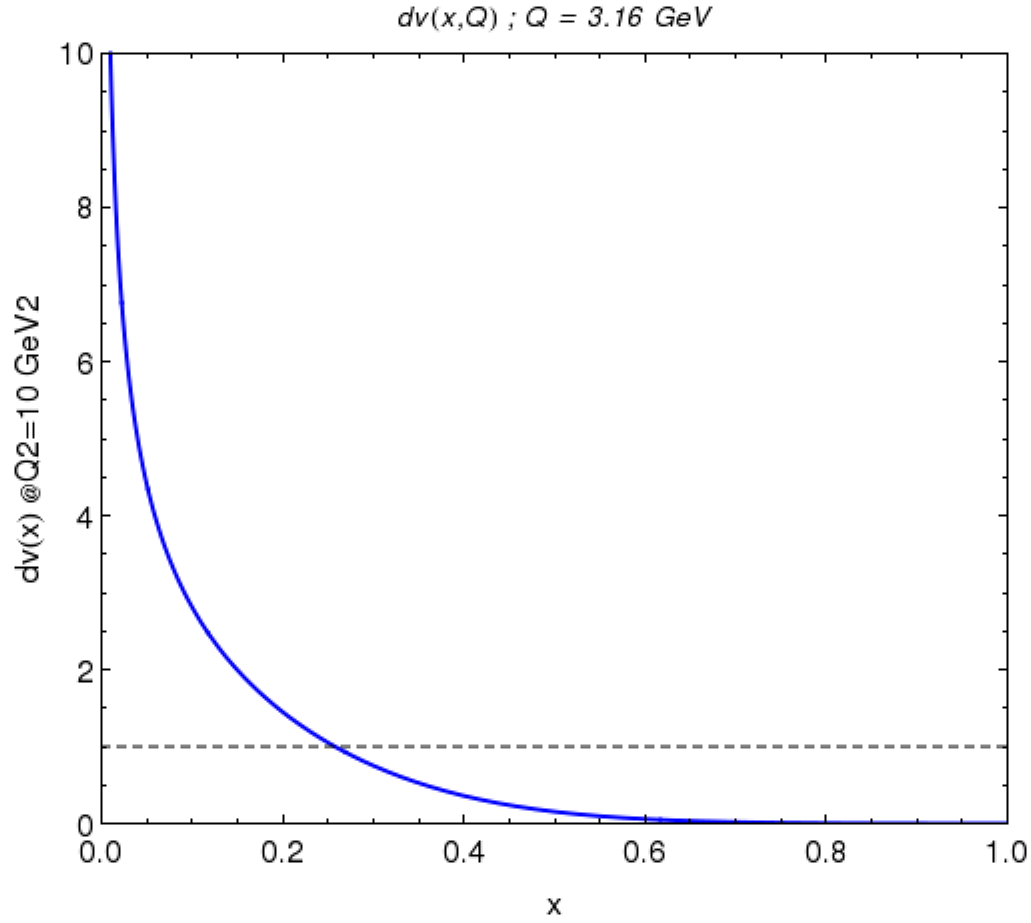
integral ≈ 0.15, 0.06, 0.10

Structure of the Proton

The D Quark

(Q^2 = 10 GeV^2; Q = 3.16 GeV)

$d_{valence} = d - \bar{d} \quad (x,Q^2)$



This linear plot demonstrates the flavor sum rule:

integral = 1

Structure of the Proton

The D and U Quarks

(Q^2 = 10 GeV^2; Q = 3.16 GeV)

