CTEQ School on
QCD Analysis and Electroweak Phenomenology

LECTURE 3

Introduction to the Parton Model and Perturbative QCD

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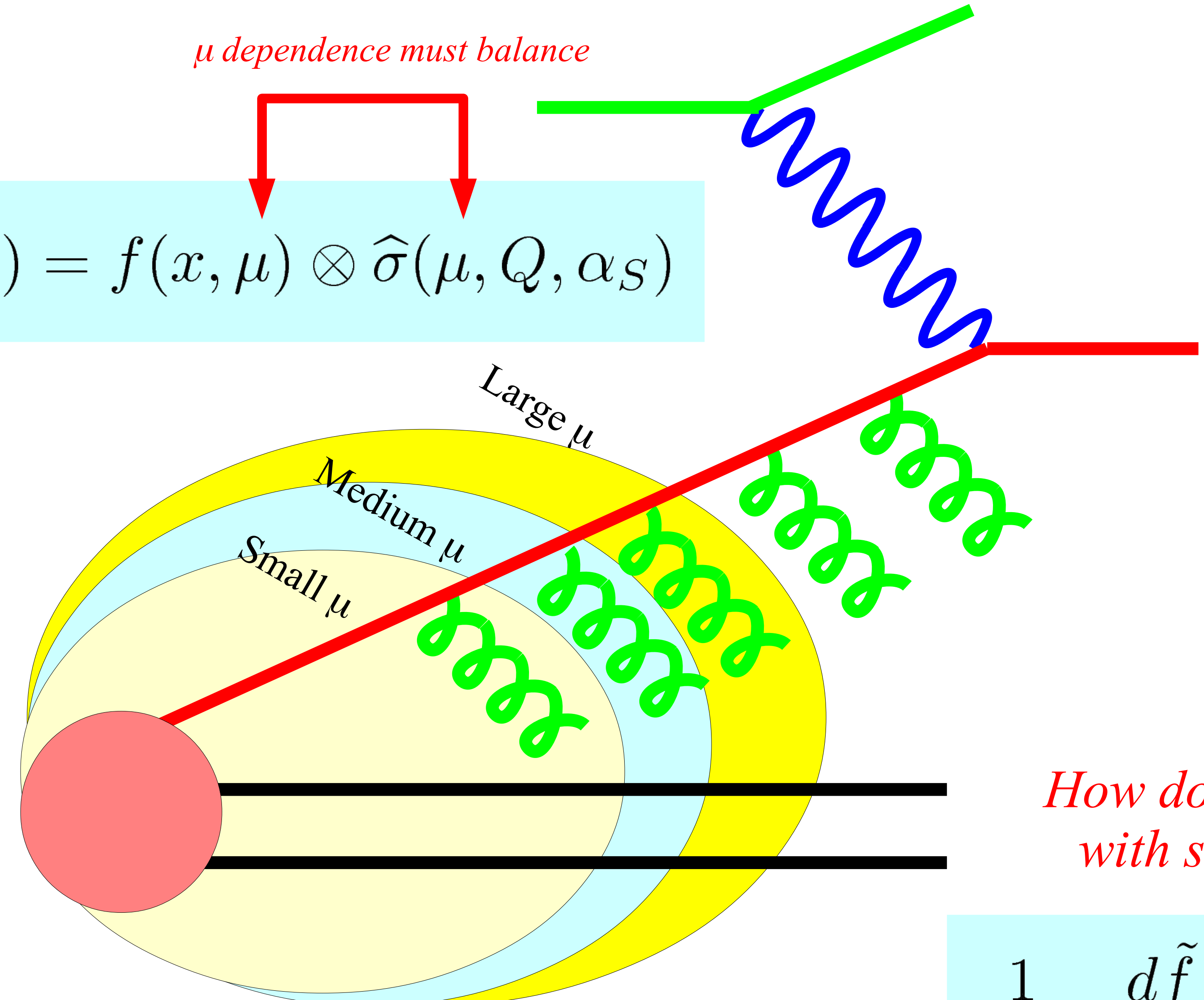
Recap: Parton Model, Factorization, Evolution

μ dependence must balance

$$\sigma(Q,x) = f(x,\mu) \otimes \widehat{\sigma}(\mu, Q, \alpha_S)$$



How does f change with scale μ???

