An Introduction to the Parton Model and Perturbative QCD

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Overview of the Lectures

Lecture I - Parton Model, QCD, and e^+e^- annihilation

- History and Background
- Parton Model and Relationship to QCD
- Examples from $e^+e^- \to hadrons$
  - Total cross section
  - Angular distribution
  - Inclusive hadron production
- Next order order correction to the total cross section
- Running Coupling
- Differential observables - thrust
- Inclusive hadron production - scale-dependent fragmentation functions

Lecture II - Deeply Inelastic Scattering (DIS)

- Kinematics
- Cross sections and structure functions
- Lowest order results - parton model
- Parton distributions functions (PDFs)
- Higher order corrections
- Factorization schemes
- PDF scale dependence and evolution equations
- QCD-improved parton model

Lecture III - Vector Boson Production

- $l^+l^-, W^\pm,$ and Z production
- Kinematics
- Observables in lowest order
- QCD improved parton model
- p_T distributions and Higher order corrections

Lecture IV - Hadron-hadron production of particles, jets, and photons

- Kinematics
- Observables in lowest order - QCD improved parton model
- Higher order corrections
- More complex observables and the need for Monte Carlo techniques
- Overview of phase space slicing methods

Resources

- Previous introductory lectures by Ellis, Olness, Sterman, Soper, and Tung are online at cteq.org
- My lectures cover more or less the same material, but in in a complementary manner
- I have included material that will be useful for understanding the applications of theory and the comparisons to data
- The Handbook of Perturbative QCD is also online.

History and Background

- 1950s and 60s saw the discovery of many hadronic resonances - hundreds!
  - 1963 – Gell-Mann and Ne'eman – "Eightfold way" applied SU(3) symmetry (flavor) to hadron spectroscopy
  - Particles/resonances organized in decuplets, octets, singlets
  - Common spin, parity C-parity for each member of a group
  - All the observed combinations could be made using members of the fundamental representation - called quarks by Gell-Mann

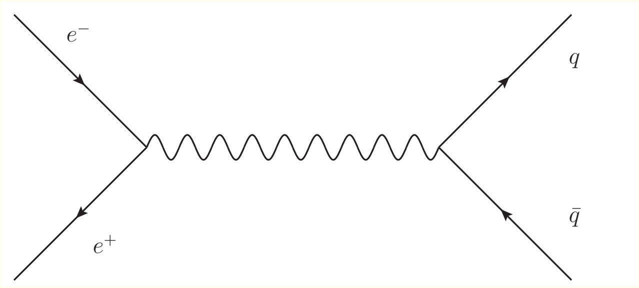
- New accelerators were being designed and built - Fermilab (200 GeV fixed target) and CERN's ISR (Intersecting Storage Ring, proton proton at $\sqrt{s} = 62$ GeV)
  - Needed new theories/models for what would be observed at the new higher energies
  - Feynman explored the consequences of hadrons being made of constituents - partons
- DIS experiments at SLAC - 1968
  - Observed unexpected scaling behavior in their data (to be discussed later)
  - Theoretical insights by J.D. Bjorken
  - Feynman's parton model offered a ready explanation of scaling
  - Gave plausible argument for the reality of quarks

- 1972 - Fritzsch, Leutwyler, and Gell-Mann formulated QCD as a gauge theory of quarks interacting with gluons
- 1973 - Polizter, Gross, and Wilczek demonstrate that QCD possesses "asymptotic freedom" wherein the strong coupling becomes weaker for large momentum transfer processes
- 1974 - Charm discovered at the SLAC e^+e^- storage ring SPEAR.
  - Further demonstration of the reality of quarks
  - e^+e^- demonstrated that parton model ideas worked well for observables involving hadronic final states

By the mid 1970s it was clear that one could use QCD to develop a "QCD improved parton model" that could be applied to large momentum transfer processes

My goal is to investigate simple parton model ideas (basically kinematics) and show how these relate to perturbative QCD calculations in lepton-lepton, lepton-hadron, and hadron-hadron scattering.

$e^+e^- \rightarrow hadrons$



- No free quarks $\longrightarrow$ $q\bar{q}$ final state completely converts to hadrons
- Lowest order result yields

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2 e_q^2}{2s}(1+\cos^2\theta)$$

where θ is the angle between the e^+e^- axis and the $q\bar{q}$ axis in the overall center-of-mass system and e_q is the fraction quark charge in units of e

- The total cross section is

$$\sigma = \frac{4\pi\alpha^2 e_q^2}{3s}$$

- The ratio to the cross section for μ pair production is

$$R \equiv \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_q e_q^2$$

where the factor of three comes from the sum over quark colors

Homework: Derive these results

Data for R confirm the lowest order prediction (up to higher order corrections)