Blue: $d(x,Q^2)$

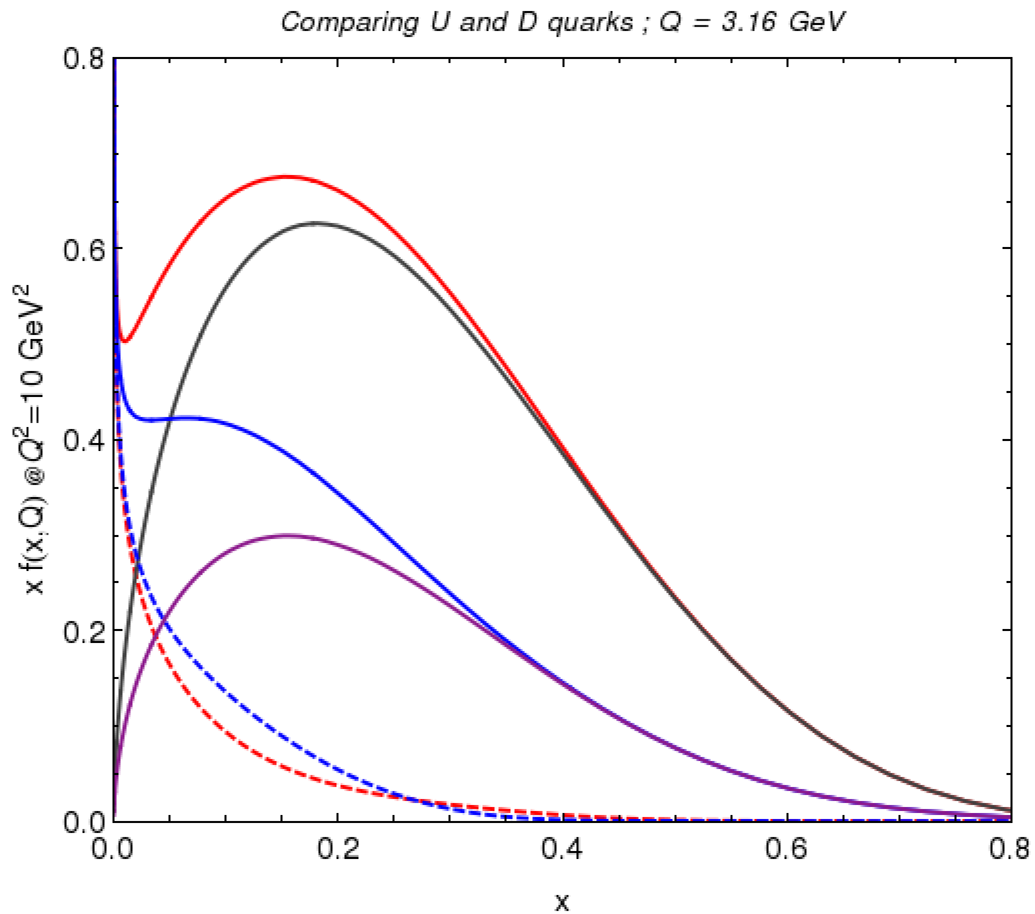
Dashed Blue: $\bar{d}(x,Q^2)$

Gray: $d_{valence}(x,Q^2)$

Red: $u(x,Q^2)$

Dashed Red: $\bar{u}(x,Q^2)$

Gray: $u_{valence}(x,Q^2)$



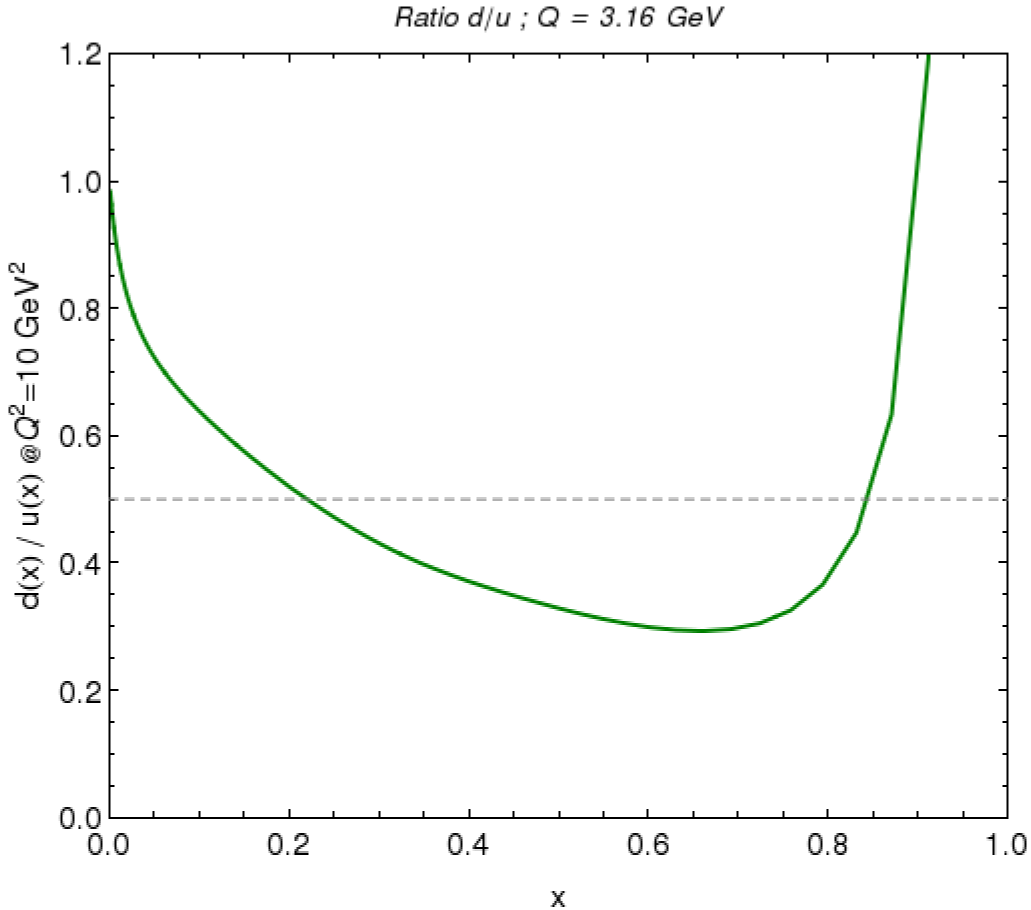
This linear plot compares U and D -- quarks and antiquarks; naively, d = 0.5 u, but it's not that simple.

