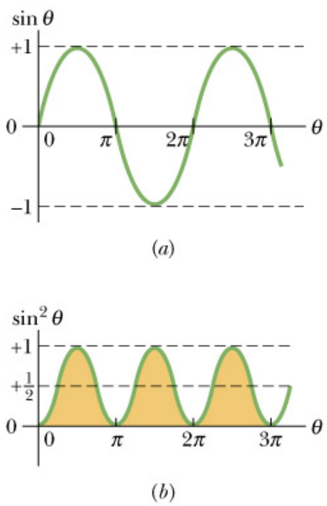
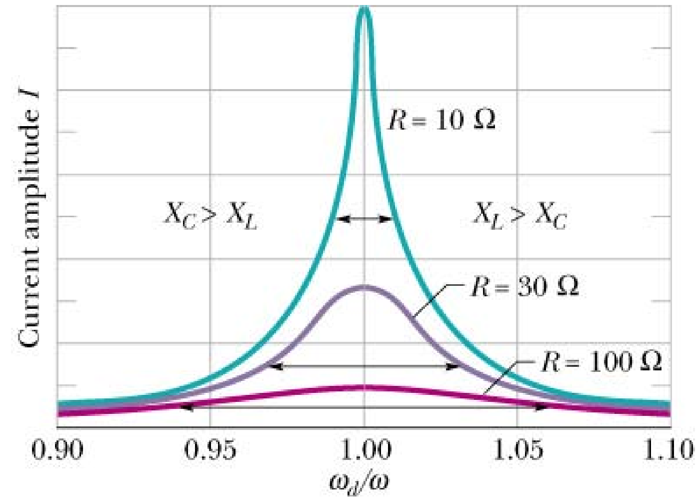
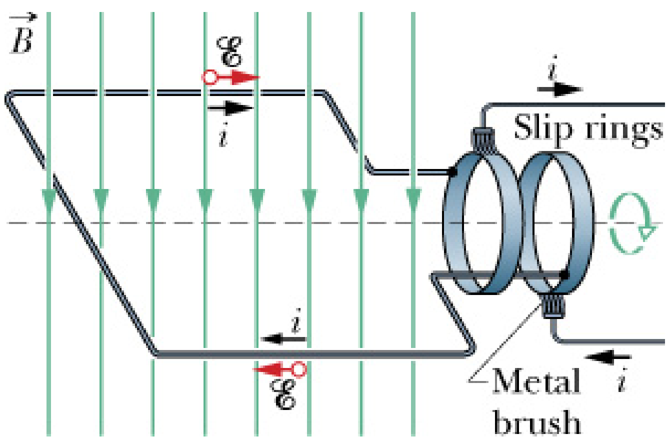
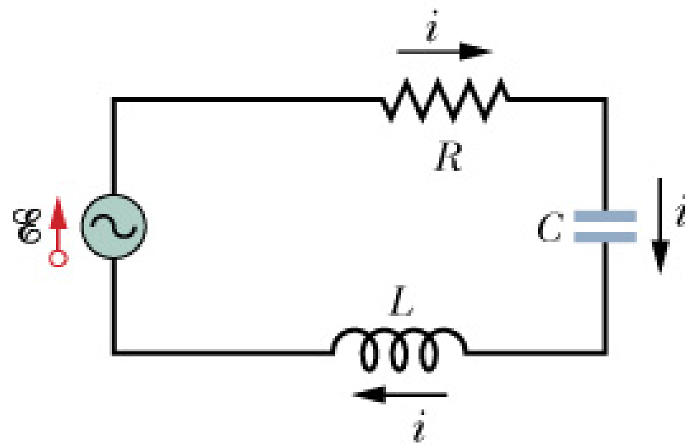
Chapter 21: RLC Circuits





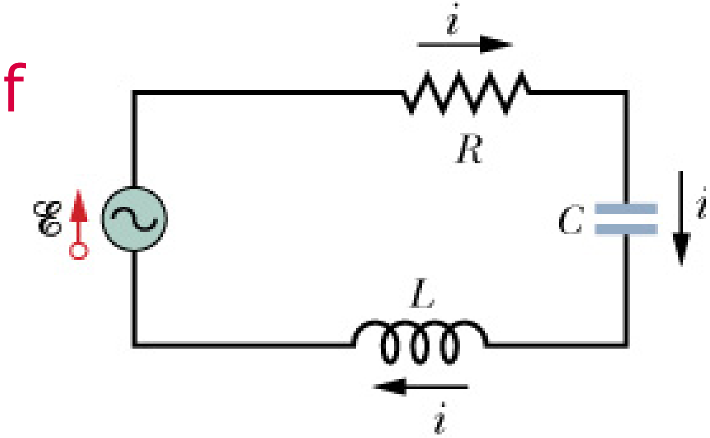




Voltage and Current in RLC Circuits

➔AC emf source: "driving frequency" f

$$\varepsilon = \varepsilon_m \sin\omega t \qquad \omega = 2\pi f$$



➔If circuit contains only R + emf source, current is simple

$$i = \frac{\varepsilon}{R} = I_m \sin(\omega t) \qquad I_m = \frac{\varepsilon_m}{R} \quad (\text{current amplitude})$$