6 40. *Plots of cross sections and related quantities*

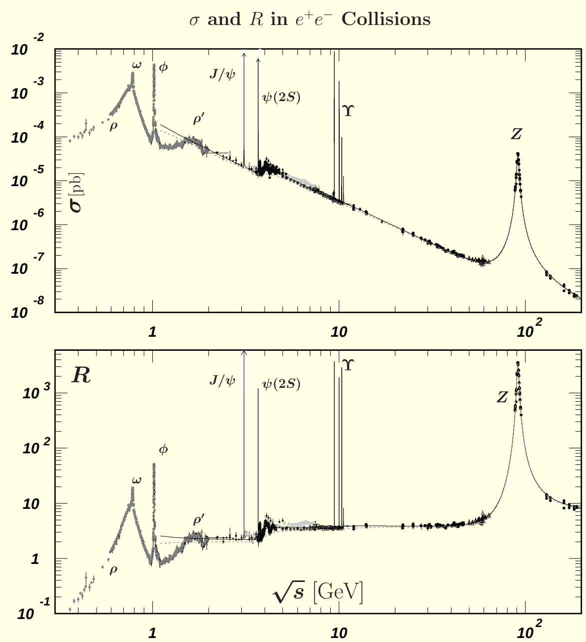
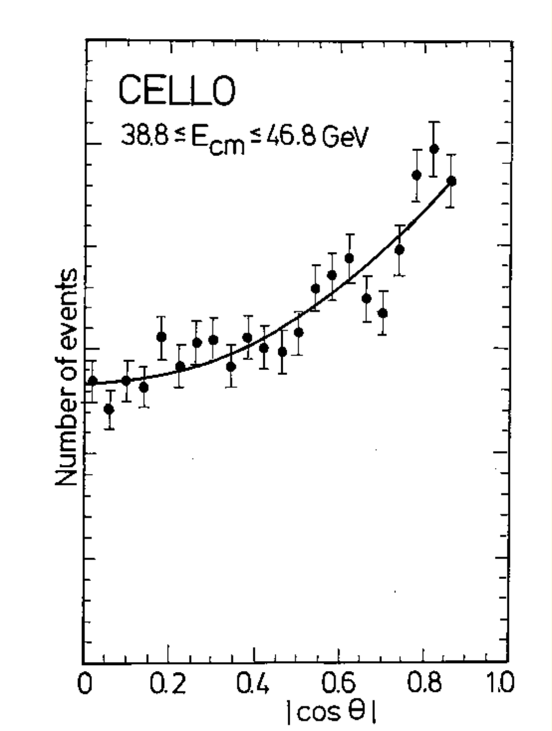


Figure 40.6: World data on the total cross section of $e^+e^- \to hadrons$ and the ratio $R(s) = \sigma(e^+e^- \to hadrons, s)/\sigma(e^+e^- \to \mu^+\mu^-, s)$. $\sigma(e^+e^- \to hadrons, s)$ is the experimental cross section corrected for initial state radiation and electron-positron vertex loops, $\sigma(e^+e^- \to \mu^+\mu^-, s) = 4\pi\alpha^2(s)/3s$. Data errors are total below 2 GeV and statistical above 2 GeV. The curves are an educative guide: the broken one (green) is a naive quark-parton model prediction and the solid one (red) is 3-loop pQCD prediction (see "Quantum chromodynamics" section of this *Review*, Eq. (9.12) or, for more details, K. G. Chetyrkin *et al.*, Nucl. Phys. B **586** (2000) 56 (Erratum *ibid.* B **634** (2002) 413). Breit-Wigner parameterizations of J/ψ, $\psi(2S)$, and $\Upsilon(nS), n = 1, 2, 3, 4$ are also shown. The full list of references to the original data and the details of the R ratio extraction from them can be found in `[arXiv:hep-ph/0312114]`. Corresponding computer-readable data files are available at http://pdg.ihep.su/xsect/contents.html. (Courtesy of the COMPAS(Protvino) and HEPDATA(Durham) Groups, August 2005. Corrections by P. Janot (CERN) and M. Schmitt (Northwestern U.))

- Expect the hadrons to form "jets" - hadrons are roughly collimated along the direction of the parent quark (more on this later)
- Can reconstruct the jet axis which should also be the $q\bar{q}$ axis in the center of mass
- Data agree with the $1 + \cos^2\theta$ prediction



Inclusive Hadron Production

Introduce a parton fragmentation functions $D_{h/c}(z)$ defined so that

$$D_{h/c}(z)\,dz$$

gives the probability of producing a hadron h from a parton c with a fraction z of the parton's momentum between z and $z + dz$.