$$\frac{1}{\tilde{f}} \, \frac{d\tilde{f}}{d\ln[\mu]} = -\gamma$$

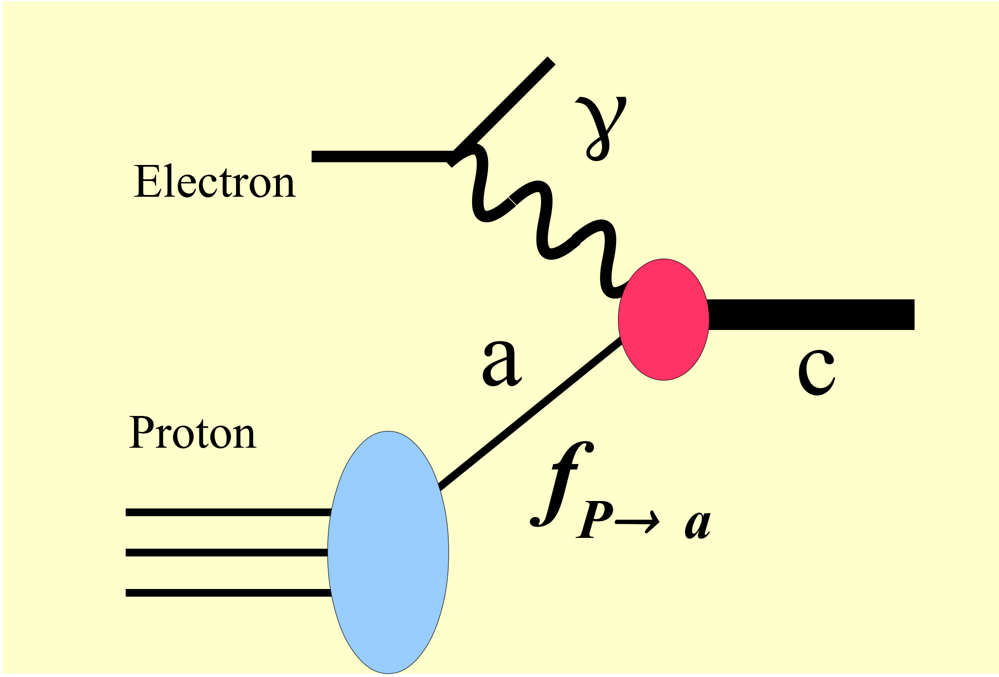
DGLAP Evolution Equation

DIS

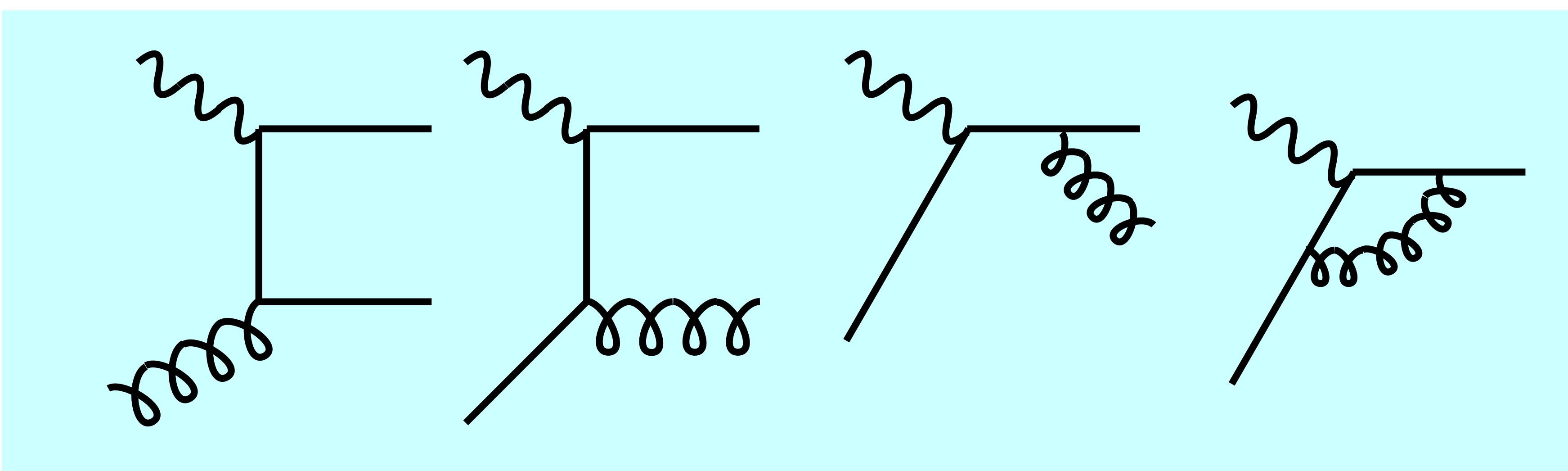
AT

NLO

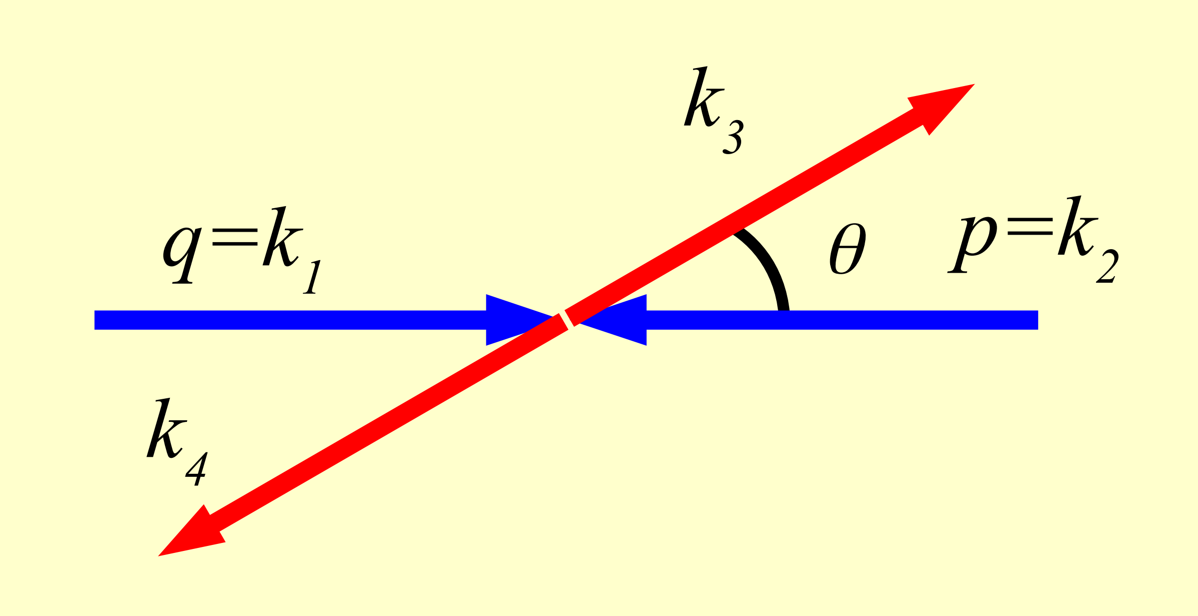
DIS at NLO



Sample NLO contributions to DIS



DIS NLO Kinematics



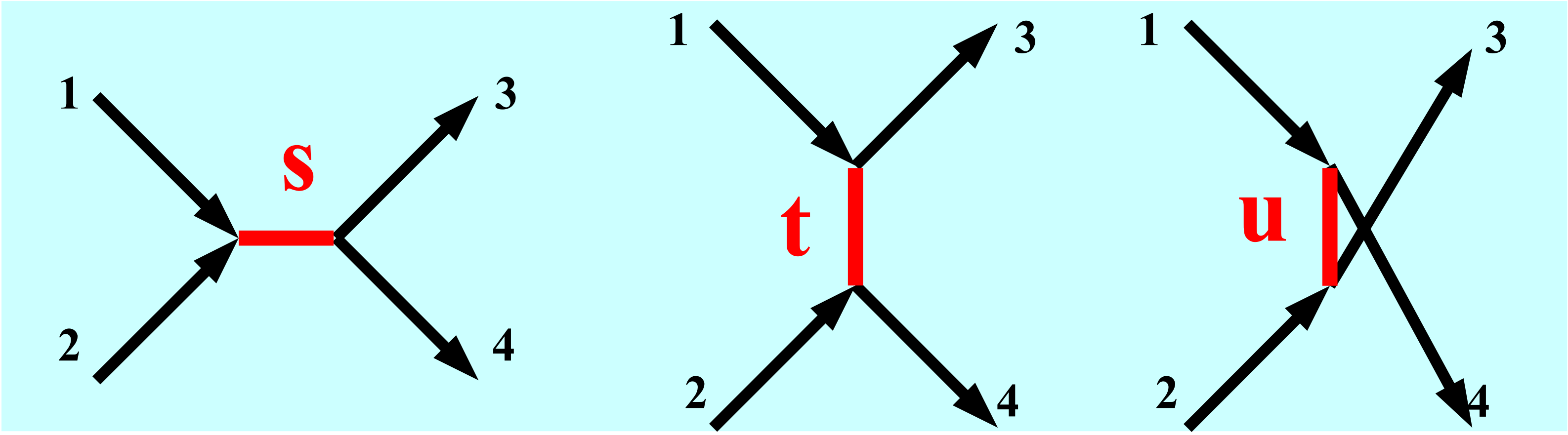
$$k_1 \equiv q^\mu \quad = \quad \left(\frac{s-Q^2}{2\sqrt{s}}, 0, 0, \frac{(s+Q^2)}{2\sqrt{s}}\right) \qquad -q^2 = Q^2 > 0$$

$$k_2 \equiv p^\mu \quad = \quad \left(\frac{s+Q^2}{2\sqrt{s}}, 0, 0, \frac{-(s+Q^2)}{2\sqrt{s}}\right) \qquad p^2 = 0$$

$$k_3^\mu \quad = \quad \tfrac{\sqrt{s}}{2}\,(1, +\sin\theta, 0, +\cos\theta) \qquad k_3^2 = 0$$

$$k_4^\mu \quad = \quad \tfrac{\sqrt{s}}{2}\,(1, -\sin\theta, 0, -\cos\theta) \qquad k_4^2 = 0$$

Mandelstam Variables {s,t,u}



$$s = (k_1 + k_2)^2 \equiv (k_3 + k_4)^2$$
$$t = (k_1 - k_3)^2 \equiv (k_2 - k_4)^2$$
$$u = (k_1 - k_4)^2 \equiv (k_2 - k_3)^2$$
$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

Exercise

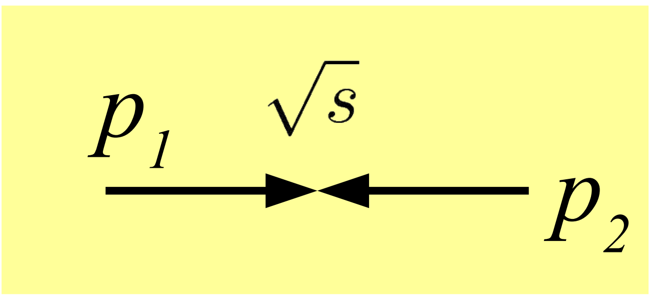
{s,t,u} are partonic

$$s = +Q^2 \frac{(1-x)}{x} \qquad t = -Q^2 \frac{(1-z)}{2x} \qquad u = -Q^2 \frac{(1+z)}{2x}$$

$$x = \frac{Q^2}{2p \cdot q} \qquad x \subset [0,1]$$

$$z \equiv \cos\theta \qquad z \subset [-1,1]$$

1) Let's work out the general 2→2 kinematics for general masses.