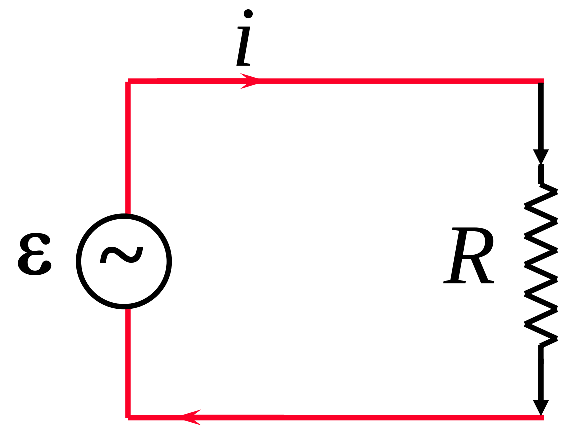
➔If L and/or C present, current is *not* in phase with emf

$$i = I_m \sin(\omega t - \phi) \qquad I_m = \frac{\varepsilon_m}{Z}$$

➔Z, ϕ shown later

AC Source and Resistor Only

➔Driving voltage is $\varepsilon = \varepsilon_m \sin \omega t$



➔Relation of current and voltage

$$i = \varepsilon / R$$

$$i = I_m \sin \omega t \qquad I_m = \frac{\varepsilon_m}{R}$$

- Current is *in phase* with voltage ($\phi = 0$)

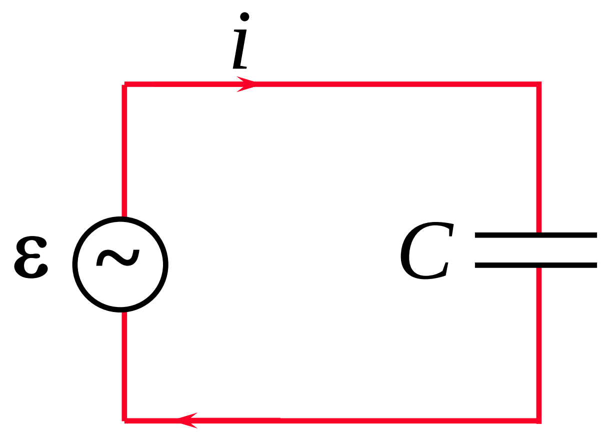
AC Source and Capacitor Only

➔Voltage is $v_C = \dfrac{q}{C} = \varepsilon_m \sin\omega t$

➔Differentiate to find current

$$q = C\varepsilon_m \sin\omega t$$

$$i = dq/dt = \omega C V_C \cos\omega t$$



➔Rewrite using phase (check this!)

$$i = \omega C V_C \sin\left(\omega t + 90^\circ\right)$$

➔Relation of current and voltage

$$i = I_m \sin\left(\omega t + 90^\circ\right) \quad I_m = \frac{\varepsilon_m}{X_C} \quad \left(X_C = 1/\omega C\right)$$

➔"Capacitive reactance": $X_C = 1/\omega C$

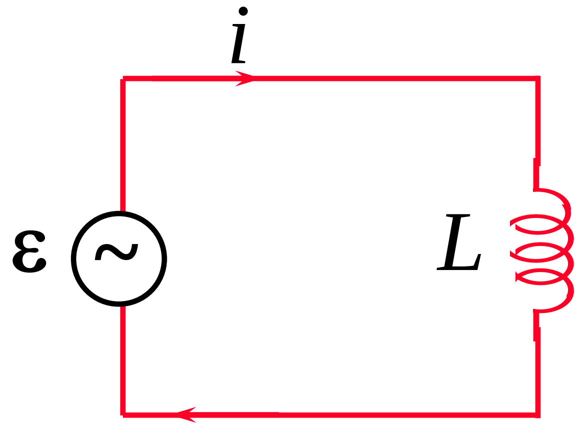
- ◆Current "leads" voltage by 90°

AC Source and Inductor Only

- ➔Voltage is $v_L = L\,di/dt = \varepsilon_m \sin\omega t$
- ➔Integrate di/dt to find current:

$$di/dt = \left(\varepsilon_m / L\right)\sin\omega t$$

$$i = -\left(\varepsilon_m / \omega L\right)\cos\omega t$$



- ➔Rewrite using phase (check this!)

$$i = \left(\varepsilon_m / \omega L\right)\sin\left(\omega t - 90°\right)$$

- ➔Relation of current and voltage

$$i = I_m \sin\left(\omega t - 90°\right) \quad I_m = \frac{\varepsilon_m}{X_L} \quad \left(X_L = \omega L\right)$$

- ➔"Inductive reactance": $X_L = \omega L$
  - ◆Current "lags" voltage by 90°

General Solution for RLC Circuit

- ➔ We *assume* steady state solution of form $i = I_m \sin(\omega t - \phi)$
  - ◆ I_m is current amplitude
  - ◆ ϕ is phase by which current "lags" the driving EMF
  - ◆ Must determine I_m and ϕ
- ➔ Plug in solution: differentiate & integrate sin(ωt-ϕ)

$$\left.\begin{aligned} i &= I_m \sin(\omega t - \phi) \\ \frac{di}{dt} &= \omega I_m \cos(\omega t - \phi) \\ q &= -\frac{I_m}{\omega}\cos(\omega t - \phi) \end{aligned}\right\} \xrightarrow{\text{Substitute}} L\frac{di}{dt} + Ri + \frac{q}{C} = \varepsilon_m \sin \omega t$$

$$\downarrow$$

$$\boxed{I_m \omega L \cos(\omega t - \phi) + I_m R \sin(\omega t - \phi) - \frac{I_m}{\omega C}\cos(\omega t - \phi) = \varepsilon_m \sin \omega t}$$