- The inclusive z distribution for a hadron h in e^+e^- annihilation is given by

$$\frac{1}{\sigma}\frac{d\sigma}{dz} = \sum_q e_q^2 \left[D_{h/q}(z) + D_{h/\bar{q}}(z)\right]$$

  where σ denotes the total cross section.

- Momentum conservation requires

$$\sum_h \int_0^1 dz z D_{h/c}(z) = 1, \quad \text{for each c}$$

QCD and the Parton Model

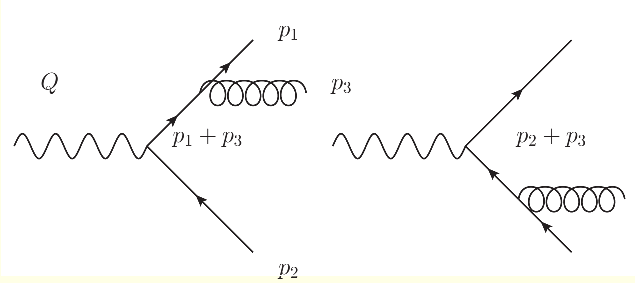
- We have looked at the total cross section, the jet angular distribution, and the inclusive hadron z distribution in the context of the parton model.
- In lowest order, QCD and the parton model give the same results

But QCD allows one to calculate higher order corrections to the basic parton model results

Let's do this for our previously derived results!

e^+e^- annihilation

First, consider the $2 \to 3$ $e^+e^- \to q\bar{q}g$ subprocess. Actually, it is easier to consider the decay of a virtual photon of 4-momentum Q as shown below:



- Kinematics - use massless quarks and gluons.
- Define $x_i = 2E_i/Q, i = 1, 2, 3$ in the overall center-of-mass system where Q denotes the total energy $\Rightarrow x_1 + x_2 + x_3 = 2$.
- $(p_1 + p_3)^2 = 2p_1 \cdot p_3 = (Q - p_2)^2 = Q^2(1 - x_2)$
- $(p_2 + p_3)^2 = 2p_2 \cdot p_3 = (Q - p_1)^2 = Q^2(1 - x_1)$
- The quark propagators from the above diagrams will give factors of $(1-x_1)$ and $(1-x_2)$ in the denominator. $x_1 \to 1$ corresponds to $\vec{p}_3 \parallel \vec{p}_2$ while $x_2 \to 1$ corresponds to $\vec{p}_3 \parallel \vec{p}_1$. Note that if both x_1 and $x_2 \to 1$ then $x_3 \to 0$.

3-body Phase Space

Exercise: Show that

$$dPS_3 = \frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}(2\pi)^4\delta^4(Q-p_1-p_2-p_3)$$
$$= \frac{Q^2}{16(2\pi)^3}\,dx_1\,dx_2$$

Using this result it is straightforward to show that the differential cross section can be written as

$$\frac{1}{\sigma}\frac{d\sigma}{dx_1\,dx_2} = C_F\frac{\alpha_s}{2\pi}\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

where C_F is a color factor equal to 4/3 for QCD.

For the total cross section, one should integrate over both x_1 and x_2. These integrations diverge when either x_1 or x_2 or both approach unity.

Partial fraction the denominators:

$$\frac{1}{(1-x_1)(1-x_2)} = \frac{1}{x_3}\left(\frac{1}{(1-x_1)} + \frac{1}{(1-x_2)}\right)$$

- This shows that the double pole when both x_1 and x_2 approach unity is due to a combination of a collinear divergence (x_1 or $x_2 \to 1$) and a soft divergence ($x_3 \to 0$).
- The problem now is how to generate a finite contribution to the total cross section.
- We shall use dimensional regularization
  - Analytically continue in the number of dimensions from $n = 4$ to $n = 4 - 2\epsilon$.
  - For the soft and collinear singularities we will take $\epsilon < 0$
  - Converts logarithmic divergences into poles in ϵ.
  - Note: we will use the substitution $g_s \to g_s\mu^\epsilon$ in order for the strong coupling to remain dimensionless in n dimensions

Phase space becomes

$$dPS_3^n = \frac{Q^2}{16(2\pi)^3}\left(\frac{Q^2}{4\pi}\right)^{-2\epsilon}\left(\frac{1-u^2}{4}\right)^{-\epsilon}\frac{1}{\Gamma(2-2\epsilon)}x_1^{-2\epsilon}dx_1\, x_2^{-2\epsilon}dx_2$$