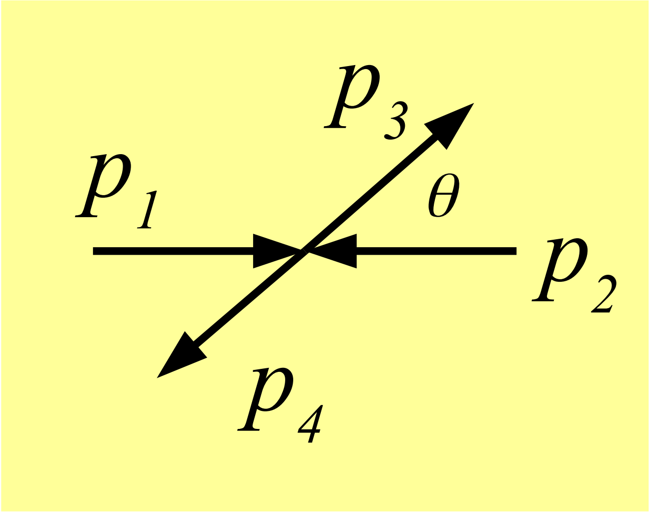
a) Start with the incoming particles.
Show that these can be written in the general form:

$$p_1 = (E_1, 0, 0, +p) \qquad p_1^2 = m_1^2$$
$$p_2 = (E_2, 0, 0, -p) \qquad p_2^2 = m_2^2$$

... with the following definitions:

$$E_{1,2} = \frac{\hat{s} \pm m_1^2 \mp m_2^2}{2\sqrt{\hat{s}}} \qquad p = \frac{\Delta(\hat{s}, m_1^2, m_2^2)}{2\sqrt{\hat{s}}}$$

$$\Delta(a,b,c) = \sqrt{a^2+b^2+c^2-2(ab+bc+ca)}$$



Note that Δ(a,b,c) is symmetric with respect to its arguments, and involves the only invariants of the initial state: s, m_1^2, m_2^2.

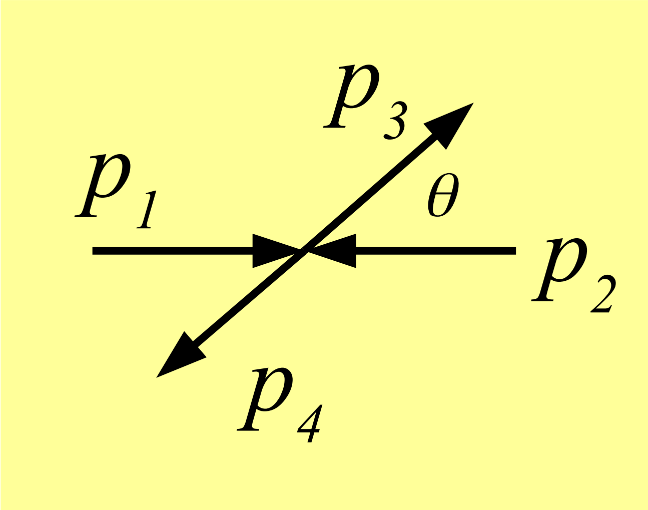
b) Next, compute the general form for the final state particles, p_3 and p_4. Do this by first aligning p_3 and p_4 along the z-axis (as p_1 and p_2 are), and then rotate about the y-axis by angle θ.

PROBLEM #2: Consider the reaction: $pp \rightarrow pp$ $(12 \rightarrow 34)$ with **CMS** scattering angle θ. The CMS energy is $\sqrt{s} = 2\,TeV$.

a) Compute the boost from the CMS frame to the rest frame of #2 (lab frame)

b) Compute the energy of #1 in the lab frame.

c) Compute the scattering angle θ_{lab} as a function of the CMS θ and invariants.



Hint: by using invariants you can keep it simple.
I.e., don't do it the way Goldstein does.

The power of invariants

Matrix element: NLO DIS

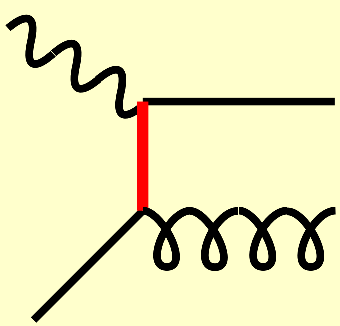
$$|\mathcal{M}|^2 = \frac{s}{-t} + \frac{-t}{s} + \frac{2uQ^2}{st}$$

For the real 2→2 graphs

$$\simeq \frac{2(1-x)}{(1-z)} + \frac{2(1-z)}{(1-x)} + \frac{2x(1+z)}{(1-x)(1-z)}$$

Singular at z=1

$$z \longrightarrow 1, \quad \cos\theta \longrightarrow 1$$
$$\theta \longrightarrow 0, \quad t \longrightarrow 0$$

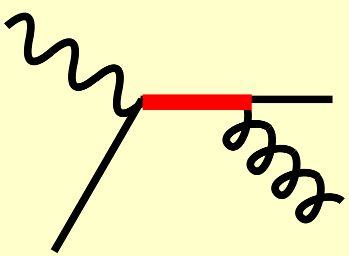


Collinear Singularity

Separate infinity, and subtract

Singular at x=1

$$x \longrightarrow 1, \quad s \longrightarrow 0$$



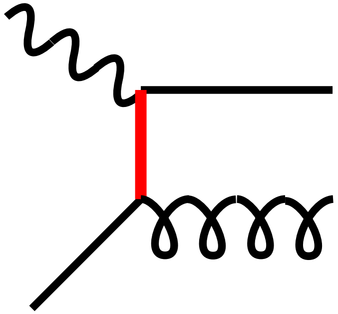
Soft Singularity

Separate infinity, cancel with virtual graphs

The Plan

Collinear Divergences

$$|\mathcal{M}|^2 \underset{z\to 1}{\longrightarrow} \frac{2}{(1-z)} \frac{(1+x^2)}{(1-x)}$$



Looks like a PDF splitting function

Plan

1) Separate ∞ at z=1

2) Subtract … *(should be part of PDF)*

Method

Need to regulate ∞

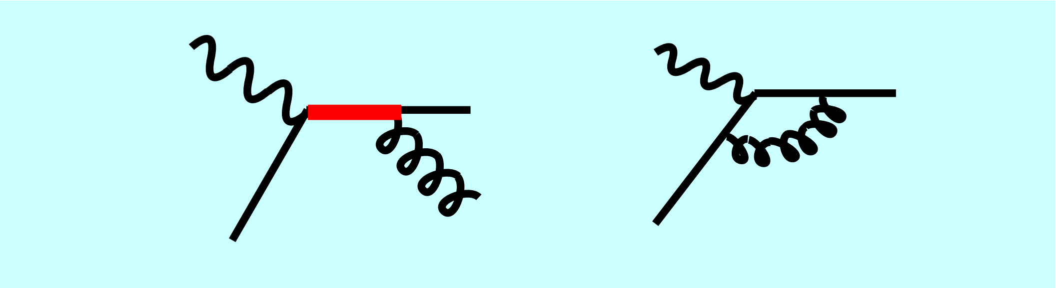
Choices

1) Dimensional Regularization

2) Quark Mass

3) θ Cut

Soft Singularities



Plan

1) Separate ∞ at x=1

2) Cancel between Real and Virtual graphs

Method

Need to regulate ∞

Choices

1) Dimensional Regularization

2) Gluon Mass

3) ...

