An Introduction to the Parton Model and Perturbative QCD

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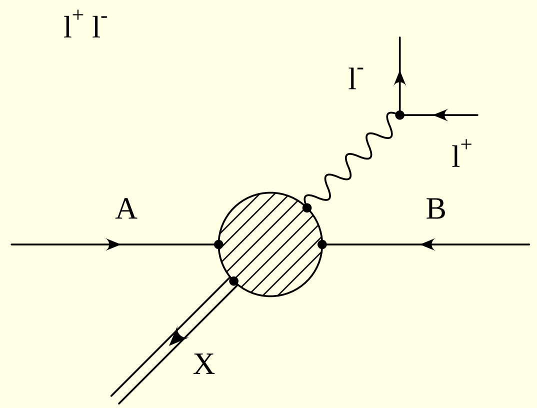
Lecture III - Outline

- $l^+l^-, W^\pm,$ and Z production
- Kinematics
- Observables in lowest order
- QCD improved parton model
- p_T distributions and Higher order corrections

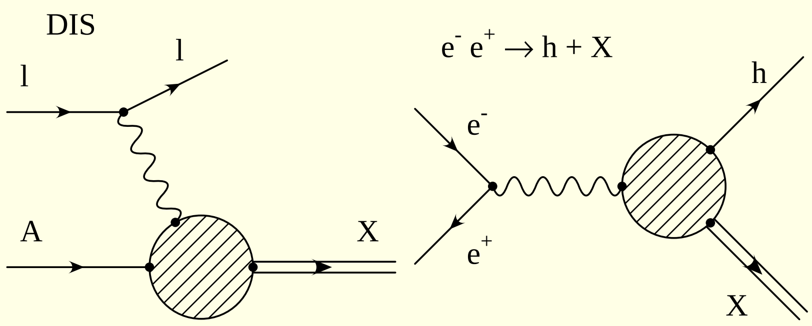
Lepton Pair Production ($l^+l^-, l^+\nu,$ or $l^-\bar{\nu}$)

Original idea due to Drell and Yan: S.D. Drell and T.-M. Yan, PRL **25**, 316 (1970)

- Electromagnetic probe of a hadron-hadron process
- Compare to
  - DIS: E-M probe of a single hadron process
  - e^+e^-: E-M probe of hadron production
- Simple description in terms of the (then new) parton model
- Mass of the pair could be varied to insure that the parton momentum fractions were neither too small nor too large (avoid problems with x near 0 or 1)

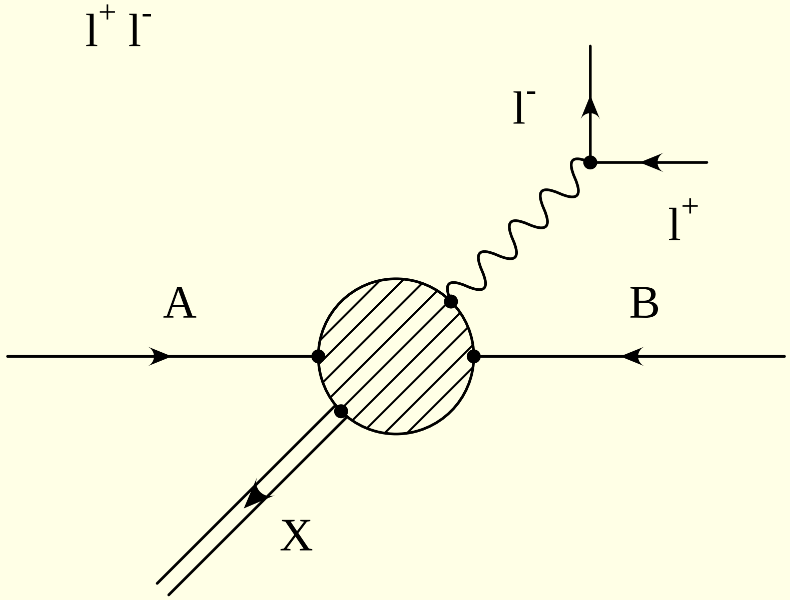


- Basic Feynman diagram for lepton pair production.
- The task is to figure out what is in the blob
- Compare to DIS and e^+e^- annihilation



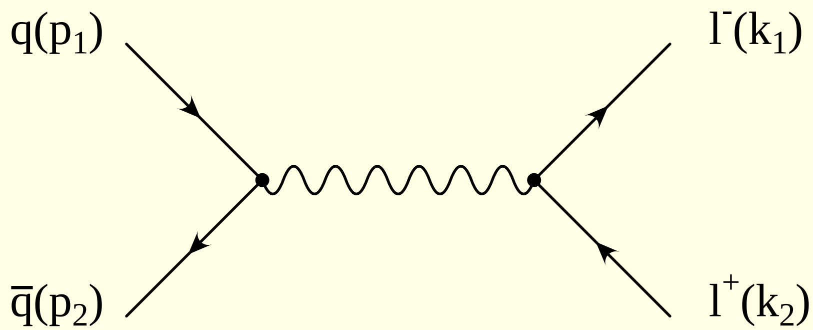
- Will see that the structure of the parton model calculation preserved in the presence of QCD corrections
- First example of a calculable hadron-hadron process in the context of the parton model
- Process is of historical interest (2 Nobel prizes)
- Pedagogical importance - one of the early calculations of higher order QCD corrections
- Important for precision Standard Model measurements
- Excellent probe of $\bar{q}$ PDFs
- Can help determine PDFs in pions and kaons
- Important roles in searches for new physics

Basic Idea



- Producing a virtual boson of mass Q
- Our task is to figure out what is in the shaded circle in the figure
- Simplest possibility: $q\bar{q} \rightarrow l^+l^-$
- Represents purely E-M process in the context of the parton model (treating the quarks as free)

Born Term



Lorentz invariant variables

$$\begin{aligned}\hat{s} &= (p_1 + p_2)^2 = (k_1 + k_2)^2 \\ \hat{t} &= (p_1 - k_1)^2 = (p_2 - k_2)^2 \\ \hat{u} &= (p_1 - k_2)^2 = (p_2 - k_1)^2\end{aligned}$$

The matrix element is $M = e_q \frac{e^2}{\hat{s}} \overline{u}(k_1)\gamma_\mu v(k_2)\overline{v}(p_2)\gamma^\mu u(p_1)$ and Appendix I shows how to obtain the parton-level cross section result

$$\sigma(q\overline{q} \to l^+ l^-) = \frac{4\pi\alpha^2}{9\hat{s}} e_q^2 \equiv \sigma_0.$$

Hadronic Cross Section

Convolute the parton-level cross section σ_0 with the appropriate quark and antiquark parton distribution functions:

$$\sigma(AB \to l^+l^- + X) = \sum_q \int dx_a dx_b \, \sigma_0 \left[q(x_a)\overline{q}(x_b) + a \leftrightarrow b \right]$$

Note: remember to symmetrize with respect to the beam and target particles. This corresponds to $\hat{t} \leftrightarrow \hat{u}$ here, so σ_0 is unchanged.