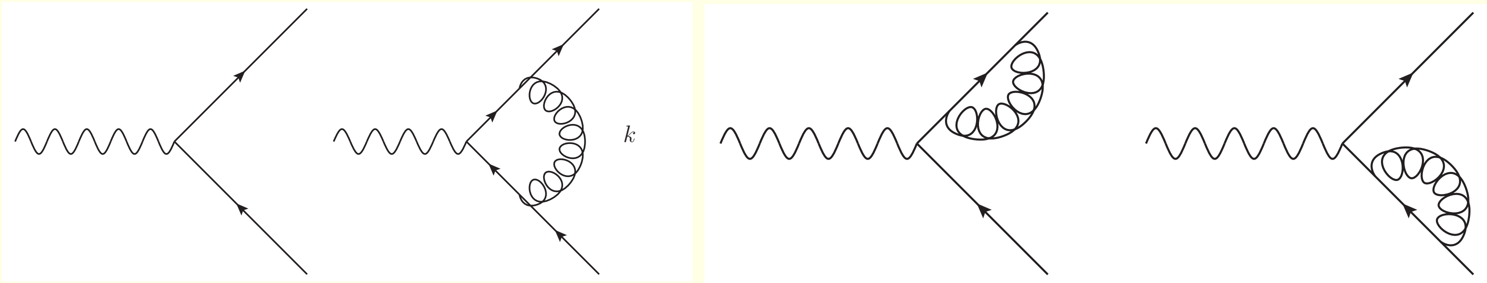
where $u = 1 - \frac{2(1-x_1-x_2)}{x_1 x_2}$

- It is not obvious how this helps until you make a substitution $x_2 = 1 - vx_1$
- The u-dependent term introduces factors of $(1-v)^{-\epsilon}$ and $(1-x_1)^{-\epsilon}$
- dx_2 becomes $x_1 dv$
- Then note that

$$\int_0^1 dx(1-x)^{-1-\epsilon} = \frac{1}{-\epsilon}(1-x)^{-\epsilon}|_0^1 = \frac{1}{-\epsilon}$$

  as long as $\epsilon < 0$.
- The logarithmic divergence has, indeed, been converted into a pole in ϵ.

$2 \longrightarrow 2$ contribution



- The loop graphs are $\mathcal{O}(\alpha_s)$ so the interference with the lowest order term gives an $\mathcal{O}(\alpha_s)$ contribution to the cross section
- For on-shell massless quarks the self-energy loop corrections are zero
- The vertex correction loop integral has a denominator of the form: $k^2(p_1+k)^2(p_2-k)^2$
- The denominator vanishes when $k \rightarrow 0$ or when k is collinear with either p_1 or p_2
- These singularities correspond to the same types as observed for the $q\overline{q}g$ final state
- Can also use dimensional regularization to evaluate the loop contribution in n-dimensions

Final Results

- After doing both of the integrations for the three-body , one arrives at

$$\sigma_3 = \frac{\alpha_s}{2\pi} C_F \sigma_0 \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \frac{2\pi^2}{3}\right]$$

  where σ_0 is the lowest order result.

- After doing the loop integral for the virtual contribution one gets

$$\sigma_v = \frac{\alpha_s}{2\pi} C_F \sigma_0 \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{2\pi^2}{3}\right]$$

- Adding the two together along with the lowest order result yields

$$\sigma = \sigma_0 (1 + \frac{\alpha_s}{\pi})$$

- The poles in ϵ have all cancelled, leaving a finite higher order correction

Running Coupling

The expression for the total cross section now depends on $\alpha_s = \frac{g_s^2}{4\pi}$. But what value should one use for α_s?

Consider a dimensionless observable R measured at some energy scale Q.

- If there are no other mass scales in the problem, R can only depend on the renormalized coupling α_s and the ratio Q/μ where μ is the renormalization scale.
- R should be independent of the choice of μ. This is expressed by a Renormalization Group Equation:

$$\mu^2 \frac{dR}{d\mu^2} = \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s}\right] R = 0$$

- Let $t = \ln\left(\frac{Q^2}{\mu^2}\right)$ and $\beta(\alpha_s) = \mu^2 \frac{\partial \alpha_s}{\partial \mu^2}$

Rewrite the RGE as

$$\left[-\frac{\partial}{\partial t} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right] R(e^t, \alpha_s) = 0$$

To solve this equation, define a running coupling $\alpha_s(Q^2)$ by

$$t = \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} \frac{dx}{\beta(x)}$$

From this definition we can derive (just take appropriate derivatives)

$$\frac{\partial \alpha_s(Q^2)}{\partial t} = \beta(\alpha_s(Q^2)) \text{ and } \frac{\partial \alpha_s(Q^2)}{\partial \alpha_s(\mu^2)} = \frac{\beta(\alpha_s(Q^2)}{\beta(\alpha_s(\mu^2)}$$

Exercise: Show that $R(1, \alpha_s(Q^2))$ is a solution of the RGE.

That is, all of the scale dependence is contained in the running coupling.