General Solution for RLC Circuit (2)

$$I_m \omega L \cos(\omega t - \phi) + I_m R \sin(\omega t - \phi) - \frac{I_m}{\omega C} \cos(\omega t - \phi) = \varepsilon_m \sin \omega t$$

➔Expand sin & cos expressions

$$\left.\begin{aligned} \sin(\omega t - \phi) &= \sin \omega t \cos\phi - \cos \omega t \sin\phi \\ \cos(\omega t - \phi) &= \cos \omega t \cos\phi + \sin \omega t \sin\phi \end{aligned}\right\} \quad \text{High school trig!}$$

➔Collect $\sin\omega t$ & $\cos\omega t$ terms separately

$$\left.\begin{aligned} (\omega L - 1/\omega C)\cos\phi - R\sin\phi &= 0 \\ I_m(\omega L - 1/\omega C)\sin\phi + I_m R\cos\phi &= \varepsilon_m \end{aligned}\right\} \quad \begin{aligned} &\cos\omega t \text{ terms} \\ &\sin\omega t \text{ terms} \end{aligned}$$

➔These equations can be solved for I_m and ϕ (next slide)

General Solution for RLC Circuit (3)

➔Solve for ϕ and I_m

$$\tan\phi = \frac{\omega L - 1/\omega C}{R} \equiv \frac{X_L - X_C}{R} \qquad I_m = \frac{\varepsilon_m}{Z}$$

➔R, X_L, X_C and Z have dimensions of resistance

$$X_L = \omega L$$ Inductive "reactance"

$$X_C = 1/\omega C$$ Capacitive "reactance"

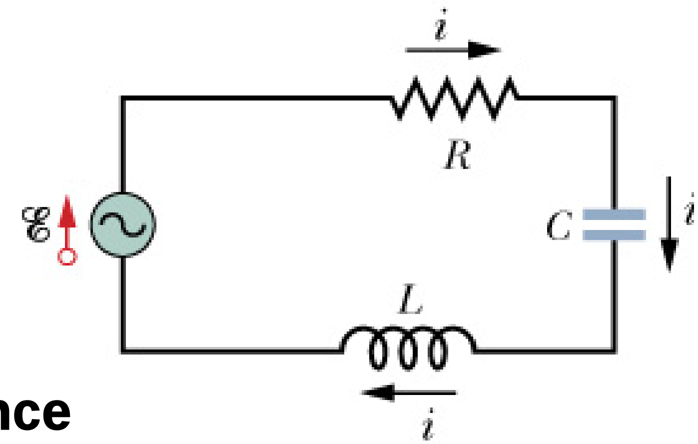
$$Z = \sqrt{R^2 + \left(X_L - X_C\right)^2}$$ Total "impedance"

➔This is where ϕ, X_L, X_C and Z come from!

AC Source and RLC Circuits

$$I_m = \frac{\mathcal{E}_m}{Z}$$ **Maximum current**

$$\tan\phi = \frac{X_L - X_C}{R}$$ **Phase angle**



$$X_L = \omega L \quad (\omega = 2\pi f)$$ **Inductive reactance**

$$X_C = 1/\omega C$$ **Capacitive reactance**

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$ **Total impedance**

ϕ= angle that current "lags" applied voltage

What is Reactance?

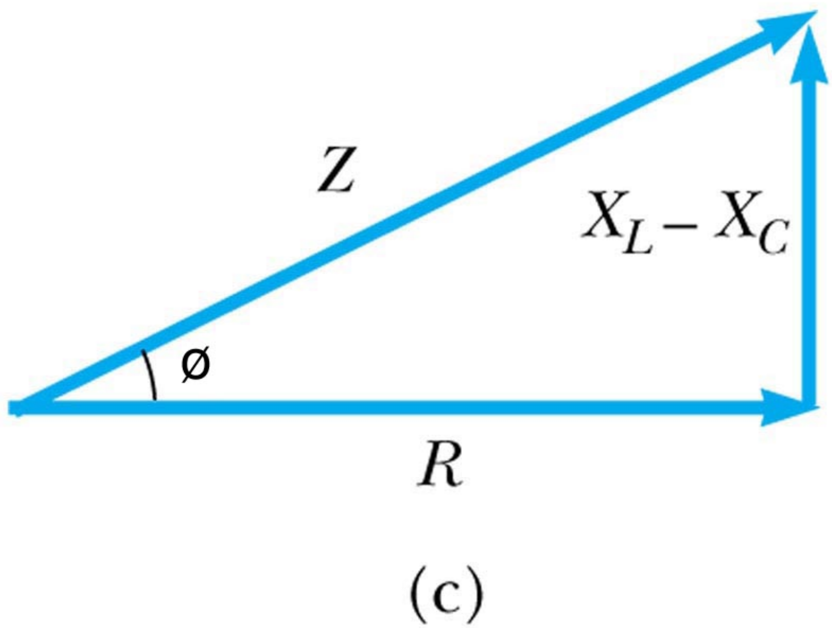
Think of it as a frequency-dependent resistance