Differential Distributions

The total cross section involves a convolution with products of parton distributions. In order to test the theory or to learn more about the parton distributions it has proven to be convenient to undo one or both of the integrations by looking a differential distributions. If we ignore external hadronic masses, we can relate the hadronic and partonic center of mass energies as follows:

$$\hat{s} = (p_1 + p_2)^2 = 2p_1 \cdot p_2 = 2(x_a p_A) \cdot (x_b p_B) = x_a x_b s$$

where it has been assumed that $p_1 = x_a p_A$ and $p_2 = x_b p_B$.

Lepton pair mass distribution

For $q\overline{q} \to l^+ l^-$ the invariant mass of the lepton pair is just $Q^2 = \hat{s}$. Thus,

$$\frac{d\sigma}{dQ^2} \;=\; \sum_q \int dx_a dx_b \, H_q(x_a, x_b) \, \sigma_0 \, \delta(Q^2 - \hat{s})$$

Here the sum over the products of parton distributions is denoted by the function $H_q(x_a, x_b)$. Next, evaluate the δ function as follows:

$$\int dx_a dx_b \delta(Q^2 - x_a x_b s) = \int \frac{dx_a}{x_a s} \delta(x_b - Q^2/x_a s)$$

Thus,

$$\frac{d\sigma}{dQ^2} = \sum_q \int \frac{dx_a}{x_a s} H_q(x_a, \frac{Q^2}{x_a s}) \frac{4\pi\alpha^2}{9Q^2} e_q^2$$

Scaling

It is convenient to define a dimensionless parameter $\tau = Q^2/s = x_a x_b$ so that $0 \leq \tau \leq 1$. Then, we can write

$$Q^4 \frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{9} \sum_q \int_\tau^1 \frac{dx_a}{x_a} \; \tau H_q(x_a, \frac{\tau}{x_a}) e_q^2$$

- $Q^4 \frac{d\sigma}{dQ^2}$ is dimensionless
- Righthand side is a function only of τ at this level of approximation
- Plot $Q^4 \frac{d\sigma}{dQ^2}$ or $(Q^3 \frac{d\sigma}{dQ})$ for different values of s
- Should lie on a universal "scaling" curve
- Approximate scaling is observed in the data

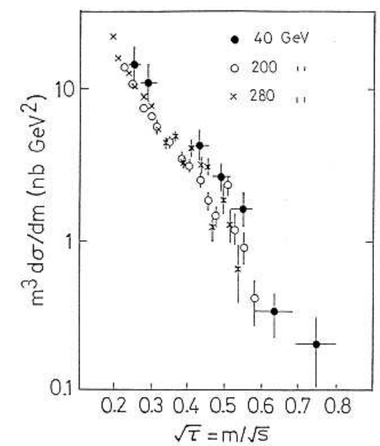


Fig. 5.13. Approximate scaling of $m^3 d\sigma/dm$ for Drell-Yan pair production in $\pi^- p$ scattering with lab momentum from 40 to 280 GeV

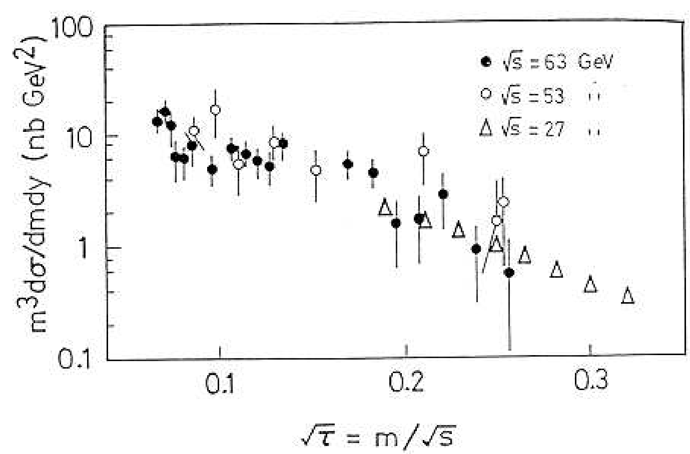


Fig. 5.14. Approximate scaling of $m^3 d\sigma/dm\,dy(y=0)$ for Drell-Yan pair production in pp scattering [Phys. Lett. 91B, 475 (1980)].

Longitudinal Momentum Distributions

- $x_F = p_l/p_{lmax} \approx 2p_l/\sqrt{s}$ where p_l is the lepton pair longitudinal momentum in the hadron-hadron cms.

- Parton 4-vectors and lepton pair energy and longitudinal momentum

$$\begin{array}{ll} p_1 = x_a \frac{\sqrt{s}}{2}(1,0,0,1) & p_2 = x_b \frac{\sqrt{s}}{2}(1,0,0,-1) \\ E = \frac{\sqrt{s}}{2}(x_a + x_b) & p_l = \frac{\sqrt{s}}{2}(x_a - x_b) \end{array}$$

- These yield $x_F = x_a - x_b$.

- One can use this to define a double differential cross section

$$\frac{d\sigma}{dQ^2 dx_F} = \frac{4\pi\alpha^2}{9Q^4} \sum_q e_q^2 \int_\tau^1 \frac{dx_a}{x_a}\ \tau H_q(x_a, \frac{\tau}{x_a}) \delta(x_F - x_a + \frac{\tau}{x_a})$$

- $\delta(x_F - x_a + \frac{\tau}{x_a})$
- The δ function constraint can be solved for x_a yielding

$$x_a = \frac{1}{2}\left(x_F + \sqrt{x_F^2 + 4\tau}\right).$$

- Using $x_b = \tau/x_a$ one derives

$$x_b = \frac{1}{2}\left(-x_F + \sqrt{x_F^2 + 4\tau}\right).$$

- The Jacobian factor from the δ function introduces a factor of $x_a/(x_a + x_b)$ so the final result can be written as

$$\frac{d\sigma}{dQ^2 dx_F} = \frac{4\pi\alpha^2}{9Q^4}\frac{1}{\sqrt{x_F^2 + 4\tau}}\tau \sum_q e_q^2 H_q(x_a, \frac{\tau}{x_a}).$$