α_s is the solution to the following equation

$$t = \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} \frac{dx}{\beta(x)}$$

The β function has a perturbative expansion

$$\beta(\alpha_s) = -b\alpha_s^2(1 + b'\alpha_s + ...)$$

with $b = \frac{(33-2n_f)}{12\pi}$ and $b' = \frac{153-19n_f}{2\pi(33-2n_f)}$
where n_f is the number of active parton flavors.

Exercise: Neglecting b', show that the running coupling is given by

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)bt}$$

Now, let's go back to the e^+e^- hadronic total cross section. The discussion above shows that choosing $\mu = Q$ puts all of the scale dependence into the running coupling:

$$\sigma(Q^2) = \sigma_0 \left(1 + \frac{\alpha_s(Q^2)}{\pi}\right)$$

Let's express this result in terms of $\alpha_s(\mu^2)$ using

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)bt} = \alpha(\mu^2) \sum_{j=0}^{\infty} (-\alpha_s(\mu^2)bt)^j$$

Then we get

$$\sigma(Q^2) = \sigma_0 \left(1 + \frac{\alpha_s(\mu^2)}{\pi} \sum_{j=0}^{\infty} (-\alpha_s(\mu^2)bt)^j\right)$$

In this case the *leading logarithms* depending on μ have been summed into the running coupling $\alpha_s(Q^2)$

- Note how in this example the terms are of the form $\alpha_s(\mu^2)^{j+1}t^j$
- For the leading-log approximation, only the one-loop term (b) is retained.
- What if we had calculated the next term in the perturbative expansion for σ? Denote this by $\sigma_2\alpha_s^2$
- Expressing this in terms of $\alpha_s(\mu^2)$ we have

$$\sigma_2\alpha_s^2(\mu^2)[1-2\alpha_s(\mu^2)bt+\ldots]$$

- Notice that there is one less logarithm per power of α_s than for the lowest order term. We have included some subleading logarithms.
- We must examine other sources of subleading logs. One such place is the expression used for the running coupling - using the two-loop expression for β (keeping the b' term) generates contributions which are down by one logarithm
- So, for next-to-leading-order (NLO) calculations the two-loop term (b') must be retained.

Infrared Safety

- This is an example which will play out over and over in the following – for a suitable defined inclusive observable there is a cancellation between the soft and collinear singularities occurring in the real contributions and those which occur in the loop contributions. We saw that in the total cross section calculation.
- It is imperative that this cancellation be allowed to occur when calculating any observable!
- Care must be taken when designing new observables to insure that they do not distinguish between a configuration of partons and the same one where a soft or collinear parton is added.
- Observables that respect this constraint are called infrared safe observables
- The requirement of infrared safety is a necessary condition for an observable to be calculable in perturbation theory.

Differential Observables

- In order to further test the theory one would like to have more information than that provided buy the total cross section
- The phase space integrations obscure a lot of information that should be tested by comparison with data
- An example of an infrared safe observable - Thrust

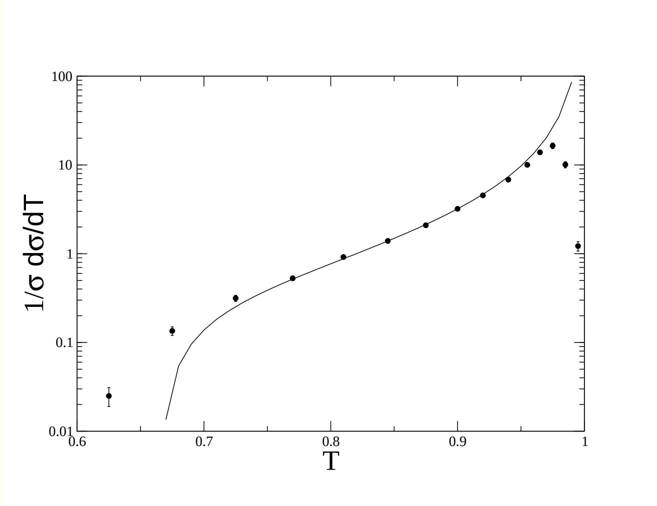
$$T = \max \vec{n} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|}$$

- Vary the choice of the thrust axis $\vec{n}$ in order to maximize T
- 2 parton final state: $\vec{n}$ lies along p_1 and T=1

- If one of the partons emits a collinear parton, then nothing changes and $T = 1$
- If a soft gluon is emitted, then in the limit of zero energy nothing changes and $T = 1$
- The various divergent contributions seen previously all lie at $T = 1$ so that the cancellations still occur
- $T \neq 1$ yields information on the relative angular distributions of the three final state partons
- Note: no jet definition is required in order to study the thrust distribution
- Note: A spherically symmetric multiparton final state: T=1/2
- The thrust distribution is easily calculable: $T = \max[x_i]$ at this order
- Integrate $d\sigma/dx_1\,dx_2$ over x_1 and x_2 subject to the above constraint