$$X_C = \frac{1}{\omega C}$$ Shrinks with increasing ω

$$X_L = \omega L$$ Grows with increasing ω

$$(\text{"}X_R\text{"} = R)$$ Independent of ω

Pictorial Understanding of Reactance



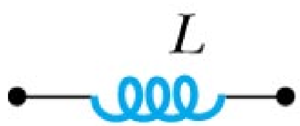
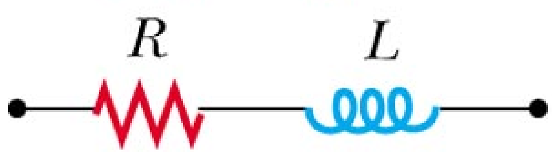
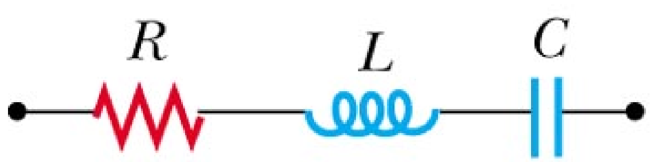
$$Z = \sqrt{R^2 + \left(X_L - X_C\right)^2}$$

$$\tan\phi = \frac{X_L - X_C}{R}$$

$$\cos\phi = \frac{R}{Z}$$

Summary of Circuit Elements, Impedance, Phase Angles

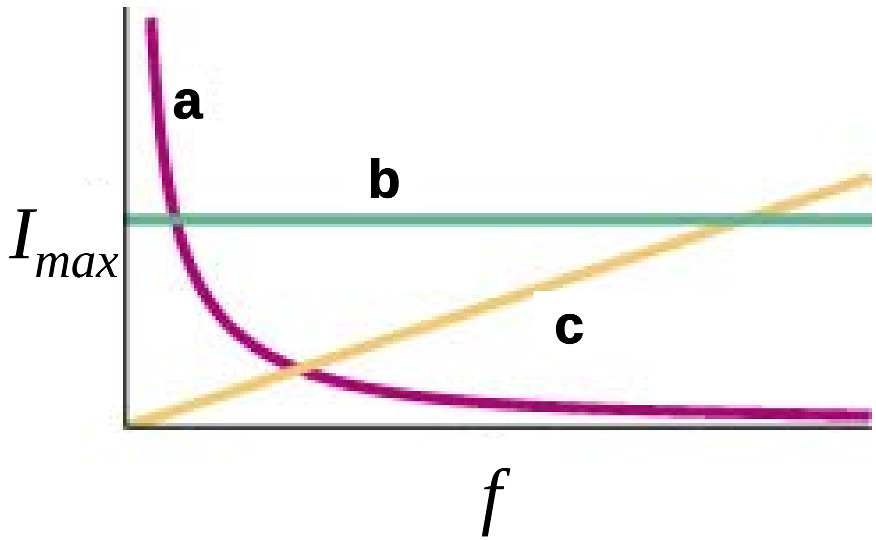
$$Z = \sqrt{R^2 + \left(X_L - X_C\right)^2} \qquad \tan\phi = \frac{X_L - X_C}{R}$$

	R	$0°$
	X_C	$-90°$
	X_L	$+90°$
	$\sqrt{R^2 + X_C^2}$	Negative, between $-90°$ and $0°$
	$\sqrt{R^2 + X_L^2}$	Positive, between $0°$ and $90°$
	$\sqrt{R^2 + (X_L - X_C)^2}$	Negative if $X_C > X_L$ Positive if $X_C < X_L$

Quiz

➔ Three identical EMF sources are hooked to a single circuit element, a resistor, a capacitor, or an inductor. The current amplitude is then measured as a function of frequency. Which one of the following curves corresponds to an inductive circuit?

- ◆ (1) a
- ◆ (2) b
- ◆ (3) c
- ◆ (4) Can't tell without more info



$$X_L = \omega L \quad (\omega = 2\pi f)$$

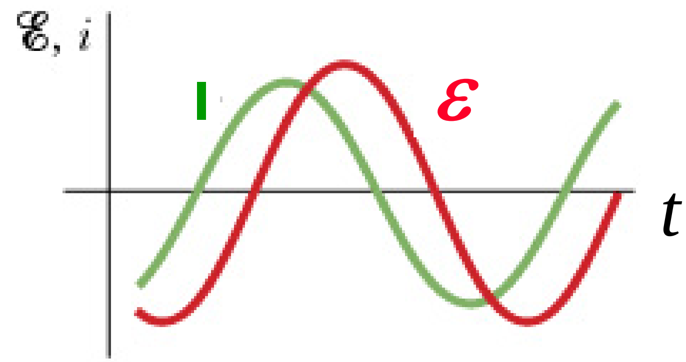
$$I_{\max} = \varepsilon_{\max} / X_L$$

For inductor, higher frequency gives higher reactance, therefore lower current

RLC Example 1

➔ Below are shown the driving emf and current vs time of an RLC circuit. We can conclude the following

- Current "leads" the driving emf ($\phi<0$)
- Circuit is capacitive ($X_C > X_L$)



RLC Example 2

➔R = 200Ω, C = 15μF, L = 230mH, ε_{max} = 36v, f = 60 Hz

- $X_L = 2\pi \times 60 \times 0.23 = 86.7\Omega$
- $X_C = 1/\left(2\pi \times 60 \times 15 \times 10^{-6}\right) = 177\Omega$

$X_C > X_L$
Capacitive circuit

- $Z = \sqrt{200^2 + \left(86.7 - 177\right)^2} = 219\Omega$
- $I_{\text{max}} = \varepsilon_{\text{max}} / Z = 36/219 = 0.164\,\text{A}$
- $\phi = \tan^{-1}\left(\dfrac{86.7 - 177}{200}\right) = -24.3°$

Current leads emf
(as expected)

$$\boxed{i = 0.164\sin\left(\omega t + 24.3°\right)}$$

Resonance

➔ Consider impedance vs frequency

$$Z = \sqrt{R^2 + \left(X_L - X_C\right)^2} = \sqrt{R^2 + \left(\omega L - 1/\omega C\right)^2}$$

➔ Z is minimum when $\omega L = 1/\omega C \quad \omega = \omega_0 = 1/\sqrt{LC}$

- This is resonance!