Rapidity

- Rapidity is defined as

$$y = \frac{1}{2} \ln \frac{E + p_l}{E - p_l} = \frac{1}{2} \ln x_a x_b$$

- Exercise: Show that $x_a = \sqrt{\tau} e^{y}$ and $x_b = \sqrt{\tau} e^{-y}$.
- Changing variables from (Q^2, x_F) to (y, τ) is done using

$$dQ^2 dx_F = dy d\tau \; s\sqrt{x_F^2 + 4\tau}$$

- This results in

$$\frac{d\sigma}{dy d\tau} = \frac{4\pi\alpha^2}{9s} \sum_q \frac{e_q^2}{\tau} H_q(x_a, x_b)$$

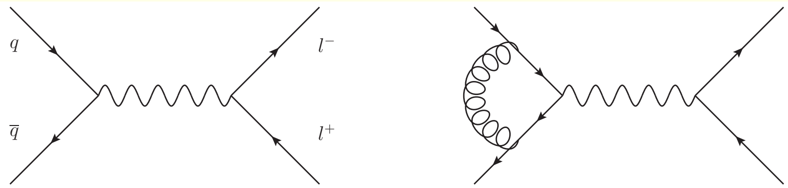
QCD improved parton model

- Following the discussion in Lecture II, there is only one change needed – Replace the PDFs with *scale-dependent* PDFs

$$q(x) \to q(x, M_f^2)$$

- But, what is the scale M_f? That depends on the observable.
- For $\frac{d\sigma}{dQ^2}$, $\frac{d\sigma}{dydQ^2}$, or $\frac{d\sigma}{dx_F dQ^2}$ there is only one large scale – Q
- This one change yields the LO QCD predictions with leading-log PDFs

Next Order Correction to $\frac{d\sigma}{dQ^2}$



Let's write the lowest order expression for the cross section as follows:

$$\frac{d\sigma}{dQ^2} = \frac{\sigma(Q^2)}{s} \int \frac{dx_a}{x_a} \frac{dx_b}{x_b} \sum e_q^2 \left[q(x_a)\overline{q}(x_b) + q \leftrightarrow \overline{q}\right] \delta(1-z)$$

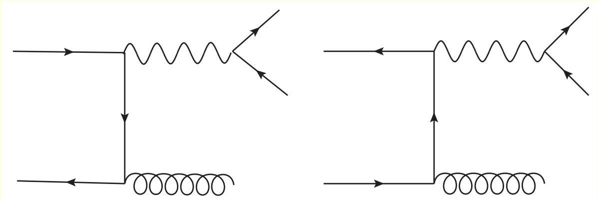
where $z = \frac{Q^2}{x_a x_b s}$

Now, consider the virtual corrections - these are the same as in the e^+e^- example. One simply replaces $\delta(1-z)$ in the above expression by

$$\delta(1-z)\left[1 + \frac{\alpha_s}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{2\pi^2}{3}\right]\right]$$

Next, we must consider the contributions from the Compton and annihilation subprocesses

Annihilation Contribution



- By now, the steps should be familiar - square the matrix element in n-dimensions, multiply by 3-body n-dimensional phase space, and divide by the flux factor
- Perform the relevant phase space integrations using the changes of variables and the "+" distributions outlined in the e^+e^- case
- Add to the preceding results for the lowest order and virtual contributions
- The ϵ^{-2} terms will cancel, as will some of the ϵ^{-1} terms, leaving some residual ϵ pole terms.
- Factorize these remaining singular terms and absorb them into the bare PDFs, leaving a residual finite $\mathcal{O}(\alpha_s)$ correction
- As before, the factorization of the initial state collinear singularities will be facilitated by the introduction of a mass factorization scale M_f

The full annihilation contribution, including the lowest order and virtual contributions, at the parton level prior to the mass factorization step is

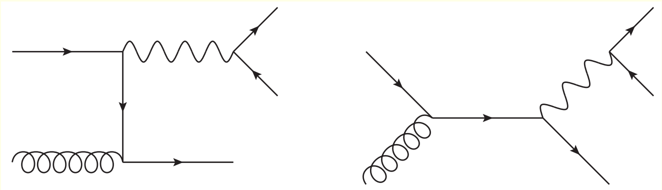
$$\delta(1-z)+\frac{\alpha_s}{2\pi}C_F\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\delta(1-z)\left(-\frac{3}{\epsilon}-8+\frac{2\pi^2}{3}\right)\right.$$
$$\left.-\frac{2}{\epsilon}\frac{1+z^2}{(1-z)_+}+4(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+-2\frac{1+z^2}{1-z}\ln z\right]$$

One can recognize the familiar splitting function $P_{qq}(z)$ in this expression. The result can be simplified to

$$\delta(1-z)-\frac{2}{\epsilon}\frac{\alpha_s}{2\pi}\left(\frac{M_f^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}P_{qq}(z)+\frac{\alpha_s}{2\pi}2P_{qq}(z)\ln\frac{Q^2}{M_f^2}+\frac{\alpha_s}{2\pi}f_q(z)$$

where $f_q(z)$ represents a finite $\mathcal{O}(\alpha_a)$ correction as in the DIS example and I have kept the factorization scale dependent term separate from f_q

Compton Subprocess



- The same procedure as outlined on the preceding slides is followed for the Compton subprocess
- This time there is no lower order term and there are no virtual corrections
- The only singularity is the collinear singularity associated with the gluon splitting vertex
- The full Compton result using the same normalization as for the annihilation result is

$$-\frac{1}{\epsilon}\frac{\alpha_s}{2\pi}P_{qg}(z)\left(\frac{M_f^2}{4\pi\mu^2}\right)^{-\epsilon}+\frac{\alpha_s}{2\pi}P_{qg}(z)\ln\left(\frac{Q^2}{M_f^2}\right)+\frac{\alpha_s}{2\pi}f_g(z)$$

- At this point the final step is to factorize the remaining collinear terms into the bare PDFs
- This is easily done using the expressions given previously as I have already isolated the appropriate subtraction terms for the $\overline{\mathrm{MS}}$ scheme.
- Restoring the full normalization for the cross section we get

$$
\begin{aligned}
\frac{d\sigma}{dQ^2} \quad = \quad & \frac{\sigma(Q^2)}{s} \int \frac{dx_a}{x_a} \frac{dx_b}{x_b} \left[\sum_q e_q^2 \left[q(x_a, M_f^2) \overline{q}(x_b, M_f^2) + a \leftrightarrow b \right] \right. \\
& \left. \cdot \left[\delta(1-z) + \frac{\alpha_s}{2\pi} \left(2P_{qq}(z) \ln\left(\frac{Q^2}{M_f^2} \right) + f_q(z) \right) \right] \right. \\
+ \quad & \sum_q e_q^2 \left[\left(q(x_a, M_f^2) + \overline{q}(x_a, M_f^2) \right) g(x_b, M_f^2) + a \leftrightarrow b \right] \\
& \left. \frac{\alpha_s}{2\pi} \left(P_{qg}(z) \ln\left(\frac{Q^2}{M_f^2} \right) + f_g(z) \right) \right]
\end{aligned}
$$