We'll use a simple example to illustrate the key points:

Dimensional Regularization

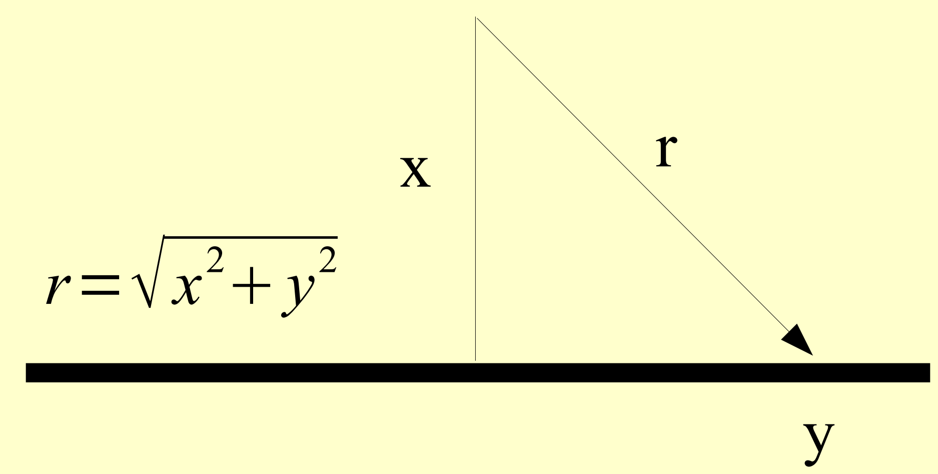
meets

Freshman E&M

M. Hans, Am.J.Phys. 51 (8) August (1983). p.694
C. Kaufman, Am.J.Phys. 37 (5), May (1969) p.560
B. Delamotte, Am.J.Phys. 72 (2) February (2004) p.170

Regularization, Renormalization, and Dimensional Analysis:
Dimensional Regularization meets Freshman E&M.
Olness & Scalise, arXiv:0812.3578 [hep-ph]

Infinite Line of Charge



$$r=\sqrt{x^2+y^2}$$

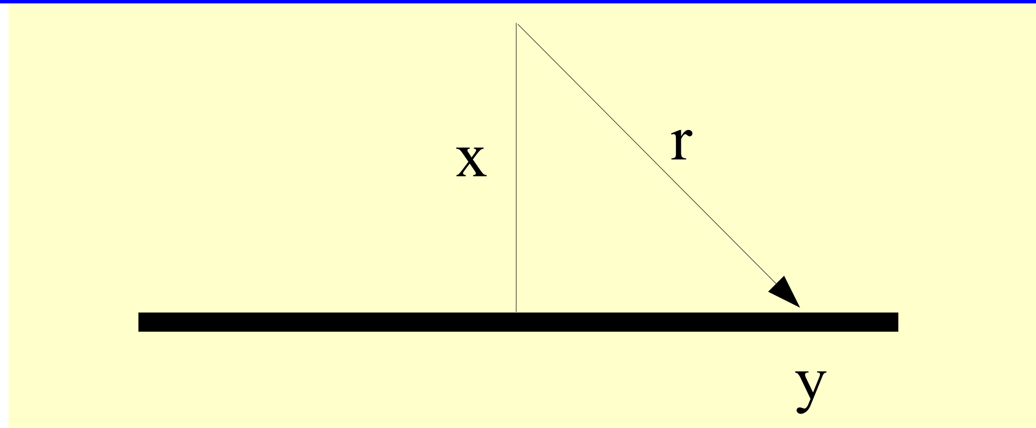
$$dV=\frac{1}{4\pi\epsilon_0}\frac{dQ}{r}$$

$$\lambda=Q/y$$

$$V=\frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}dy\frac{1}{\sqrt{x^2+y^2}}=\infty$$

Note: ∞ can be very useful

Scale Invariance



$$V(kx) =$$
$$= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dy \frac{1}{\sqrt{(kx)^2 + y^2}}$$
$$= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} d\left(\frac{y}{k}\right) \frac{1}{\sqrt{x^2 + (y/k)^2}}$$
$$= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dz \frac{1}{\sqrt{x^2 + z^2}}$$
$$= V(x)$$

$$V(kx) = V(x)$$

Naively Implies:
V(kx) – V(x) = 0

Note: **∞ + c = ∞**

∴ ∞ - ∞ = c

How do we distinguish this from

∞ - ∞ = c+17

$$V = \frac{\lambda}{4\pi\epsilon_0} \int_{-L}^{+L} dy \; \frac{1}{\sqrt{x^2 + y^2}}$$

$$V = \frac{\lambda}{4\pi\epsilon_0} \log\left[\frac{+L + \sqrt{L^2 + x^2}}{-L + \sqrt{L^2 + x^2}}\right]$$

V(x) depends on artificial regulator L

We cannot remove the regulator L

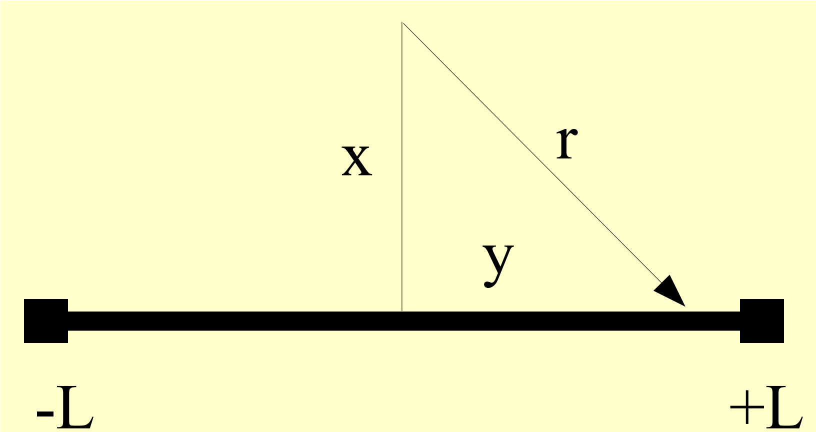
All physical quantities are independent of the regulator:

Electric Field $$E(x) = \frac{-dV}{dx} = \frac{\lambda}{2\pi\epsilon_0 \; x} \; \frac{L}{\sqrt{L^2 + x^2}} \; \rightarrow \; \frac{\lambda}{2\pi\epsilon_0 \; x}$$

Energy $$\delta V = V(x_1) - V(x_2) \underset{L\rightarrow\infty}{\rightarrow} \frac{\lambda}{4\pi\epsilon_0} \log\left[\frac{x_2^2}{x_1^2}\right]$$

Problem solved at the expense of an extra scale L
AND we have a broken symmetry: translation invariance

Broken Translational Symmetry



Shift: y → y' = y – c

y=[+L+c, -L+c]

$$V = \frac{\lambda}{4\pi\epsilon_0} \int_{-L+c}^{+L+c} dy \frac{1}{\sqrt{x^2+y^2}}$$

$$V = \frac{\lambda}{4\pi\epsilon_0} \log\left[\frac{+(L+c)+\sqrt{(L+c)^2+x^2}}{-(L-c)+\sqrt{(L-c)^2+x^2}}\right]$$

V(r) depends on “y” coordinate!!!