Structure of the Proton

The D and U Quarks

(Q^2 = 10 GeV^2; Q = 3.16 GeV)

$d(x,Q^2) / u(x,Q^2)$ v x



$d(x,Q^2) / u(x,Q^2)$ is interesting.

***Exercise**:* What scattering processes could provide information on this ratio?

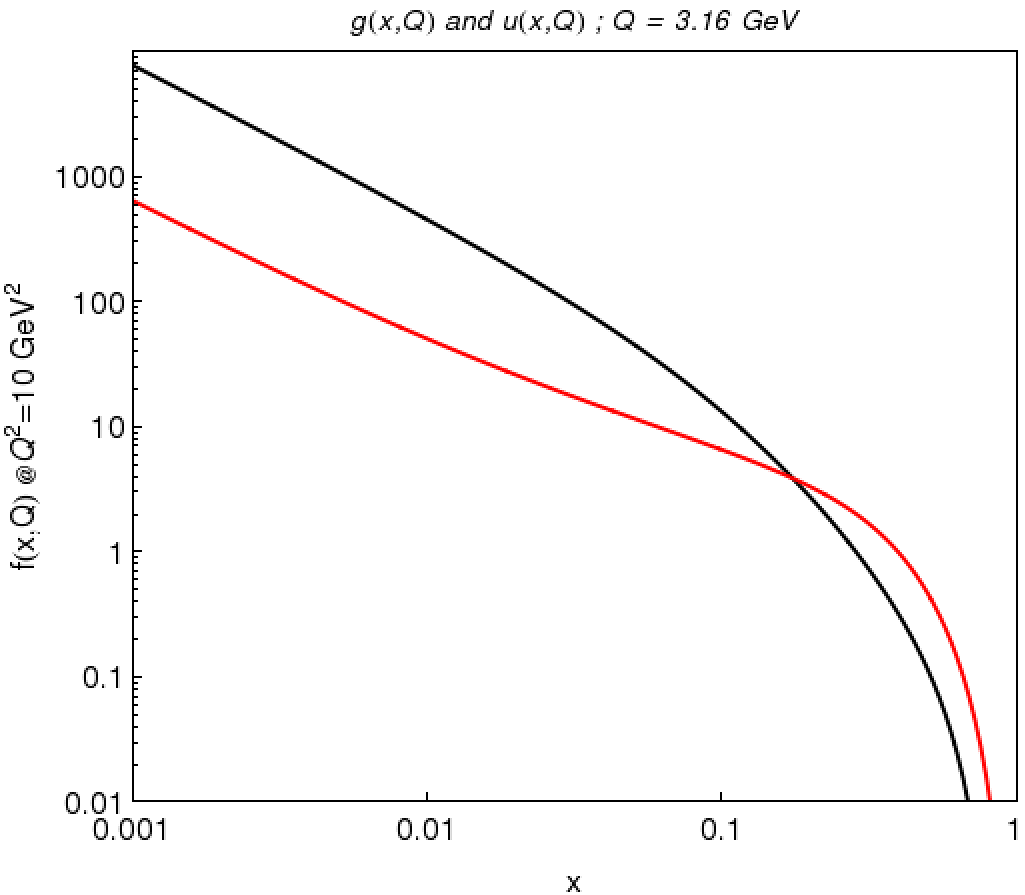
Structure of the Proton

Gluons

(Q^2 = 10 GeV^2; Q = 3.16 GeV)

black : $g(x,Q^2)$

red : $u(x,Q^2)$



The gluon dominates at small x.