The relatively simple expression on the previous page contains many of the elements that characterize NLO calculations in general

- The chosen form strongly suggests the choice $M_f = Q$ which follows from the fact that the phase space factor $\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}$ is given in terms of Q^2 which sets the natural scale for the process
- There is explicit M_f dependence in the NLO term which partially cancels that contained in the lowest order term
- To see this, take a derivative with respect to $\ln M_f^2$ of the cross section expression
  - The derivative of $q(x_a, M_f^2)$ gives a contribution of

    $$\frac{\alpha_s}{2\pi}\left[P_{qq} \otimes q + P_{qg} \otimes g\right]$$

    where $\otimes$ is shorthand for the convolution of the PDF and splitting function. This follows from the DGLAP equations for the scale dependence of the PDFs
  - The derivative of the NLO correction gives a similar term, but with a minus sign coming from the $\ln\left(\frac{Q^2}{M_f^2}\right)$ factors

- The cancellation is not exact, but is correct up to the next order in α_s (Exercise: Show this)
- This is a feature which is typical of NLO calculations and is one of the reasons for why they are important - they generally, but not always, feature a decreased scale dependence relative to the leading-order calculation

For completeness here are the remaining factors in the NLO calculation

$$\begin{aligned} f_q(z) &= C_F \Big[\delta(1-z)(-8+\frac{2\pi^2}{3}) \\ &\quad +4(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - 2\frac{1+z^2}{1-z}\ln z \Big] \end{aligned}$$

and

$$f_g(z) = \frac{1}{2}\left[\ln\frac{1+z^2}{z}\left[z^2+(1-z)^2\right] + \frac{1}{2} + 3z - 7z^2\right]$$

Comments

- If you compare the expressions for the fs in the Lepton Pair Production case to those in the DIS case for F_2 you will see some features in common, but also some differences
  - In both cases there are combinations of delta function terms, plus regulators, and other z-dependent terms
  - The dependence on the plus regulators is different
- The differences stem from the fact that in both case we are integrating over an additional parton in the final state, but the phase space is different in the two cases - we have an spacelike photon in the initial state for one and a timelike photon in the final state for the other
- Remember that the plus regulators are related to the limitations placed on parton emission near threshold and these constraints are different in the two cases

- The change from space-like to time-like Q^2 also affects terms involving $\ln Q^2$ since the argument will be negative for one of the processes.
- $Re(-1)^{-\epsilon} = Re\exp(-i\pi\epsilon) = 1 - \epsilon^2\pi^2 + \cdots$
- This multiplies ϵ^{-2} and so generates a contribution proportional to π^2
- Historically, the existence of these π^2 terms played an important role in understanding the QCD description of these two processes

This brings us to the idea of the infamous "K Factors"

Aside: Why infamous? Because they don't have a unique definition and they aren't true factors!!

K factors

- The idea of K factors started innocently enough. In the 1980s the early Lepton Pair Production experimental results were compared with existing predictions based on leading order PDFs and the lowest order hard scattering expressions
- The results were given as the ratio of the data to the predictions and this ratio was called the K factor, *i.e.* the amount one would have to multiply the theoretical predictions by in order to describe the data
- The early comparisons showed that this result was about 2, which seemed like a real problem for QCD
- The explanation came when NLO calculations became available
- Almost all of the NLO correction is associated with a large contribution proportional to $\delta(1-z)$
- To understand this requires several steps...

- First, consider that leading order PDFs fitted to DIS data (that was all we had at first) essentially have all of the higher order corrections absorbed into the PDFs themselves
- This would be equivalent to using the DIS factorization convention where f_2^q and f_2^g are absorbed into the PDFs
- But then, when one calculates the Lepton Pair Production cross section these DIS corrections must be removed from the PDFs. In this DIS scheme we must replace f_q by $f_q - f_2^q$ and similarly for the gluon terms
- The coefficient of the delta function term is then

$$\frac{\alpha_s}{2\pi} C_F \left(1 + \frac{4\pi^2}{3}\right)$$

  relative to which the lowest order term is just 1 (Exercise: Show this)
- For $Q \approx 5$ GeV this correction changes the lowest order term by about 1.8, *i.e.,* a K factor of nearly 2!
- In this case, the bulk of the correction comes from the π^2 terms which appear in the delta function term and so the correction is roughly a constant times the lowest order results
- And, so, the idea of a K factor has been with us ever since

Comments

- One should be worried that the next order correction is so large - is perturbation theory converging?
- Feynman:

$$\frac{1}{1-x} = 1 + x + \ldots \text{ and } e^x = 1 + x + \ldots$$

  For $x \approx 1$ the first diverges while the second gives about 2.7
- As it turns out a significant part of the correction term exponentiates so the second example is closer to what is happening
- In this example the kinematics of the NLO correction delta function piece is the same as for the lowest order, so the correction is essentially a multiplicative constant
- This does not happen very often
- In the more usual case there are many different subprocesses in the NLO calculation and they can have very different dependences on the process kinematics
- Various phase space factors can cause the higher order parton emission contributions to contribute differently in different regions of phase space

- So, in general the ratio of the NLO to LO calculations (the so-called theoretical K factor) will depend on the kinematic variables and *will not be a constant*
- Furthermore, there is the issue of the scale dependence. The NLO and LO terms have different scale dependences. They partially cancel each other, which is a good thing.
- This has the effect that the ratio of the two terms will depend strongly on the chosen factorization scale
- But if the so-called K factor depends on the scale choice, then how can it be a uniquely defined "factor"?

It Can't!

I'll have more examples of this in a later lecture

Time to get on the Soap Box

Some people are fond of saying something like "QCD says that in lowest order the lepton pair is produced with no transverse momentum." This statement is false. Let's see why.

- It is true, that for the $q\overline{q} \to l^+l^-$ subprocess, the lepton pair has the same transverse momentum as the $q\overline{q}$ initial state
- It is also true that we have used kinematics in which we treat the initial partons as being collinear with the beam
- However, the PDFs in the cross section expressions are the scale-dependent PDFs and carry an argument M_f^2.
- This dependence on the factorization scale comes from integrating over the p_T of the additional partons emitted from the initial state (either radiated gluons or quarks and antiquarks created by gluons)
- Thus, QCD radiation causes the incoming partons to have non-zero transverse momenta, but these are integrated out when the scale-dependent PDFs are used
- We make an approximation when we treat the partons given by the integrated PDFs as having zero transverse momenta, and this is appropriate for longitudinal momentum distributions.