In QFT, gauge symmetries are important. E.g., Ward identies

Dimensional Regularization

Compute in n-dimensions

$$dy \to d^n y = \frac{d\Omega_n}{2} \; y^{n-1} dy$$

$$\Omega_n = \int d\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

$$\Omega_{1,2,3,4} = \{2, 2\pi, 4\pi, 2\pi^2\}$$

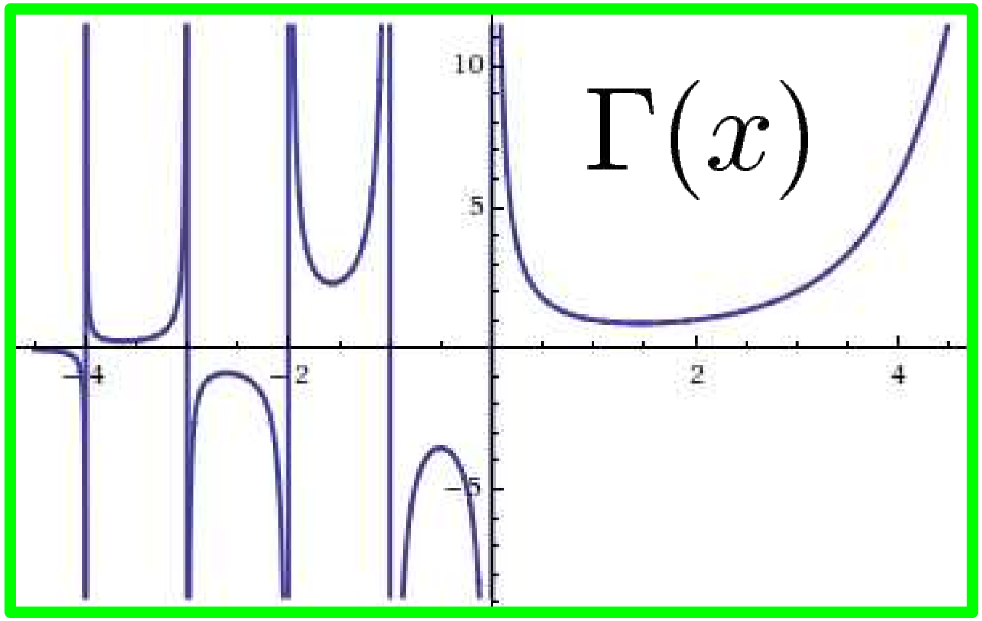
$$V = \frac{\lambda}{4\pi\epsilon_0} \int_0^{+\infty} d\Omega_n \; \frac{y^{n-1}}{\mu^{n-1}} \; \frac{dy}{\sqrt{x^2+y^2}}$$

Each term is individually dimensionaless

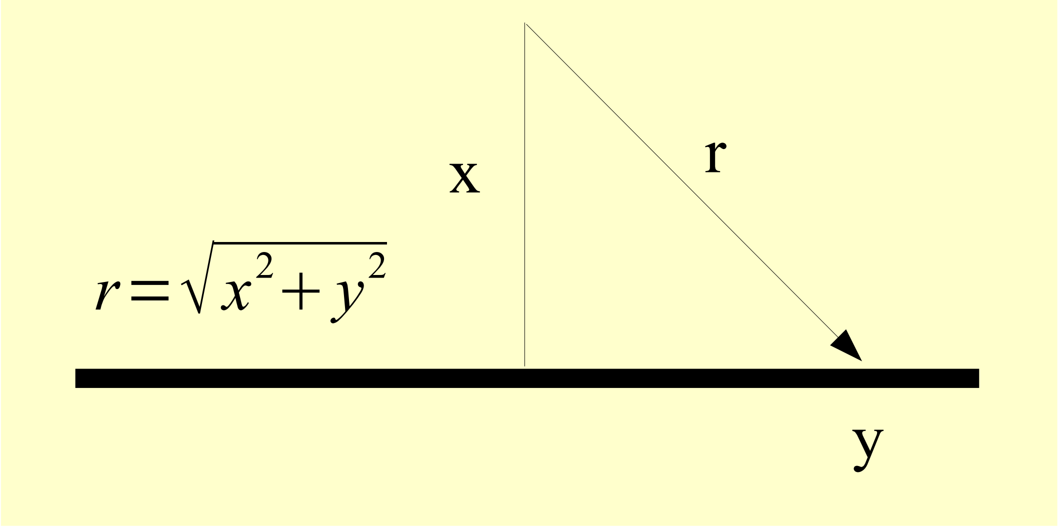
New scale μ

$$n = 1 - 2\epsilon$$

$$V = \frac{\lambda}{4\pi\epsilon_0} \left(\frac{\mu^{2\epsilon}}{x^{2\epsilon}} \frac{\Gamma[\epsilon]}{\pi^{\epsilon}} \right)$$



Why do we need an extra scale μ ???



$$dV=\frac{1}{4\pi\epsilon_0}\,\frac{dQ}{r} \qquad \lambda=Q/y$$

$$V=\frac{\lambda}{4\pi\epsilon_0}\,f(x)$$

All physical quantities are independent of the regulators:

Electric Field $$E(x)=\frac{-dV}{dx}=\frac{\lambda}{4\pi\epsilon_0}\left[\frac{2\epsilon\mu^{2\epsilon}\Gamma[\epsilon]}{\pi^{\epsilon}x^{1+2\epsilon}}\right]\underset{\epsilon\to 0}{\to}\frac{\lambda}{2\pi\epsilon_0}\frac{1}{x}$$

Energy $$\delta V=V(x_1)-V(x_2)\underset{\epsilon\to 0}{\to}\frac{\lambda}{4\pi\epsilon_0}\log\left[\frac{x_2^2}{x_1^2}\right]$$

Problem solved at the expense of an extra scale μ **AND** regulator ε

Translation invariance is preserved!!!

Dimensional Regularization respects symmetries

$$V \to \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon}+\ln\left[\frac{e^{-\gamma_E}}{\pi}\right]+\ln\left[\frac{\mu^2}{x^2}\right]\right]$$ Original

$$V \to \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon}+\ln\left[\frac{e^{-\gamma_E}}{\pi}\right]+\ln\left[\frac{\mu^2}{x^2}\right]\right]$$ MS

$$V \to \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon}+\ln\left[\frac{e^{-\gamma_E}}{\pi}\right]+\ln\left[\frac{\mu^2}{x^2}\right]\right]$$ MS-Bar

Physical quantities are independent of renormalization scheme!

$$V_{\overline{MS}}(x_1)-V_{\overline{MS}}(x_2)=\delta V=V_{MS}(x_1)-V_{MS}(x_2)$$

But only if performed consistently:

$$V_{\overline{MS}}(x_1)-V_{MS}(x_2)\neq\delta V\neq V_{MS}(x_1)-V_{\overline{MS}}(x_2)$$

Connection to QFT

This was the potential from our "Toy" calculation:

$$V \to \frac{\lambda}{4\pi\,\epsilon_0}\left[\frac{1}{\epsilon}+\ln\left[\frac{e^{-\gamma_E}}{1\ \pi}\right]+\ln\left[\frac{\mu^2}{x^2}\right]\right]$$

This is a partial result from
a <u>real</u> NLO Drell-Yan Calculation:
Cf., B. Potter

$$\frac{D(\epsilon)}{\epsilon} = \left(\frac{4\pi\,\mu^2}{Q^2}\right)\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \to \left[\frac{1}{\epsilon}+\ln\left[\frac{e^{-\gamma_E}}{4\pi}\right]+\ln\left[\frac{\mu^2}{Q^2}\right]\right]$$

Recap

Regulator provides unique definition of V, f, ω

Cutoff regulator L:
simple, but does NOT respect symmetries

Dimensional regulator ε:
respects symmetries: translation, Lorentz, Gauge invariance
introduces new scale μ