The valence quarks dominate at large x.

Structure of the Proton

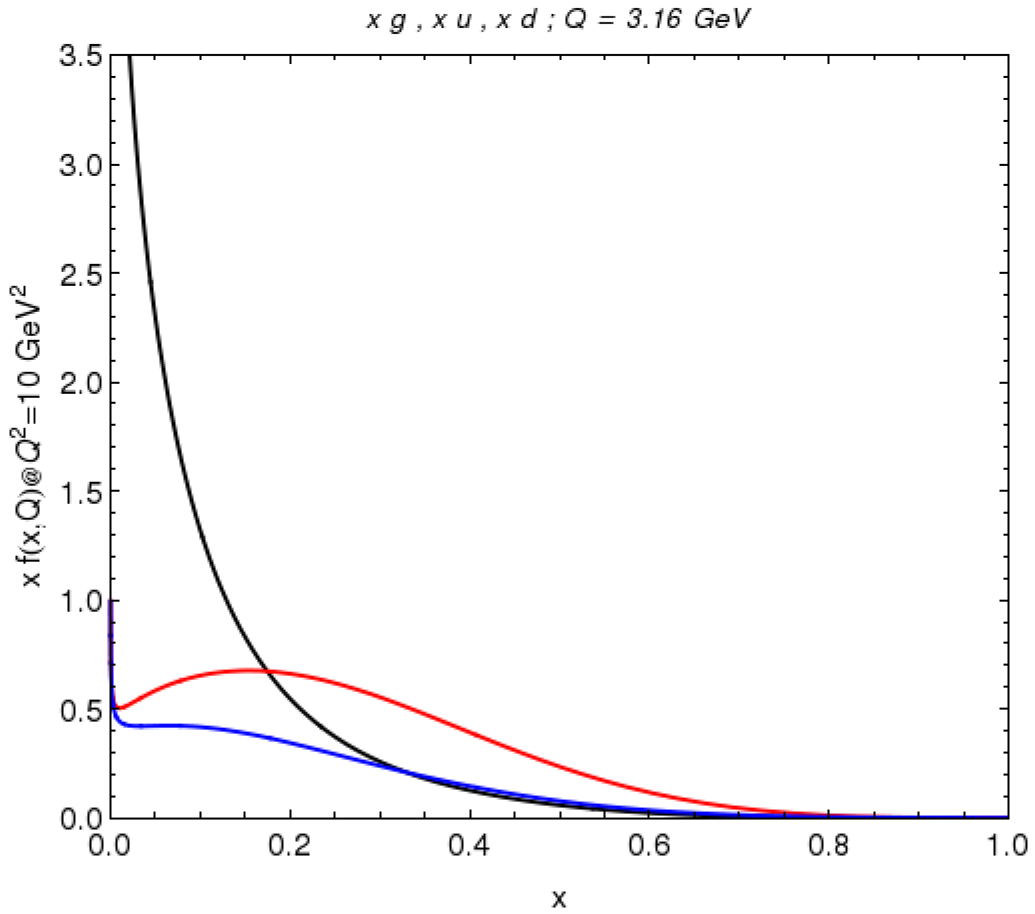
Gluons

(Q^2 = 10 GeV^2; Q = 3.16 GeV)