- Thus, QCD predicts that the lepton pair will have a transverse momentum distribution, but we have integrated over it (even if we didn't realize it) when we use the expressions given previously.
- Then, how do we undo the integration? And what value should we use for M_f^2?

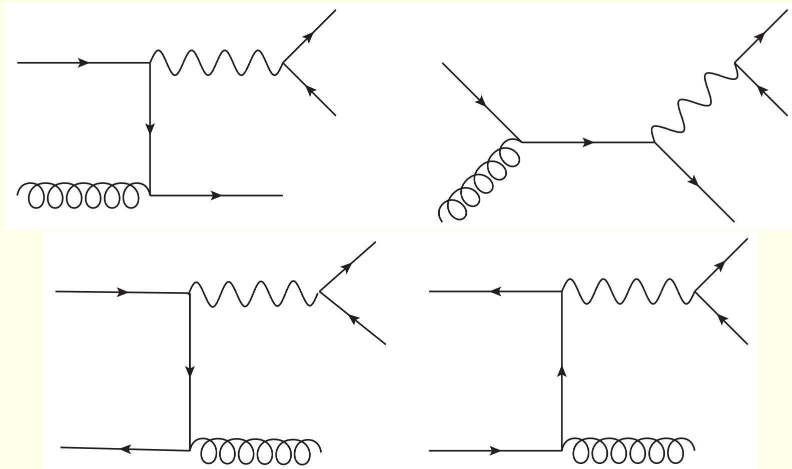
Choosing the Factorization Scale

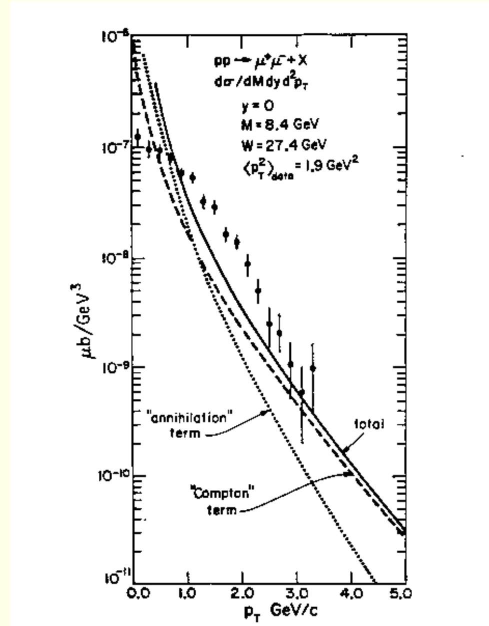
- The factorization scale M_f can be understood as setting the upper limit on the integration over the transverse momenta of the partons emitted in the initial state evolution
- The leading-log contributions from higher order subprocesses have been included in the scale-dependent PDFs
- So, if one wants to calculate the cross section for producing a lepton pair of mass Q then a choice of $M_f \approx Q$ would be appropriate.
- This is not exact, since the true upper limit of the transverse momentum integration would be given by a more complicated expression involving a function of τ and y multiplying Q. But in the leading-log approximation the choice Q is acceptable.
- Of course, any constant times Q is equally acceptable as long as the constant isn't too large (or too small) since then one would generate spurious large logarithms

p_T Distribution

So, given the preceding discussion, how does one calculate the lepton pair p_T distribution? Answer - Go to higher order!

- To calculate the p_T spectrum we will have to consider having the lepton pair recoil against at least one parton. The subprocesses are
  - Compton process: $qg \to l^+l^-q$
  - Annihilation process $q\overline{q} \to l^+l^-g$
- Using these subprocesses one can calculate a p_T distribution, but it will be a *leading order* prediction for the p_T dependence
- There is still the issue of the scale choice for the PDFs and the running coupling.
- If one is interested in the high-p_T region where $p_T \sim Q$ then there is only one large scale and either p_T or Q or some combination is appropriate.
- If one is interested in the region $p_T \ll Q$ then one has a two scale problem and logs of p_T/Q may become important
- This situation requires resummation...





The $\mathcal{O}(\alpha_s)$ subprocesses both give contributions which diverge as p_T^{-2} as p_T goes to zero
These divergent terms are factorized and included in the scale dependent PDFs

We want to do a better calculation in the low p_T region

- Have to figure out what to do with the low p_T radiated partons
- Have to figure out what the scale should be for the PDFs

Comments on Resummation

- Want to include the effects of multiple parton emission
- Divergent pieces from each emission factorize
- Need to insure transverse momentum conservation
- Insert a δ function to enforce it
- Use the Dirac representation of the δ function

$$\delta^2(\vec{p}_T - \vec{k}_{T_1} - \cdots - k_{T_n}) = \frac{1}{(2\pi)^2} \int d^2 b e^{-i\vec{b}\cdot(\vec{p}_T - \vec{k}_{T_1} - \cdots - \vec{k}_{T_n})}$$

- In this form the δ function factorizes
- Can sum the effects of multiple emissions in impact parameter space
- This concept was developed by Collins, Soper, and Sterman (Nucl.Phys.B250,199(
- See my lecture in the 2010 CTEQ Summer School for more details

- Need to take into account the transverse momentum of the incoming quarks
  - Normally integrated over, leading to the scale dependence of the PDFs
  - Factorization scale usually chosen to be on the order of the single hard scale
  - Now, the lepton pair p_T will reflect the p_Ts of the incoming quarks
  - PDF scale is chosen to be of the order of $1/b$ where b is the impact parameter seen above
  - b and p_T are conjugate variables - large $p_T \leftrightarrow$ small b
  - A scale of $1/b$ is large for large p_T and small for small p_T
- Classic application is to the lepton pair, W, or Z p_T distributions