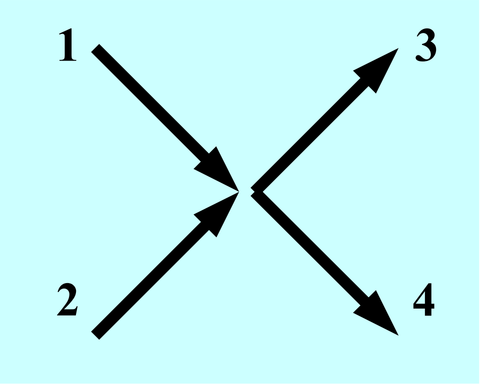
All physical quantities (E, dV, σ) are independent of the regulator
AND the new scale μ
Renormalization group equation: $d\sigma/d\mu=0$

We can define renormalized quantities (V,f,ω)
Renormalized (V,f,ω) are scheme dependent and arbitrary
Physical quantities (E,dV, σ) are unique and scheme independent
if we apply the scheme consistently

Apply Dimensional Regularization to QFT

D-Dimensional Phase Space

$$d\sigma = \frac{1}{2s}\, |\mathcal{M}|^2 \, d\Gamma$$



$$d\Gamma_i = \frac{d^D k_i}{(2\pi)^D}\, (2\pi)\, \delta(k_i^2)$$

1-particle

$$d\Gamma = d\Gamma_3\, d\Gamma_4\, (2\pi)^D\, \delta^D(k_1 + k_2 - k_3 - k_4)$$

Final state

$$d\Gamma = \frac{1}{16\pi} \left(\frac{s}{16\pi}\right)^{-\epsilon} \frac{(1-z^2)^{-\epsilon}}{\Gamma[1-\epsilon]}\, dz$$

Final state

$$g \to g\,\mu^{\epsilon}$$

Enter, μ scale

All the pieces

$$d\Gamma = \frac{1}{16\pi} \left(\frac{16\pi\mu^2}{Q^2}\right)^{+\epsilon} \frac{1}{\Gamma[1-\epsilon]}\, \frac{x^{\epsilon}}{(1-x)^{\epsilon}}\, (1-z^2)^{-\epsilon}\, dz$$

#1) Show:

$$\frac{d^3 p}{(2\pi)^3 2E} = \frac{d^4 p}{(2\pi)^4} (2\pi)\, \delta^+(p^2 - m^2)$$

This relation is often useful as the RHS is manifestly Lorentz invariant

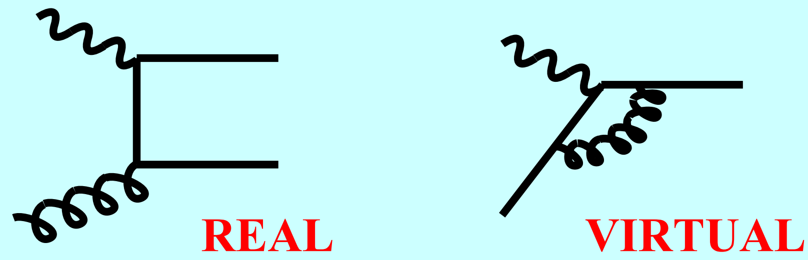
#2) Show that the 2-body phase space can be expressed as:

$$d\Gamma = \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_4} (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) = \frac{d\cos(\theta)}{16\pi}$$

Note, we are working with massless partons, and θ is in the partonic CMS frame

Soft Singularities

Soft Singularities



$$\frac{x^{\epsilon}}{(1-x)^{\epsilon}} \frac{1}{(1-x)} = \frac{1}{(1-x)_{+}} - \frac{1}{\epsilon}\,\delta(1-x)$$

- $\frac{x^{\epsilon}}{(1-x)^{\epsilon}}$: From phase space
- $\frac{1}{(1-x)}$: Soft Singularity
- $\frac{1}{(1-x)_{+}}$: Finite remainder
- $\delta(1-x)$: To be canceled by virtual diagram

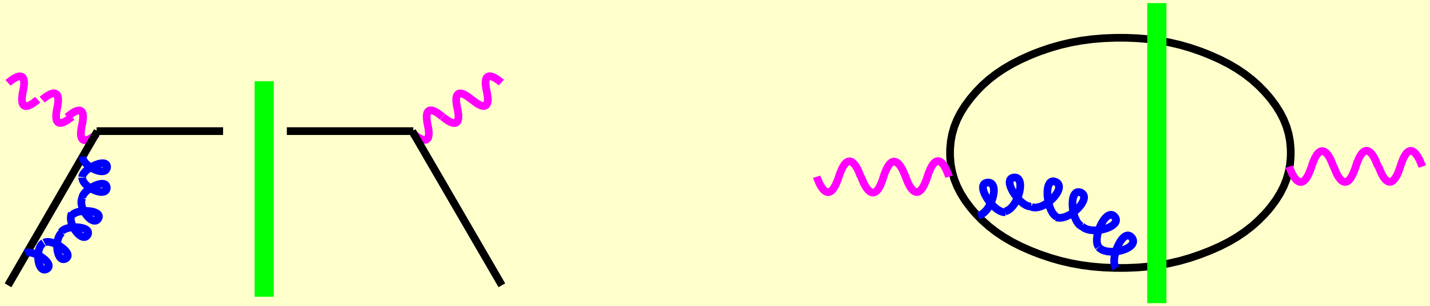
This only makes sense under the integral

$$\frac{f(x)}{(1-x)_{+}} = \frac{f(x)-f(1)}{(1-x)}$$

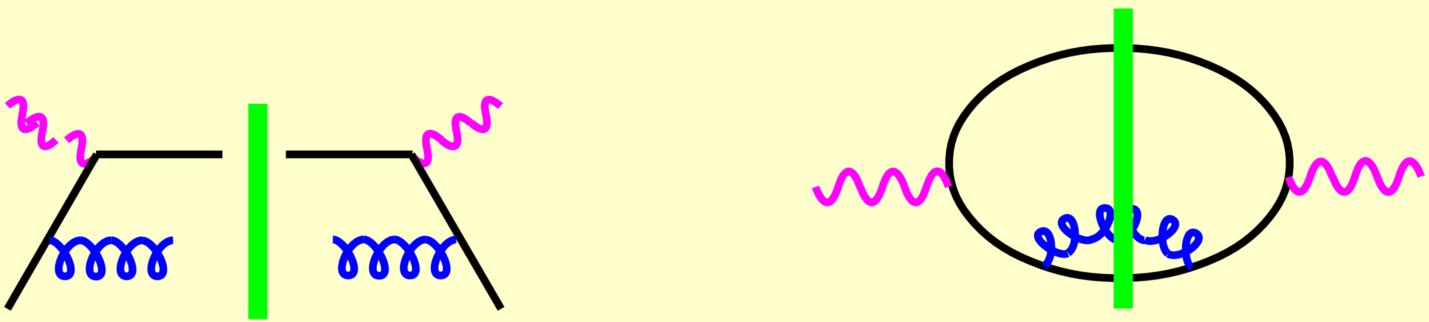
$$\int_0^1 dx\, f(x)\, \frac{x^{\epsilon}}{(1-x)^{1+\epsilon}} = \int_0^1 dx\, \frac{f(x)-f(1)}{(1-x)} - \frac{1}{\epsilon}\int_0^1 dx\, \delta(1-x)\, f(x)$$