black : $g(x,Q^2)$

red : $u(x,Q^2)$

blue: $d(x,Q^2)$



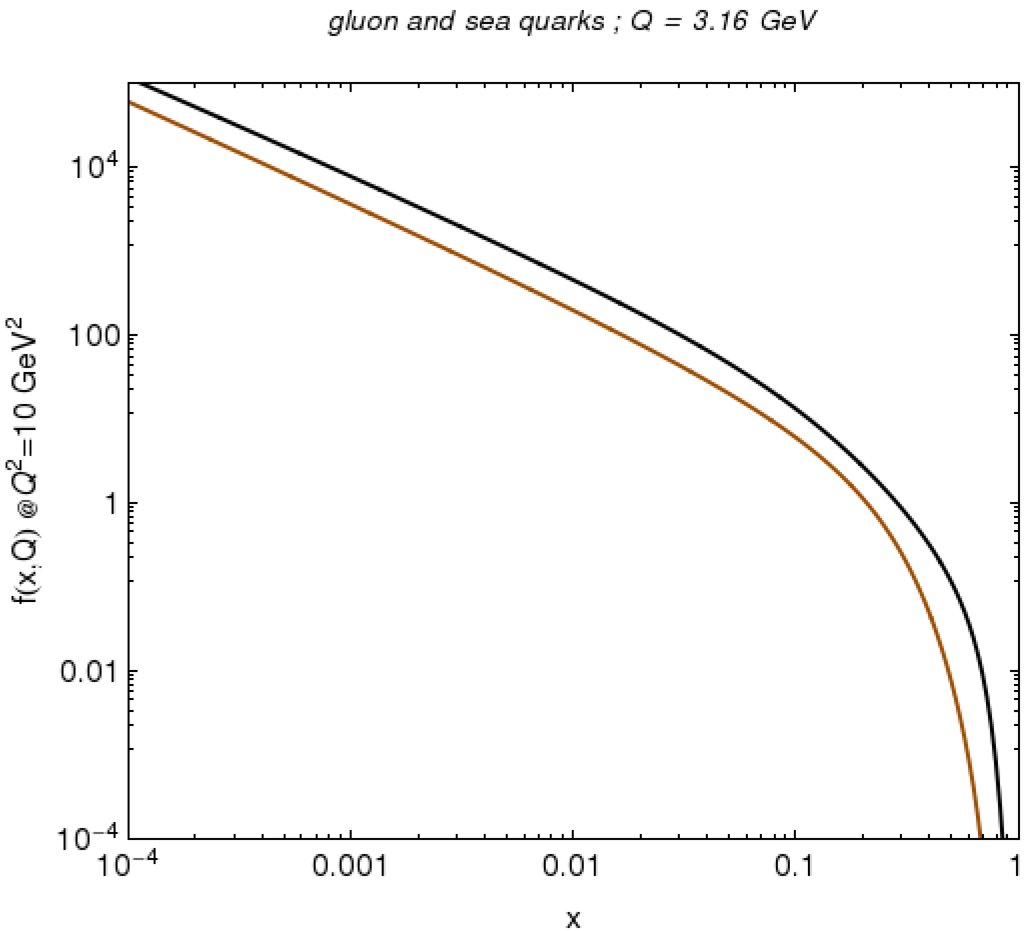
The gluon dominates at small x.

The valence quarks dominate at large x.

Structure of the Proton
Sea Quarks and Gluons
(Q^2 = 10 GeV^2; Q = 3.16 GeV)