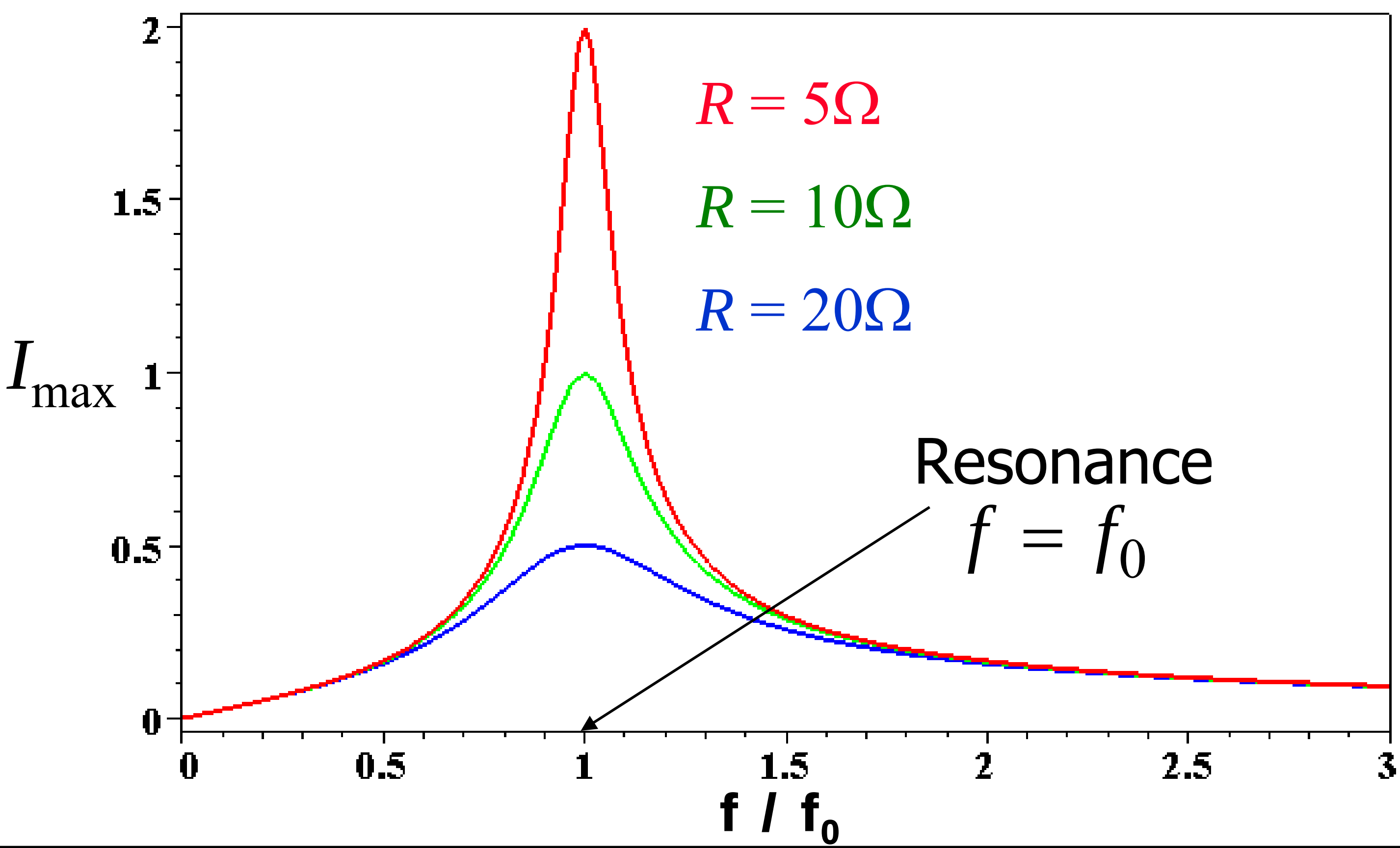
➔ At resonance

- Impedance = Z is minimum
- Current amplitude = I_m is maximum

I_{max} vs Frequency and Resonance

- ➔Circuit parameters: C = 2.5μF, L = 4mH, ε_{max} = 10v
  - $f_0 = 1 / 2\pi(LC)^{1/2} = 1590$ Hz
  - Plot I_{max} vs f

$$I_{max} = 10 / \sqrt{R^2 + (\omega L - 1/\omega C)^2}$$



Power in AC Circuits

- Instantaneous power emitted by circuit: $P = i^2 R$

$$P = I_m^2 R \sin^2(\omega_d t - \phi)$$ ← Instantaneous power oscillates

- More useful to calculate power averaged over a cycle
  - Use <…> to indicate average over a cycle

$$\langle P \rangle = I_m^2 R \left\langle \sin^2(\omega_d t - \phi) \right\rangle = \tfrac{1}{2} I_m^2 R$$

- Define RMS quantities to avoid ½ factors in AC circuits

$$\boxed{I_{\text{rms}} = \frac{I_m}{\sqrt{2}}} \qquad \boxed{\varepsilon_{\text{rms}} = \frac{\varepsilon_m}{\sqrt{2}}} \qquad \boxed{P_{\text{ave}} = I_{\text{rms}}^2 R}$$