KLN (Kinoshita, Lee, Nauenberg) Theorem

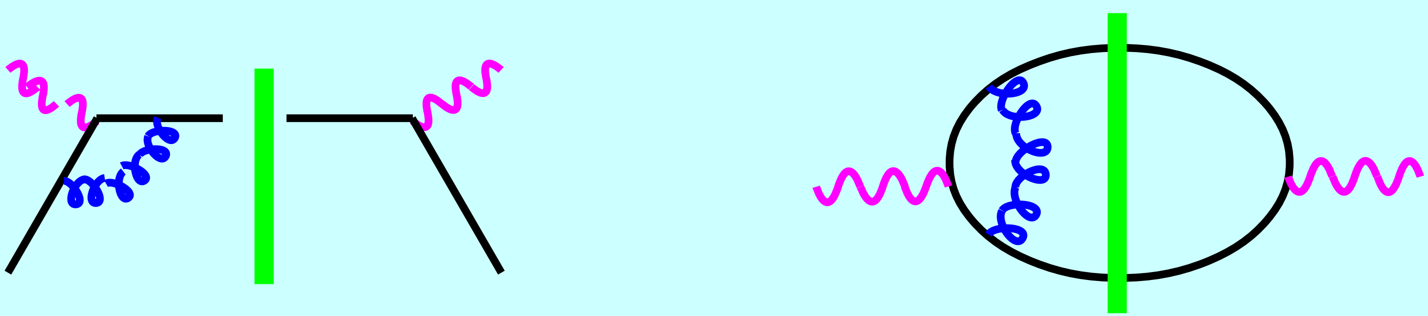
virtual



real



virtual



real

Collinear Singularities

Collinear Singularity

$$\int_{-1}^{1} dz\,(1-z^2)^{-\epsilon}\,|\mathcal{M}|^2 \simeq -\frac{1}{\epsilon}\,\frac{(1+x^2)}{(1-x)} + \frac{1-4x+4(1+x^2)\,\ln 2}{2(1-x)}$$

This looks like part of the PDF

This is finite for z=[-1,1]

... looks like a splitting kernel

Key Points

1) Subtract

2) This is defined by the scheme

3) Need to match schemes of ω and PDF

... MS, MS-Bar, DIS, ...

4) Note we have regulator ε and extra scale μ

How do we know what goes in ω and PDFs ???

Compute NLO Subtractions

for a partonic target

Application of Factorization Formula at Leading Order (LO)

Basic Factorization Formula

$$\sigma = f \otimes \omega + \mathcal{O}(\Lambda^2/Q^2)$$

Higher Twist

At Zeroth Order:

$$\sigma^0 = f^0 \otimes \omega^0 + O(\Lambda^2/Q^2)$$

Use: $f^0 = \delta$ for a parton target.

Therefore:

$$\sigma^0 = f^0 \otimes \omega^0 = \delta \otimes \omega^0 = \omega^0$$

$$\sigma^0 = \omega^0$$

f^0 f^1

for parton target

Warning: *This trivial result leads to many misconceptions at higher orders*

Application of Factorization Formula at NLO

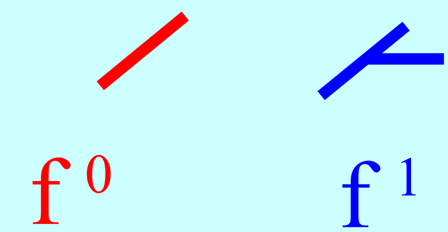
Basic Factorization Formula

$$\sigma = f \otimes \omega + \mathcal{O}(\Lambda^2/Q^2)$$

At First Order:

$$\sigma^1 = f^1 \otimes \omega^0 + f^0 \otimes \omega^1$$

$$\sigma^1 = f^1 \otimes \sigma^0 + \omega^1$$

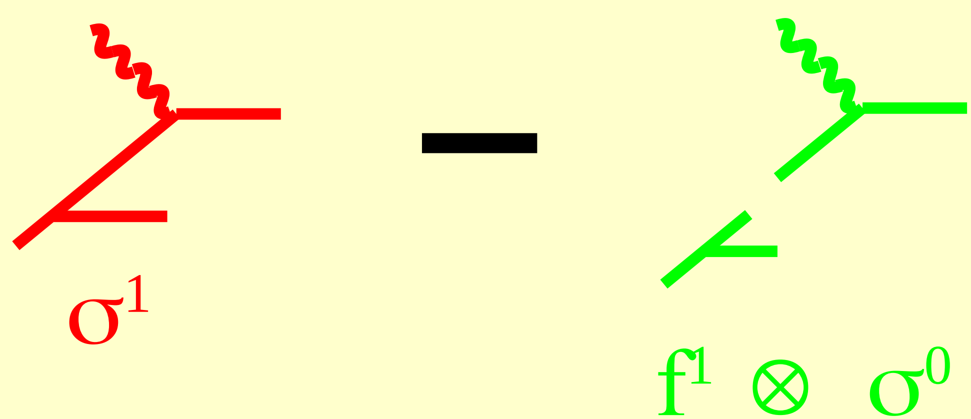


We used: $f^0 = \delta$ for a parton target.

Therefore:

$$\omega^1 = \sigma^1 - f^1 \otimes \sigma^0$$

$\omega^1 =$



$$f^1 \sim \frac{\alpha_s}{2\pi} P^{(1)}$$

$P^{(1)}$ defined by scheme choice

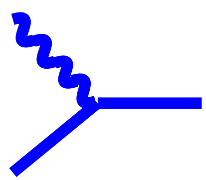
Combined Result:

Complete NLO Term: ω^1

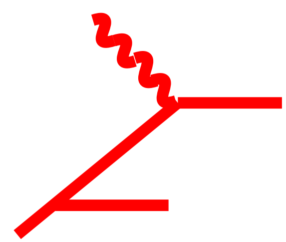
$$\underbrace{\omega^0 + \omega^1}_{\text{TOT}} = \sigma^0 + \overbrace{\sigma^1 - \underbrace{f^1 \otimes \sigma^0}_{\text{SUB}}}^{\text{Complete NLO Term: } \omega^1}$$

TOT

LO

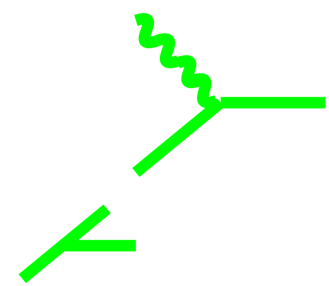


NLO



SUB

Subtraction



TOT = LO + NLO - SUB

Use the Basic Factorization Formula

$$\sigma = f \otimes \omega \otimes d + \mathcal{O}(\Lambda^2/Q^2)$$

At Second Order *(NNLO):*

Therefore:

$$\sigma^2 = f^2 \otimes \omega^0 \otimes d^0 + \ldots + f^1 \otimes \omega^1 \otimes d^0 + \ldots$$

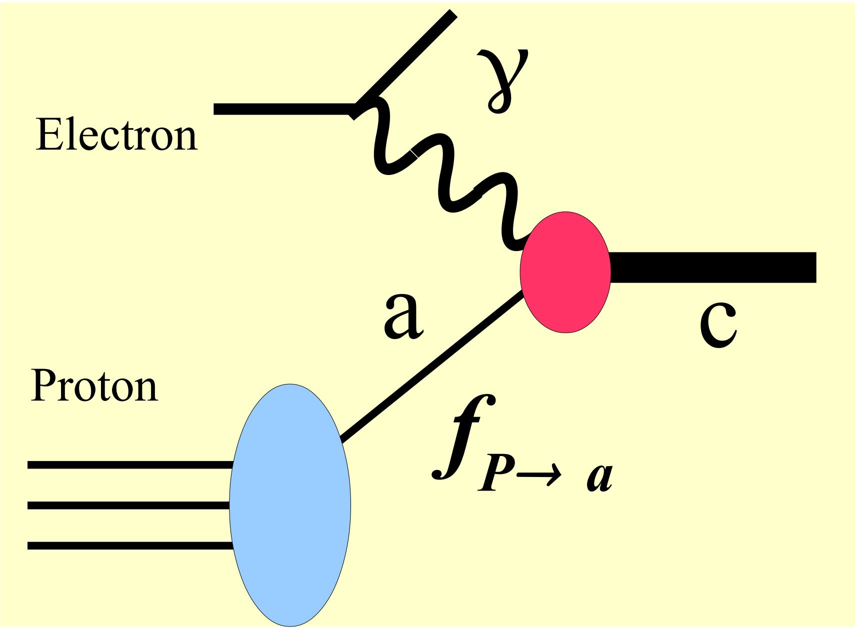
$$\omega^2 = ???$$

Include Fragmentation Functions *d*

Compute ω^2 at second order.
Make a diagrammatic representation of each term.