sea(x,Q^2) and g(x,Q^2) vs x

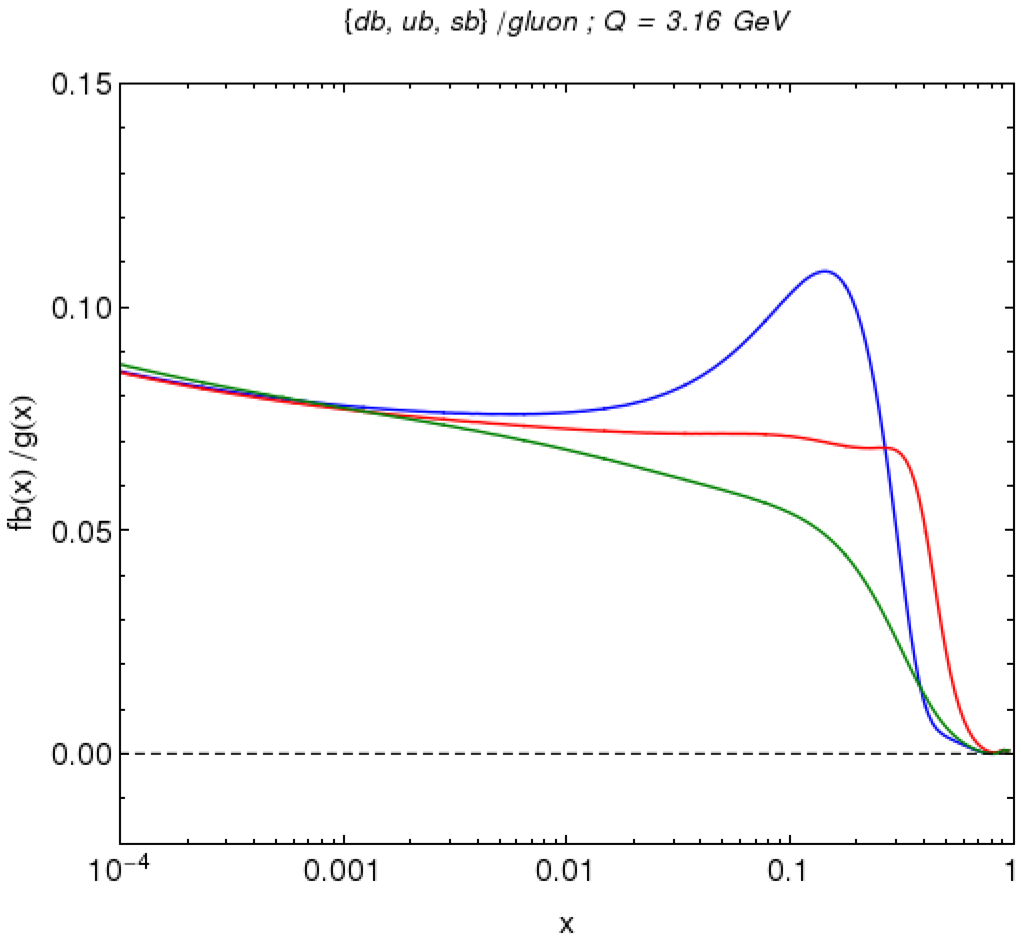
black: g(x,Q^2)
brown: sea(x,Q^2)



sea = 2 (db + ub + sb)

Structure of the Proton
(Q^2 = 10 GeV^2; Q = 3.16 GeV)