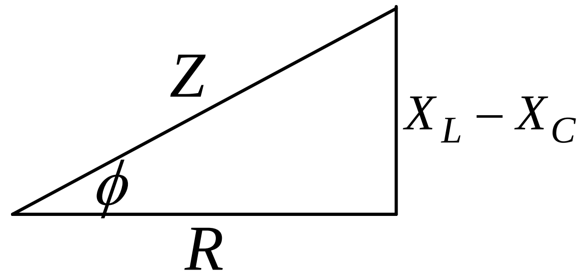
- House current
  - $V_{\text{rms}} = 110\text{V} \Rightarrow V_{\text{peak}} = 156\text{V}$

Power in AC Circuits

➔ Power formula $P_{\text{ave}} = I_{\text{rms}}^2 R \quad I_{\text{rms}} = I_{\max} / \sqrt{2}$

➔ Rewrite using $I_{\text{rms}} = \dfrac{\varepsilon_{\text{rms}}}{Z} \qquad P_{\text{ave}} = \dfrac{\varepsilon_{\text{rms}}}{Z} I_{\text{rms}} R = \varepsilon_{\text{rms}} I_{\text{rms}} \cos\phi$

$$\boxed{P_{\text{ave}} = \varepsilon_{\text{rms}} I_{\text{rms}} \cos\phi} \qquad \boxed{\cos\phi = \frac{R}{Z}}$$



➔ cosϕ is the "power factor"

- To maximize power delivered to circuit ⇒ make ϕ close to zero
- Max power delivered to load happens at resonance
- E.g., too much inductive reactance (X_L) can be cancelled by increasing X_C (e.g., circuits with large motors)

Power Example 1

➔R = 200Ω, X_C = 150Ω, X_L = 80Ω, ε_{rms} = 120v, f = 60 Hz

- $Z = \sqrt{200^2 + (80-150)^2} = 211.9\Omega$
- $I_{\text{rms}} = \varepsilon_{\text{rms}} / Z = 120/211.9 = 0.566\,\text{A}$
- $\phi = \tan^{-1}\left(\dfrac{80-150}{200}\right) = -19.3°$ ← Current leads emf Capacitive circuit
- $\cos\phi = 0.944$
- $P_{\text{ave}} = \varepsilon_{\text{rms}} I_{\text{rms}} \cos\phi = 120 \times 0.566 \times 0.944 = 64.1\,\text{W}$
- $P_{\text{ave}} = I_{\text{rms}}^2 R = 0.566^2 \times 200 = 64.1\,\text{W}$