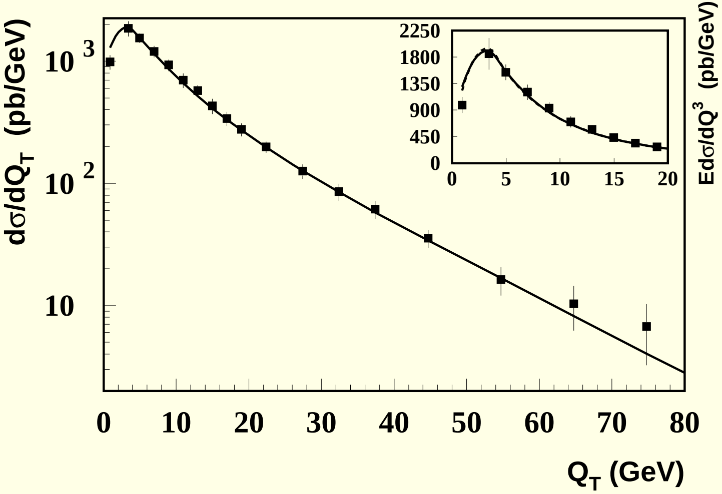
CSS Resummed Result

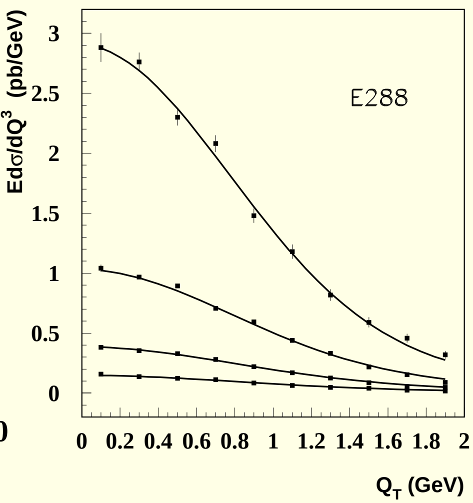
The resummed CSS result takes a relatively simple form with an exponentiation in impact parameter space and a convolution with PDFs evaluated at a scale $1/b$

$$
\begin{aligned}
\frac{d\sigma}{dQ^2\,dy\,dk_T^2} \quad &= \quad \frac{4\pi^2\alpha^2}{9Q^2 s}(2\pi)^{-2}\int d^2b e^{i\vec{k}_T\cdot\vec{b}}\sum_j e_j^2 \\
&\quad \sum_a \int_{x_a}^1 \frac{d\xi_a}{\xi_a} G_{a/A}(\xi_a, 1/b) \sum_b \int_{x_b}^1 \frac{d\xi_b}{\xi_b} G_{b/B}(\xi_b, 1/b) \\
&\quad e^{-S(Q^2,b)} C_{ja}(\frac{x_a}{\xi_a}, g(1/b)) C_{\bar{j}b}(\frac{x_b}{\xi_b}, g(1/b)) \\
&+ \quad \frac{4\pi^2\alpha^2}{9Q^2 s} Y(k_T, Q, x_a, x_b)
\end{aligned}
$$

with $S(Q^2,b) = exp\left[-\int_{1/b^2}^{Q^2} \frac{d\overline{\mu}^2}{\overline{\mu}^2}\left[\ln\left(\frac{Q^2}{\overline{\mu}^2}\right) A(g(\overline{\mu})) + B(g(\overline{\mu}))\right]\right]$

- The Y piece is the residual NLO non-log contribution
- In the expression for S, the A term sums the leading logarithms while the B term sums the next-to-leading logs
- Here are some typical resummed results (from J. Qiu and X. Zhang, Phys. Rev. D63:114011,2001) compared to data (D0 and Fermilab E-288)





- Note that by exponentiating in impact parameter space $\frac{d\sigma}{dk_T^2}$ has a non-zero intercept at $k_T = 0$
- The D0 data are shown as $\frac{d\sigma}{dk_T}$ which has a kinematic zero at $k_T = 0$
- For both plots, however, the tree level calculation would diverge as $k_T \to 0$, whereas the b-space exponentiation describes the data nicely

Other Resummation Examples

Logarithms of variables other than k_T can also occur - it depends on the type of distribution one is calculating. The logs come from the same basic vertices in the Feynman diagrams - they just appear in different ways and require different types of treatments. Another example is provided by the threshold logs we encountered previously in Lecture II

Threshold Resummation – Basic Physics

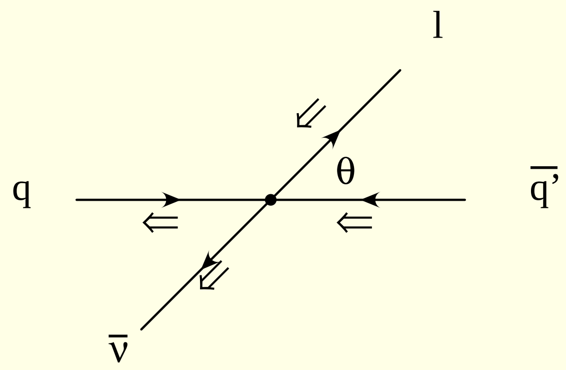
- For inclusive calculations, singularities from soft real gluon emission cancel against infrared singularities from virtual gluon emission
- Limitations on real gluon emission imposed by phase space constraints can upset this cancellation
- Singular terms still cancel, but there can be large logarithmic remainders
- Applications include high mass lepton pair production, high-p_T particle production, the fragmentation component of direct photon production,...
- All are examples of where the high mass or high-p_T limits the phase space available for gluon emission

$W^\pm$ and Z production

There are several minor changes (see Appendix II)

- The Feynman graphs are the same, but the couplings are different
- Massive propagators are used
- Due to these factors, the angular distribution of the final state leptons is different
  - l^+l^- : $1+\cos^2\theta$
  - $l^-\bar{\nu}$: $(1+\cos\theta)^2$

The source of the $(1+\cos\theta)^2$ form is easy to understand. The W couples to left-handed particles and right-handed antiparticles.



When $\theta \to \pi$ the cross section must vanish since angular momentum would not be conserved. However, $\theta = 0$ is allowed. The $(1 + \cos\theta)$ factor ensures this.

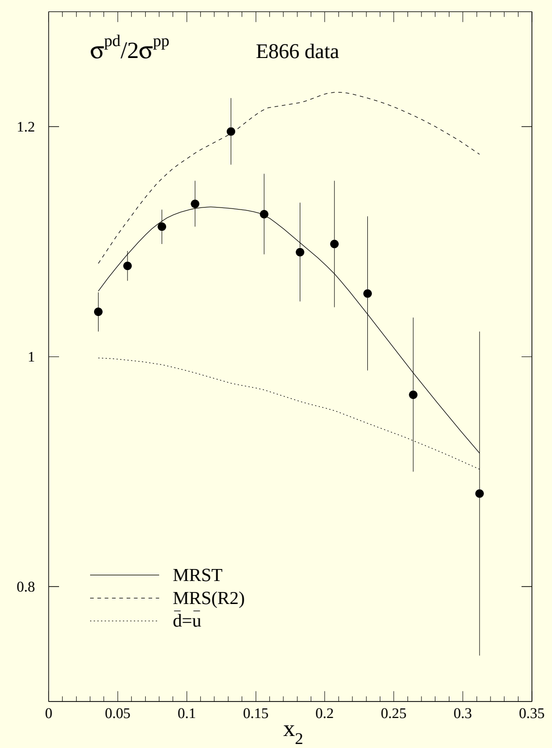
Some Phenomenology

- Fermilab experiment E866 measured the cross sections for lepton pair production in pp and pd interactions
- Using $\frac{d\sigma}{dQ^2 dx_F}$ one can extract the cross section ratio as a function of x_1 and x_2
- Let $u_1 = u(x_1, Q^2)$, etc and use isospin to relate the PDFs in a neutron to those in a proton.
- Exercise: For $x_1 \gg x_2$ show that