Do we get different answers with different schemes???

Pictorial Demonstration of Scheme Consistency

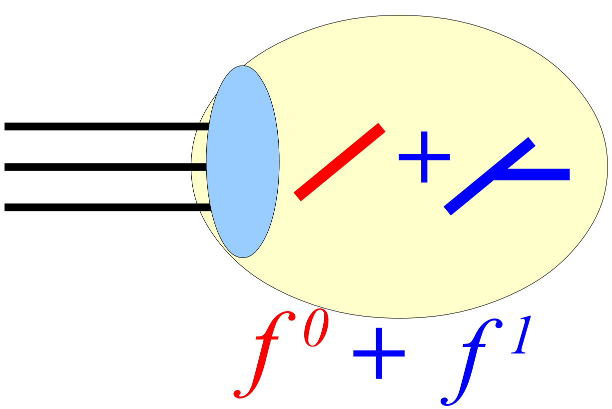


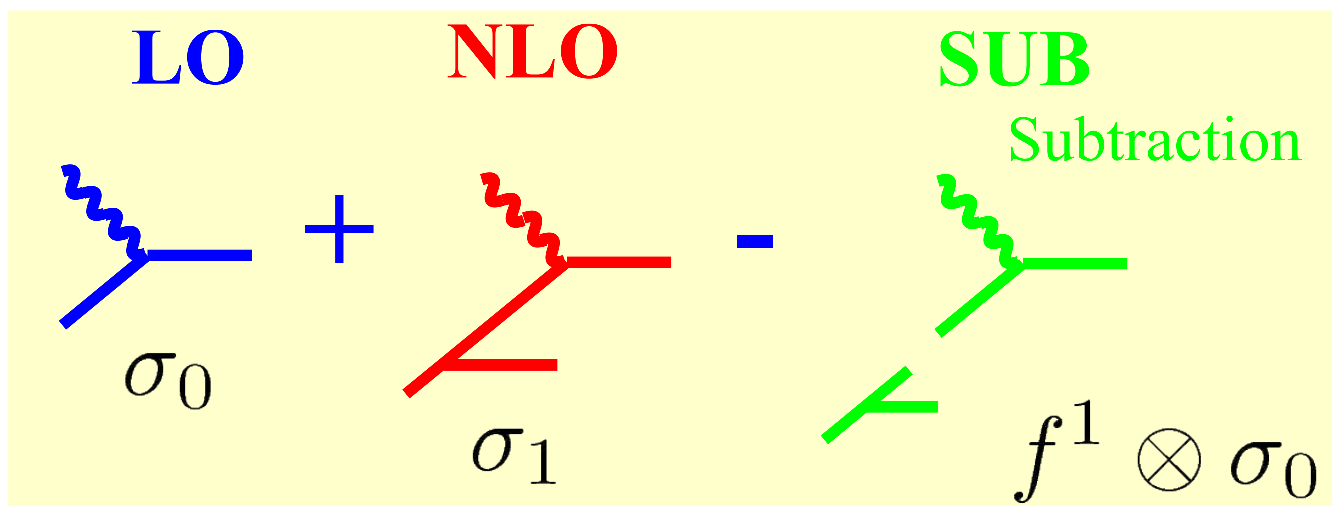
Parton Model

$$\sigma(Q^2) = f(\mu, \alpha_s) \otimes \widehat{\omega}(Q^2, \mu^2, \alpha_s)$$

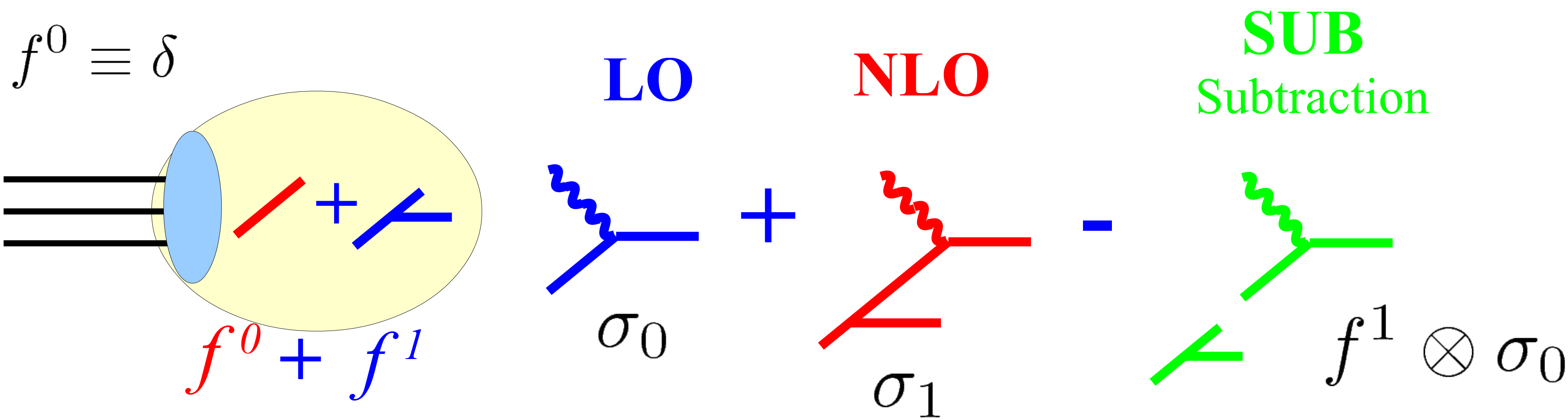
Evolution Equation

$$\frac{df}{\ln[\mu^2]} = P \otimes f$$





Pictorial Demonstration of Scheme Consistency



Complete NLO Term

$$\left[\delta + f^1\right] \otimes \left[\overbrace{\sigma_0 + \sigma_1 - f^1 \otimes \sigma_0}\right]$$

$$f^1 \sim \frac{\alpha_s}{2\pi} P^{(1)}$$

$P^{(1)}$ defined by scheme choice

$$\sigma_0 + \sigma_1 + f^1 \otimes \sigma_0 - f^1 \otimes \sigma_0 + \mathcal{O}(\alpha_s^2)$$

Contains BOTH collinear and non-collinear region

From PDF Evolution

From NLO Subtraction

QCD is Bullet-proof

Do we get different answers with different schemes???

NO !!!

End of lecture 3: Recap

NLO Theoretical Calculations:

- Essential for accurate comparison with experiments

We encounter singularities:

- Soft singularities: cancel between real and virtual diagrams
- Collinear singularities: "absorb" into PDF

Regularization and Renormalization:

- Regularize & Renormalize intermediate quantities
- Physical results independent of regulators (e.g., L, or μ and ε)
- Renormalization introduces scheme dependence (MS-bar, DIS)

Factorization works:

- Hard cross section $\widehat{\sigma}$ or ω is not the same as σ
- Scheme dependence cancels out (if performed consistently)

END OF LECTURE 3