Ratios :
(db, ub, sb)/g

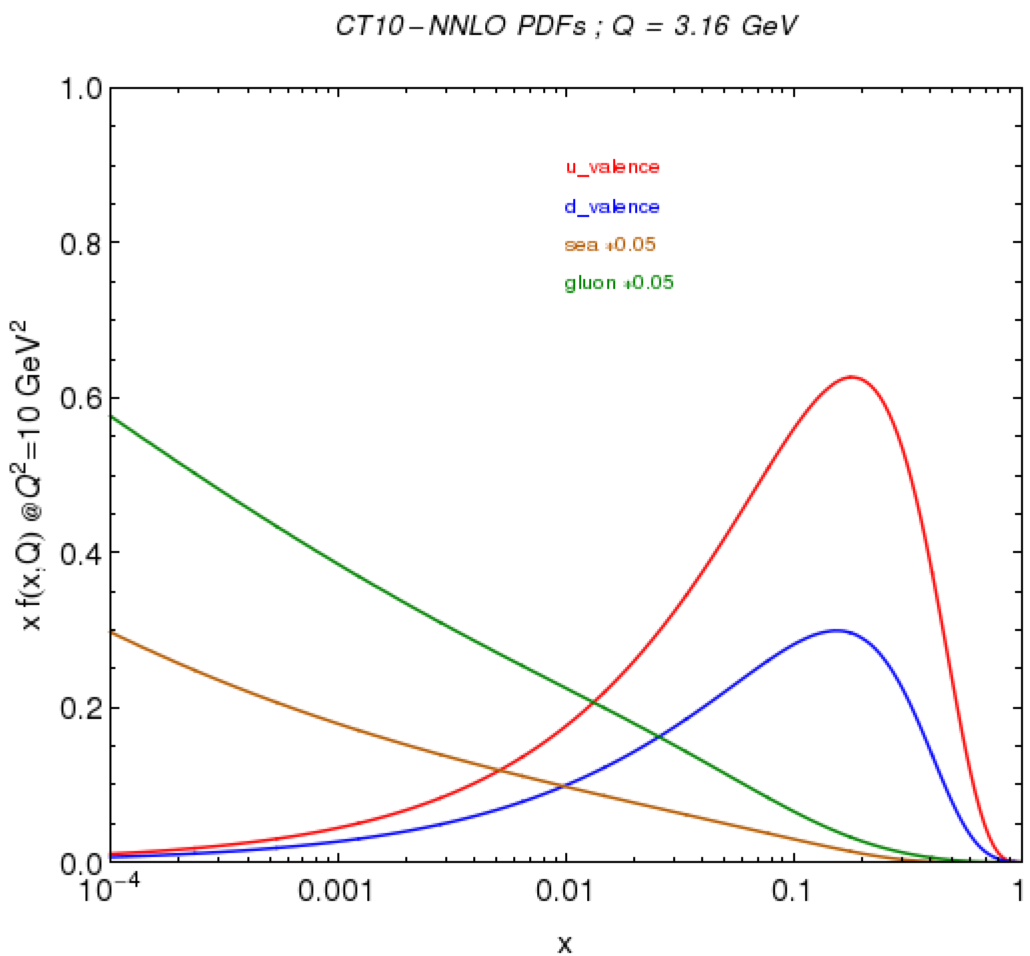


Exercises:
(1) Identify db, ub, sb.
(2) Why are they all equal at small x?

Structure of the Proton
(Q^2 = 10 GeV^2; Q = 3.16 GeV)