- Exercise: show that

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = C_F\frac{\alpha_s}{2\pi}\left[\frac{2(3T^2-3T+2)}{T(1-T)}\ln\left(\frac{2T-1}{1-T}\right) - \frac{3(3T-2)(2-T)}{(1-T)}\right]$$



- Expected divergence as $T \longrightarrow 1$ is evident. Perturbative corrections become large in this region - a better treatment is needed
- The minimum value of T is $2/3$ at this order - need higher order corrections to get to smaller T

Comments on Jet Algorithms

- At lowest order, one associates final state partons with jets. One might therefore expect that the $\mathcal{O}(\alpha_s)$ calculation would shed information on both the 2- and 3-jet cross sections
- In order to define a jet cross section, one needs an infrared safe jet definition
- Such a definition must not distinguish between a parton and two collinear partons or between a parton and a parton plus a soft parton
- Examples include Sterman-Weinberg jets, cone jets, k_T algorithms, and many more
- The key point I wish to make is that whatever algorithm is used, it must allow for the cancellation between the soft and collinear singularities from the real emission graphs for an $n-$body process and those from the $(n-1)$-body virtual graphs
- This point will become of great importance when we discuss NLO programs based on phase space slicing techniques

Fragmentation Functions

- What if one wants to study the hadronic composition of the final state?
- What if you don't want to use a jet observable which depends on choosing a specific jet algorithm?
- What if your detector is optimized for particle detection, but not for reconstructing jets?
- One solution is to introduce Fragmentation Functions (FFs) as was discussed earlier. We saw that

  $D_{h/c}(z)\,dz$ is the probability of getting a hadron h from a parton c with a fraction of the parton's momentum fraction between z and $z + dz$

- Our previous result for the lowest order form for $e^+e^- \longrightarrow hadrons$ is

$$\frac{1}{\sigma}\frac{d\sigma^h}{dz} = \sum_q e_q^2\left[D_{h/q}(z) + D_{h/\overline{q}}(z)\right]$$

- How can we extend this concept to include the $\mathcal{O}(\alpha_s)$ corrections that we have been studying?

Virtual Contributions

- These are easy, as we have already done the work! The virtual contribution, σ_v, calculated previously has the same final state structure as the lowest order term.
- It can be included along with the lowest order term by just multiplying the previous expression by $1 + \frac{\sigma_v}{\sigma_0}$

Three-body Contribution

- This one is more complicated - there are now three partons in the final state and each can give rise to hadrons
- Not only do we have quark and antiquark FFs, but now we also have to include a possible gluon FF.
- The basic structure should be familiar:

$$\frac{d\sigma}{dz} = \frac{1}{2Q}\,(\text{PhaseSpace})\,(\text{Squared matrix elements})\,(\text{FFs})$$

- Of course, this is all done in n dimensions in order to regularize the soft and collinear divergences

- Bear with me - this is going to get complicated, but there is a reason for all of this
- Plugging the appropriate terms into the above expression yields

$$
\begin{aligned}
\frac{d\sigma}{dz} &= \frac{1}{2Q}\frac{Q^2}{16(2\pi)^3}\left(\frac{Q^2}{4\pi\mu^2}\right)^{-2\epsilon}\int\int\left(\frac{1-u^2}{4}\right)^{-\epsilon}\frac{1}{\Gamma(2-2\epsilon)}x_1^{-2\epsilon}dx_1\,x_2^{-2\epsilon}dx_2 \\
&\frac{8(e\,g\,\mu^{2\epsilon})^2}{(1-x_1)(1-x_2)}[(n-2)(x_1^2+x_2^2)+2(n-4)(n-2)(2(1-x_1-x_2)+x_1x_2)] \\
&e_q^2\left[D_{h/q}(y)\delta(z-yx_1)+D_{h/\overline{q}}(y)\delta(z-yx_2)+D_{h/g}(y)\delta(z-yx_3)\right]dy
\end{aligned}
$$

- We recognize some familiar structures from the total cross section calculation, but the structure is more complex
- How can we do the integrations with the unknown FFs in the integrands?
- How can we ensure that the proper soft and collinear cancellations take place?
- Proceed as in the total cross section case. Consider the first term and make the substitution $x_2 = 1 - vx_1$.