$$\frac{\sigma_{pd}}{2\sigma_{pp}} = \frac{1}{2} \frac{1 + \frac{1}{4} R_{du}}{1 + \frac{1}{4} R_{du} r_2} (1 + r_2)$$

  where $R_{du} = \frac{d_1}{u_1}$ and $r_2 = \frac{\bar{d}_2}{\bar{u}_2}$ and contributions from strange and heavier quarks have been neglected

- If $r_2 = 1$ then the cross section ratio is also one.



- The data clearly show the $\bar{d} \neq \bar{u}$
- The line for $\bar{d} = \bar{u}$ is not constant at one because the approximation of $x_1 \gg x_2$ has been relaxed
- Exercise: repeat the derivation from the previous slide, but do not discard terms like $\bar{d}_1$ or $\bar{u}_1$

W rapidity asymmetry

- Consider the asymmetry

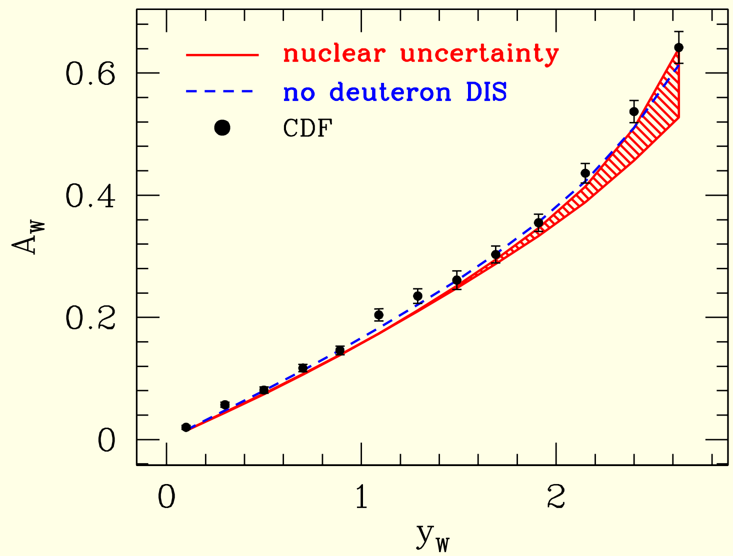
$$A_W = \frac{\sigma(W^+) - \sigma(W^-)}{\sigma(W^+) + \sigma(W^-)}$$

- Exercise: Show that for the weak mixing angle $\theta_W \approx 0$ for $\bar{p}p$ interactions one can write

$$A_W \approx \frac{R_{du}(x_2) - R_{du}(x_1)}{R_{du}(x_2) + R_{du}(x_1)}$$

  where $x_{\frac{1}{2}} = \frac{M_W}{\sqrt{s}} e^{\pm y}$

- Therefore, the asymmetry measured at the Tevatron tells us something about the d/u ratio



- These are some results from the CJ (CTEQ-Jefferson Lab) Collaboration
- Different models for nuclear corrections for deuterium DIS give different d/u ratios and A_W is sensitive to this

- The CDF extraction of A_W requires some model-dependent assumptions since one can not detect the outgoing ν's longitudinal momentum

- Often one measures the charged lepton rapidity asymmetry instead of the W asymmetry.

- The V-A nature of the decay angular distribution reduces the asymmetry (the l^- from W^- decay is pushed to higher rapidity while the l^+ from W^+ decay is pushed to lower rapidity)

- The lepton asymmetry depends on the same physics, but the sensitivity to d/u at large values of x is reduced.

Summary

- We've seen the basics of parton model calculations for vector boson production and how to incorporate QCD effects at LO and NLO
- We've seen now the collinear singularities associated with initial state radiation can be factorized and absorbed into the PDFs
- We've seen examples of the convention dependence associated with the process of factorization
- We've seen some examples of how to chose the renormalization and factorization scales
- We've seen a bit of phenomenology associated with vector boson production
- The next step is to investigate how to handle more complicated observables in processes involving more partons in the final state

Appendix I - Lepton Pair Production Born Term Calculation

Matrix element $M = e_q \frac{e^2}{\hat{s}} \overline{u}(k_1)\gamma_\mu v(k_2)\overline{v}(p_2)\gamma^\mu u(p_1)$

Spin/color averaged matrix element squared

$$\overline{\sum}|M|^2 = \frac{e_q^2 e^4}{\hat{s}^2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)3\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\mathrm{Tr}\left[\not{p}_1\gamma^\nu\not{p}_2\gamma^\mu\right]\mathrm{Tr}\left[\not{k}_2\gamma_\nu\not{k}_1\gamma_\mu\right]$$

$$= \frac{4}{3}\frac{e_q^2 e^4}{\hat{s}^2}\left[p_1^\nu p_2^\mu + p_1^\mu p_2^\nu - g^{\mu\nu} p_1 \cdot p_2\right]\left[k_{2\nu}k_{1\mu} + k_{2\mu}k_{1\nu} - g_{\mu\nu}k_1 \cdot k_2\right]$$

Red factors are for the spin average and blue factors are for the color average. Forming the indicated dot products yields

$$\begin{aligned}\overline{\sum}|M|^2 &= \frac{4}{3}\frac{e_q^2 e^4}{\hat{s}^2}\left[2p_1 \cdot k_2 p_2 \cdot k_1 + 2p_1 \cdot k_1 p_2 \cdot k_2\right] \\ &= \frac{2}{3}\frac{e_q^2 e^4}{\hat{s}^2}\left[\hat{t}^2 + \hat{u}^2\right]\end{aligned}$$