All in One Plot

u_valence
d valence
*sea * 0.05*
*gluon * 0.05*

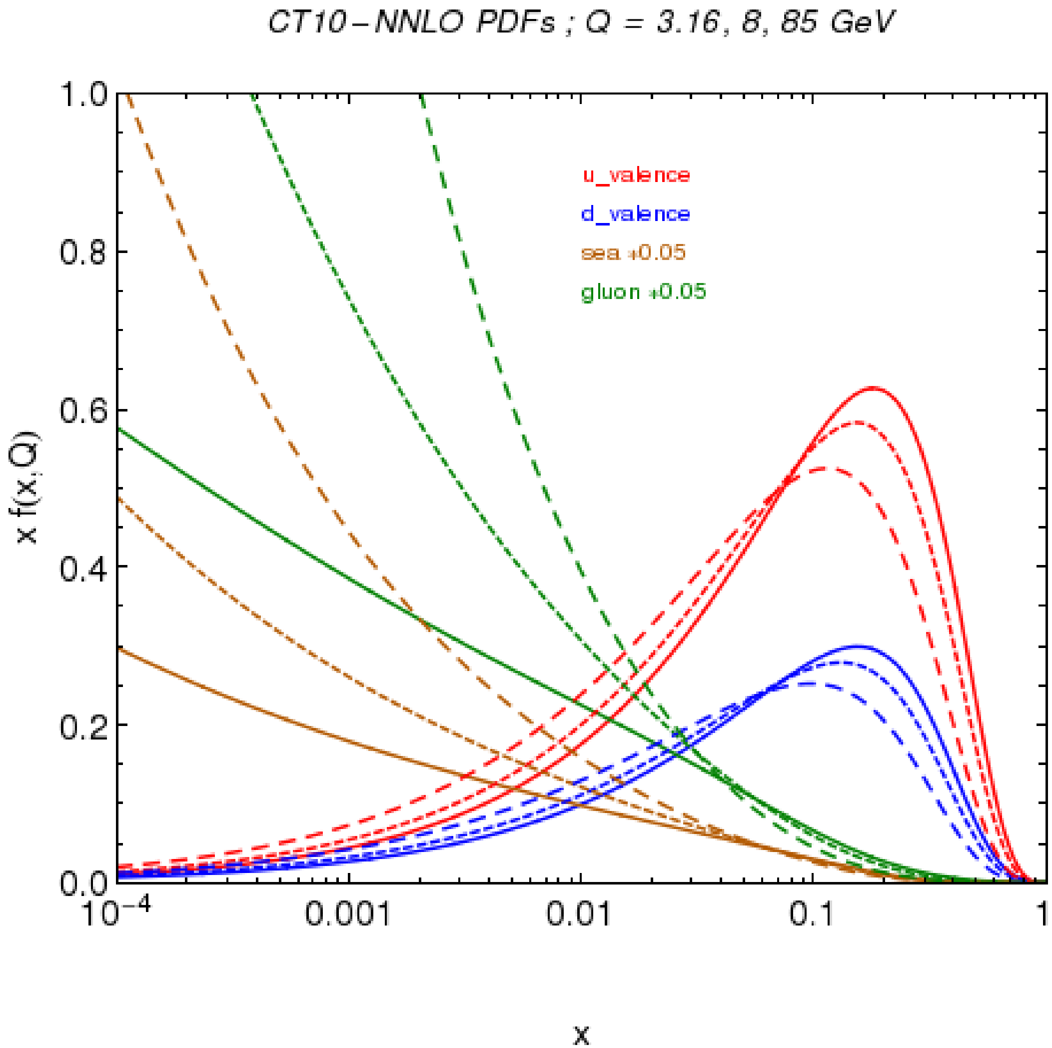


Sea = 2 (db + ub + sb)
u_valence = u – ub ;
u = u_valence + u_sea
ub = u_sea

Structure of the Proton
… as a function of Q

All in One Plot

u_valence
d valence
*sea * 0.05*
*gluon * 0.05*

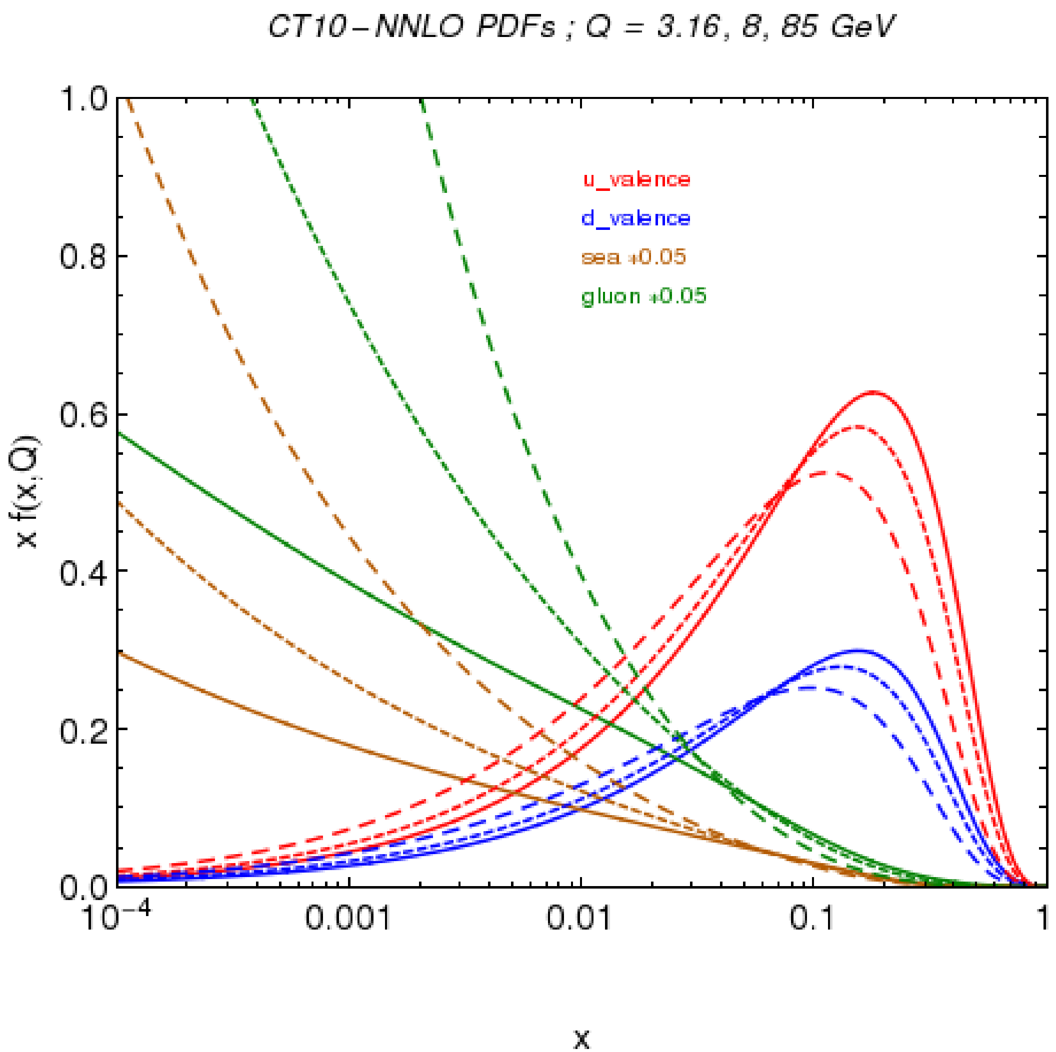


Sea = 2 (db + ub + sb)
u_valence = u – ub ;
u = u_valence + u_sea
ub = u_sea

Exercise:

(A) Use the Durham Parton Distribution Generator (online PDF calculator) to reproduce these graphs.

(B) Show the Q^2 evolution.



next …