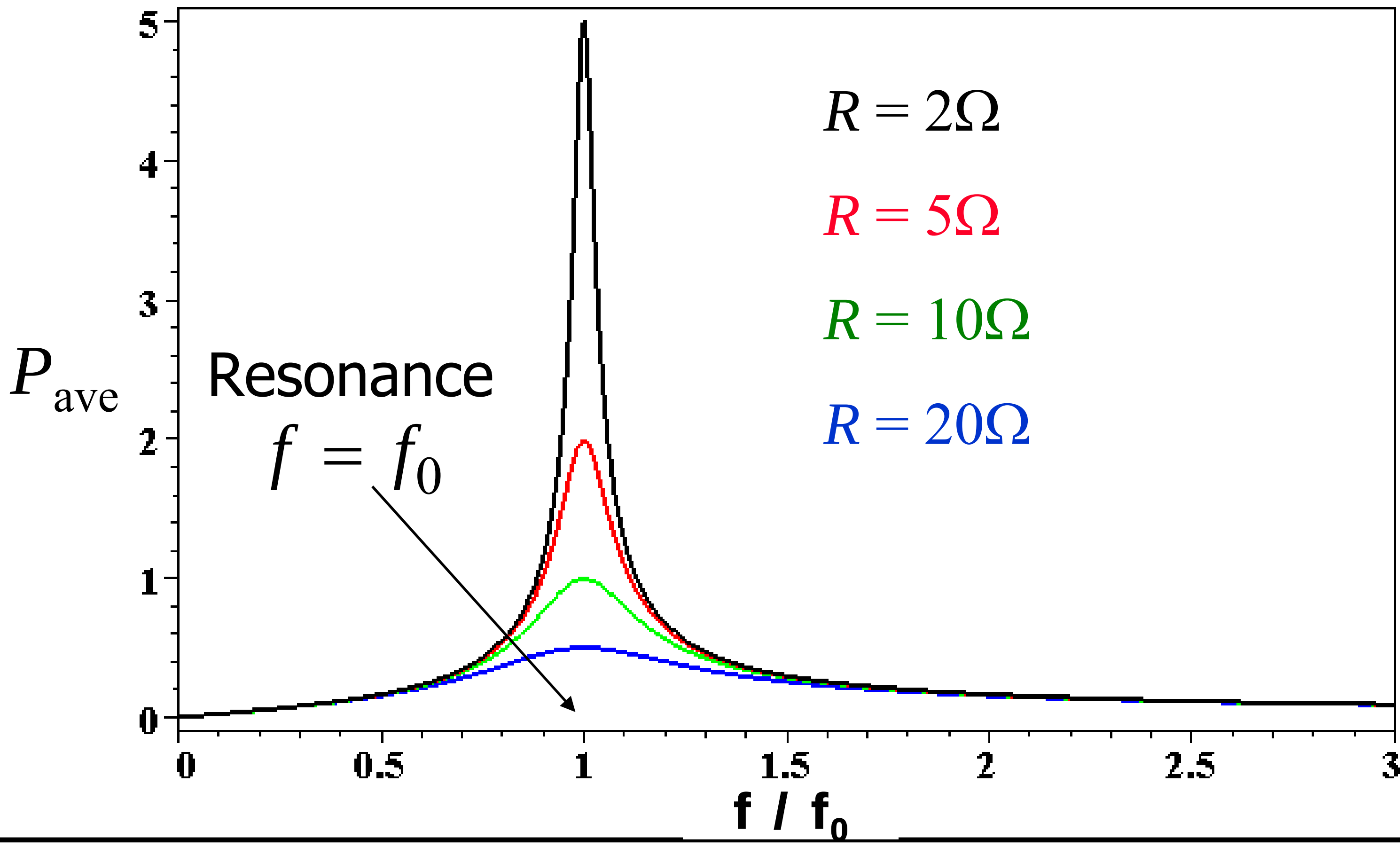
Same

Power Example 1 (cont)

- ➔ $R = 200\Omega$, $X_C = 150\Omega$, $X_L = 80\Omega$, $\varepsilon_{rms} = 120$v, $f = 60$ Hz
- ➔ How much capacitance must be added to maximize the power in the circuit (and thus bring it into resonance)?
  - Want $X_C = X_L$ to minimize Z, so must decrease X_C
  - $X_C = 150\Omega = 1/2\pi fC$ $\quad C = 17.7\,\mu\text{F}$
  - $X_{C\,\text{new}} = X_L = 80\Omega$ $\quad C_{\text{new}} = 33.2\,\mu\text{F}$
  - So we must add 15.5μF capacitance to maximize power

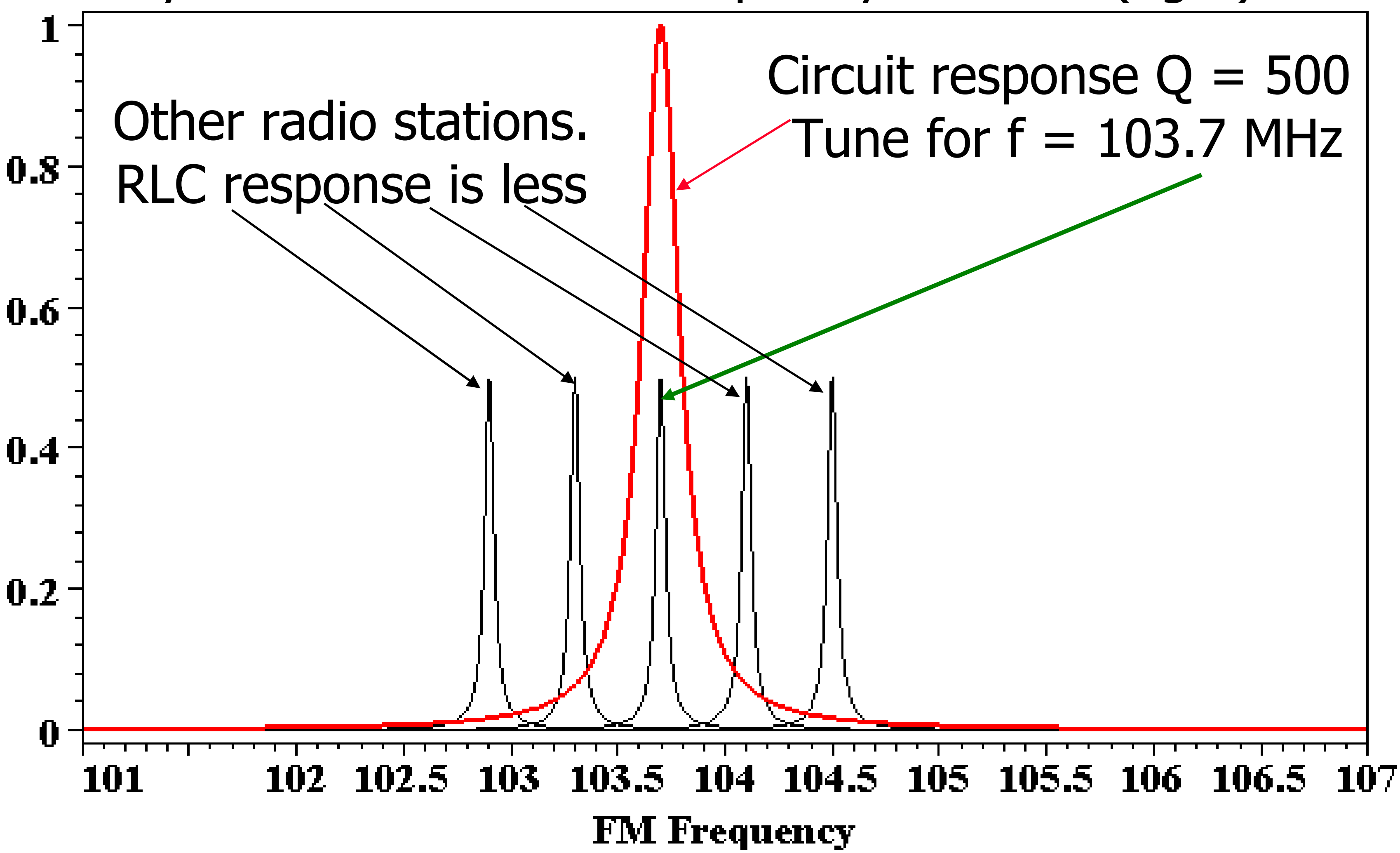
Power vs Frequency and Resonance

- Circuit parameters: C = 2.5μF, L = 4mH, ε_{max} = 10v
  - $f_0 = 1 / 2\pi(LC)^{1/2}$ = 1590 Hz
  - Plot P_{ave} vs f for different R values



Resonance Tuner is Based on Resonance

Vary C to set resonance frequency to 103.7 (ugh!)