- This introduces factors of $(1-x_1)^{-\epsilon}$ and $(1-v)^{-\epsilon}$
- The v integrations can be done using

$$\int_0^1 dv\, v^{n-1}\,(1-v)^{m-1} = B(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}$$

- This generates explicit poles in ϵ through terms like

$$B(-\epsilon, 1-\epsilon) = \frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} = -\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}$$

- Can do the y integration using the δ function

- We are left with terms like

$$\int_z^1 dx_1\, x_1^{n-1}\, (1-x_1)^{m-1} D_{h/q}(z/x_1)$$

- Note the non-zero lower limit on the integral which is forced by the argument of the FF
- How are we to do this integral in order to pull out the singular terms when we don't know the analytic form for the FF?
- Enter the "+" distribution!
- This distribution will enable us to extract the poles in ϵ from integrals of the above form

Consider

$$
\begin{aligned}
I &= \int_0^1 dw\,(1-w)^{-1-\epsilon}\,f(w) \\
&= \int_0^1 dw\,(1-w)^{-1-\epsilon}\,[f(1)+(f(w)-f(1))] \\
&= -\frac{f(1)}{\epsilon} + \int_0^1 dw\,\frac{f(w)-f(1)}{1-w}\,\left[1-\epsilon\ln(1-w)+\mathcal{O}(\epsilon^2)\right] \\
&= -\frac{f(1)}{\epsilon} + \int_0^1 dw\,\frac{f(w)-f(1)}{1-w} - \epsilon\int_0^1 dw\,\frac{\ln(1-w)}{1-w}\,[f(w)-f(1)] + \mathcal{O}(\epsilon^2) \\
&\equiv -\frac{f(1)}{\epsilon} + \int_0^1 dw\,\frac{f(w)}{(1-w)_+} - \epsilon\int_0^1 dw\,\left(\frac{\ln(1-w)}{1-w}\right)_+ f(w) + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

This last expression allows us to make the following identification

$$
(1-w)^{-1-\epsilon} = -\frac{\delta(1-w)}{\epsilon} + \frac{1}{(1-w)_+} - \epsilon\left(\frac{\ln(1-w)}{1-w}\right)_+
$$

- The astute reader will no doubt have noticed that the previous derivation involved integrals extending from zero to one. What if the lower limit is non-zero?
- The derivation can be repeated and the only difference will be in the δ-function term. There we will get (recall that $\epsilon < 0$)

$$\begin{aligned}
\frac{1}{\epsilon}(1-w)^{-\epsilon}\big|_a^1 &= -\frac{1}{\epsilon}(1-a)^{-\epsilon} \\
&= -\frac{1}{\epsilon}\left[1-\epsilon\ln(1-a)+\frac{\epsilon^2}{2}\ln^2(1-a)+\dots\right] \\
&= -\frac{1}{\epsilon}+\ln(1-a)-\frac{\epsilon}{2}\ln^2(1-a)+\dots
\end{aligned}$$

- The regulators under the integral signs behave the same way as when the lower limit was zero.

- Schematically we can write

$$\frac{1}{(1-w)_+} = \frac{1}{(1-w)_a} + \ln(1-a)\delta(1-w)$$

and

$$\left(\frac{\ln(1-w)}{1-w}\right)_+ = \left(\frac{\ln(1-w)}{1-w}\right)_a + \frac{1}{2}\ln^2(1-a)\delta(1-w)$$

There are several important points to notice about these regulators

- We derived these expressions by adding and subtracting $f(1)$ and then rearranging the integrations. When the lower limit is non-zero, the cancellation between these two terms with $f(1)$ is no longer exact and there is a remainder involving logs of $(1-a)$
- As the lower limit, a, approaches 1 these logs can become large.
- This could happen with the fragmentation functions if we were interested in the region of large z.

- These logs are called “threshold” logs and physically what is happening is that the phase space for additional gluon radiation is being limited by the requirement that z be large. These large logs must be resummed via a procedure referred to as “soft gluon” or “threshold” resummation
- Remember the idea of incomplete cancellation between the virtual and real contributions with a finite remainder consisting of potentially large logarithms
- After this interlude, we can go back to the fragmentation calculation

- We have done the y and v integrations, leaving integrals of the form

$$\int_z^1 dx_1\, x_1^{n-1}\, (1-x_1)^{m-1} D_{h/q}(z/x_1)$$

- Terms with $m = -\epsilon$ will give poles proportion to $\delta(1-x_1)$
- Doing the x_1 integration will give pole terms proportional to $D_{h/q}(z)$ which can now be combined with the lowest order terms
- In this way we can extract the divergent pieces needed for the cancellation with the virtual contributions