Center of mass frame: the 4-vectors are

$$
\begin{aligned}
p_1 &= \frac{\sqrt{\hat{s}}}{2}(1,0,0,1) \quad p_2 = \frac{\sqrt{\hat{s}}}{2}(1,0,0,-1) \\
k_1 &= \frac{\sqrt{\hat{s}}}{2}(1,\sin(\theta),0,\cos(\theta)) \quad k_2 = \frac{\sqrt{\hat{s}}}{2}(1,-\sin(\theta),0,-\cos(\theta))
\end{aligned}
$$

yielding the Lorentz scalars

$$
\hat{t} = -\frac{\hat{s}}{2}(1-\cos(\theta)) \text{ and } \hat{u} = -\frac{\hat{s}}{2}(1+\cos(\theta))
$$

with

$$
\hat{t}^2 + \hat{u}^2 = \frac{s^2}{2}(1+\cos^2(\theta))
$$

Inserting these relations into our result yields the answer we seek:

$$
\overline{\sum}|M|^2 = \frac{e_q^2 e^4}{3}(1+\cos^2(\theta))
$$

To make use of this result we need to convert it to a cross section. For this we need the two-body Lorentz invariant phase space factor:

$$\begin{aligned} PS^{(2)} &= \frac{d^3k_1}{(2\pi)^3 2E_1}\frac{d^3k_2}{(2\pi)^3 2E_2} \\ &\quad \times (2\pi)^4\delta^4(p_1+p_2-k_1-k_2) \\ &= \frac{d^3k}{16\pi^2 E_1 E_2}\delta(\sqrt{\hat{s}}-E_1-E_2). \end{aligned}$$

In the center-of-momentum frame we have $k = |\vec{k}_1| = |\vec{k}_2|$ so that in this frame we can write

$$\begin{aligned} d(E_1+E_2) &= d\sqrt{\hat{s}} \\ &= kdk\left(\frac{1}{E_1}+\frac{1}{E_2}\right) \\ &= kdk\frac{E_1+E_2}{E_1E_2}. \end{aligned}$$

with $k = \sqrt{\hat{s}}/2$.

This allows the phase space factor to be written as

$$\begin{aligned} PS^{(2)} &= \frac{k^2 dk d\Omega}{16\pi^2 E_1 E_2}\delta(\sqrt{\hat{s}} - E_1 - E_2) \\ &= \frac{k d\sqrt{\hat{s}} d\Omega}{16\pi^2 \sqrt{\hat{s}}}\delta(\sqrt{\hat{s}} - E_1 - E_2) \\ &= \frac{d\Omega}{32\pi^2} \\ &= \frac{d\cos(\theta)}{16\pi} \end{aligned}$$

To get a cross section we multiply the phase space factor times the spin and color averaged squared matrix element and multiply that by a flux factor of $1/2\hat{s}$ yielding

$$\begin{aligned}
\sigma(q\overline{q} \to l^+ l^-) &= \frac{1}{2\hat{s}} \int_{-1}^{1} \frac{d\cos(\theta)}{16\pi} \frac{e_q^2 e^4}{3} (1 + \cos^2(\theta)) \\
&= \frac{e_q^2}{3} \frac{(4\pi\alpha)^2}{16\pi} \frac{1}{2\hat{s}} \frac{8}{3}
\end{aligned}$$

where the fine structure constant $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$.

The final result for the parton-level cross section is

$$\sigma(q\overline{q} \to l^+ l^-) = \frac{4\pi\alpha^2}{9\hat{s}} e_q^2 \equiv \sigma_0.$$

Appendix II W and Z Production

- W and Z production involve subprocesses which are very similar to lepton pair production, e.g., $q\overline{q}' \to W$ and $q\overline{q} \to Z$.
- Use the narrow width approximation, i.e., the vector bosons will be treated as stable particles of fixed mass. All of the previous lepton pair production results can be easily utilized, providing that we make some changes in the couplings.

Consider $q(p_1)\overline{q}'(p_2) \to W(p)$, for which the matrix element is

$$M = -iV_{qq'}\frac{g}{\sqrt{2}}\epsilon_\alpha \overline{v}(p_2)\gamma^\alpha \frac{1}{2}(1-\gamma_5)u(p_1)$$

where $V_{qq'}$ is the appropriate element of the CKM matrix.

The spin/color averaged squared matrix element is given by

$$\begin{aligned}\overline{\sum}|M|^2 &= |V_{qq'}|^2 \frac{g^2}{96} 2\mathrm{Tr}\left[\not{p}_1(1-\gamma_5)\not{p}_2(1-\gamma_5)\right] \\ &= |V_{qq'}|^2 \frac{g^2}{24} \mathrm{Tr}[\not{p}_1\not{p}_2] = |V_{qq'}|^2 \frac{g^2}{6} p_1 \cdot p_2 \\ &= |V_{qq'}|^2 \frac{G_F M_W^4}{\sqrt{2}} \frac{2}{3}\end{aligned}$$

where $g^2 = \frac{8G_F M_W^2}{\sqrt{2}}$. The hadronic cross section σ is given by convoluting the parton level cross section $\hat{\sigma}$ with the appropriate PDFs:

$$\begin{aligned}\sigma &= \int dx_a dx_b \sum_{qq'} q(x_a)\overline{q}'(x_b)\hat{\sigma} \\ \hat{\sigma} &= \frac{1}{2\hat{s}} \frac{2}{3} \frac{G_F M_W^4}{\sqrt{2}} |V_{qq'}|^2 \int \frac{d^3p}{(2\pi)^3 2E} (2\pi)^4 \delta^4(p - p_1 - p_2).\end{aligned}$$

The integrand of the phase space integral can be rewritten as

$$2\pi \; d^4p \; \delta^4(p - p_1 - p_2)\delta(\hat{s} - M_W^2)$$

This yields

$$\hat{\sigma} = \frac{2\pi}{3}|V_{qq'}|^2 \frac{G_F M_W^2}{\sqrt{2}} \delta(\hat{s} - M_W^2).$$

Compare this to our lepton pair production result

$$\hat{\sigma}_{\gamma^*} = \frac{4\pi^2\alpha}{3} e_q^2 \delta(\hat{s} - Q^2)$$

which shows that

$$4\pi\alpha e_q^2 \leftrightarrow 2|V_{qq'}|^2 \frac{G_F M_W^2}{\sqrt{2}}.$$

Z Production

Here the subprocess is $q(p_1)\overline{q}(p_2) \to Z(p)$, with the matrix element given by

$$M = -ig\epsilon_\alpha \overline{v}(p_2)\gamma^\alpha(g_V + g_A\gamma_5)u(p_1).$$

The partonic cross section is given by

$$\hat{\sigma}_Z = \frac{8\pi}{3}\frac{G_F M_W^2}{\sqrt{2}}(g_V^2 + g_A^2)\delta(\hat{s} - M_Z^2)$$

where

$$g_V^2 + g_A^2 = \frac{1}{8}(1 - 4|e_q|\sin^2\theta_W + 8e_q^2\sin^4\theta_W).$$

Apart from changing the coupling, we can treat W and Z production just like lepton pair production at a fixed value of Q^2.