- The intermediate answer for the next order contribution after adding the virtual contribution to the three real fragmentation pieces is as follows

$$
\begin{aligned}
\frac{d\sigma}{dz} = \frac{\alpha_s}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \\
\sum_q e_q^2 \int dx\, dy\, \delta(z-xy) \left(D_{h/q}(y) + D_{h/\overline{q}}(y)\right) \\
\left[\delta(1-x)\left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + \frac{2}{\epsilon^2} + \frac{3}{2\epsilon}\right) - \frac{1}{\epsilon}\frac{1+x^2}{(1-x)_+} + \tilde{f}_q(x)\right] \\
+ \quad \frac{\alpha_s}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \sum_q e_q^2 \int dx\, dy\, \delta(z-xy)\, 2\, D_{h/g}(y) \\
\left(-\frac{1}{\epsilon}\frac{1+(1-x)^2}{x} + \tilde{f}_g(x)\right)
\end{aligned}
$$

- See that the $\frac{1}{\epsilon^2}$ terms cancel, but that there are some remaining $\frac{1}{\epsilon}$ pieces
- How should these be interpreted and what can we do about them?

- These are residual collinear singularities associated with the quark propagators going on shell in the collinear quark+gluon configuration
- On-shell propagators are associated with long range physics and should not be associated with the hard scattering correction that we are calculating.
- Factorize the remaining collinear singularities and absorb them into the bare FFs.
- We need to define a scheme to tell us how much of the finite contributions to subtract along with the ϵ pole terms. Use

$$\frac{1}{\epsilon}\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} = \frac{1}{\epsilon} + \ln(4\pi) - \gamma_E - \ln\frac{M_f^2}{\mu^2} - \ln\frac{Q^2}{M_f^2}$$
$$+ \ldots$$

- The MS scheme (Minimal Subtraction) says to subtract only the pole term
- The $\overline{\mathrm{MS}}$ scheme (Modified Minimal Subtraction) says to also subtract the $\ln(4\pi) - \gamma_E$ terms

- In addition, I have introduced a factorization scale M_f and I will subtract the $\ln \frac{M_f^2}{\mu^2}$ term, as well. Technically, each choice of M_f defines a new scheme, but we usually refer to all of them as being the $\overline{\mathrm{MS}}$ scheme
- In order to absorb the collinear singularities in the bare FFs, introduce a scale-dependent FF

$$
\begin{aligned}
D_{h/q}(z, M_f^2) &= D_{h/q}(z) + \\
&\frac{\alpha_s}{2\pi} \int dx\, dy\, \delta(z - xy) \left(-\frac{1}{\epsilon} \right) \left(\frac{M_f^2}{4\pi\mu^2} \right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \\
&\left[C_F \left(\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right) D_{h/q}(y) + C_F \frac{1+(1-x)^2}{x} D_{h/g}(y) \right]
\end{aligned}
$$

- Replacing the bare FFs by the above scale-dependent FFs will cancel the remaining collinear singularities

- The final result takes the form

$$\frac{1}{\sigma}\frac{d\sigma}{dz} =$$

$$\sum_q e_q^2 \left[D_{h/q}(z, M_f^2) + D_{h/\overline{q}}(z, M_f^2)\right]$$

$$+ \quad \frac{\alpha_s}{2\pi} \sum_q e_q^2 \int_z^1 \frac{dx}{x} \left[D_{h/q}(\frac{z}{x}, M_f^2) + D_{h/\overline{q}}(\frac{z}{x}, M_f^2)\right] \left[\ln \frac{Q^2}{M_f^2} P_{qq}(x) + \tilde{f}_q\right]$$

$$+ \quad \frac{\alpha_s}{2\pi} \sum_q e_q^2 \int_z^1 \frac{dx}{x} \, 2D_{h/g}(\frac{z}{x}, M_f^2) \left[\ln \frac{Q^2}{M_f^2} P_{gq}(x) + \tilde{f}_g\right]$$

- Here I have used two splitting functions defined as

$P_{qq}(x) = C_F \left(\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right)$ and $P_{gq}(x) = C_F \left(\frac{1+(1-x)^2}{x}\right)$

- Note that I have substituted the scale-dependent FFs on the right hand side - this is allowed to this order
- Note that the results simplify considerably if we choose $M_f = Q$
- In this case the $\ln \frac{Q^2}{M_f^2}$ terms disappear and all of the logs have been absorbed in the scale-dependent FFs

Summary

In this lecture we have seen the following

- The typical ingredients for the hard scattering subprocesses include
  - The lowest order expressions for the relevant subprocesses
  - The 1-loop virtual corrections to these subprocesses
  - The expressions for the relevant next order subprocesses
- The real processes generally have both soft and collinear singularities
- After renormalization, the loop graphs also contribute soft and collinear singularities
- For suitable observables these singularities cancel, leaving finite higher order corrections
- Observables for which this occurs are said to be infrared safe
- If one wants to study specific details of the hadronic final state, then Fragmentation Functions can be introduced.
- In the next order there will be uncancelled collinear singularities which can be absorbed into the bare FFs by defining scale-dependent FFs