Quiz

➔A generator produces current at a frequency of 60 Hz with peak voltage and current amplitudes of 100V and 10A, respectively. What is the average power produced if they are in phase?

- (1) 1000 W
- (2) 707 W
- (3) 1414 W
- (4) 500 W
- (5) 250 W

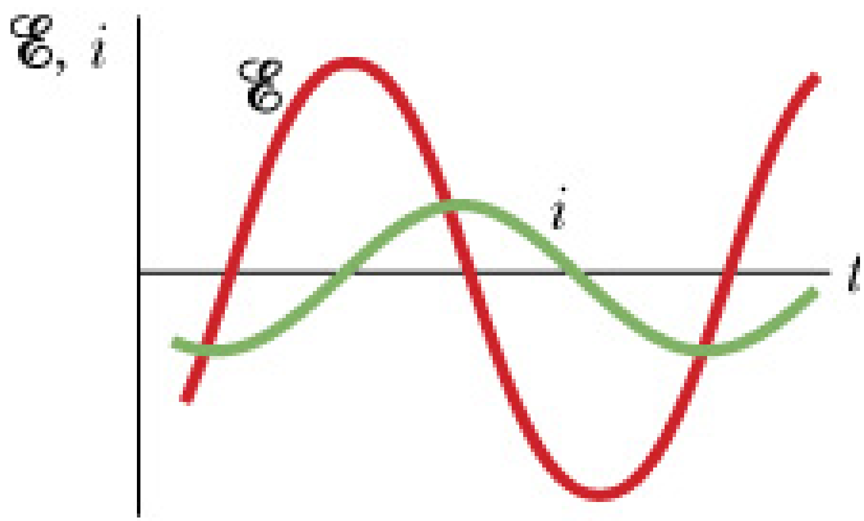
$$P_{\text{ave}} = \frac{1}{2}\varepsilon_{\text{peak}} I_{\text{peak}} = \varepsilon_{\text{rms}} I_{\text{rms}}$$

Quiz

➔The figure shows the current and emf of a series RLC circuit. To increase the rate at which power is delivered to the resistive load, which option should be taken?

- (1) Increase R
- (2) Decrease L
- (3) Increase L
- (4) Increase C

$$\tan \phi = \frac{X_L - X_C}{R}$$



Current lags applied emf ($\phi > 0$), thus circuit is inductive. Either
(1) Reduce X_L by decreasing L or
(2) Cancel X_L by increasing X_C (decrease C).

Example: LR Circuit

➔ Variable frequency EMF source with ε_m=6V connected to a resistor and inductor. R=80Ω and L=40mH.

- At what frequency f does $V_R = V_L$?

$$X_L = \omega L = R \Rightarrow \omega = 2000 \qquad f = 2000/2\pi = 318\,\text{Hz}$$

- At that frequency, what is phase angle ϕ?

$$\tan\phi = X_L/R = 1 \Rightarrow \phi = 45^\circ$$

- What is the current amplitude and RMS value?

$$I_{\max} = \varepsilon_{\max}/\sqrt{80^2+80^2} = 6/113 = 0.053\,\text{A}$$

$$I_{\text{rms}} = I_{\max}/\sqrt{2} = 0.037\,\text{A}$$

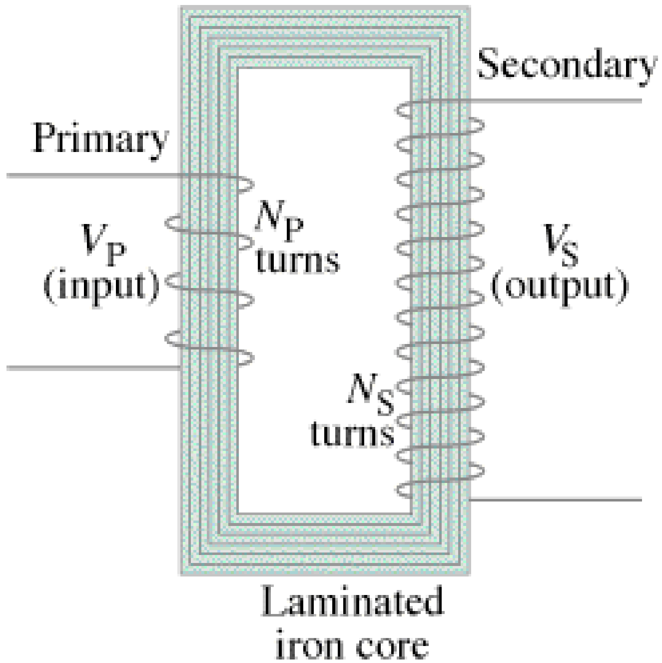
$$\boxed{i = 0.053\sin\left(\omega t - 45^\circ\right)}$$

Transformers

➔Purpose: change alternating (AC) voltage to a bigger (or smaller) value

Input AC voltage in the "primary" turns produces a flux

Changing flux in "secondary" turns induces an emf



$$V_p = N_p \frac{\Delta\Phi_B}{\Delta t}$$

$$V_s = N_s \frac{\Delta\Phi_B}{\Delta t}$$

$$\boxed{V_s = V_p \frac{N_s}{N_p}}$$

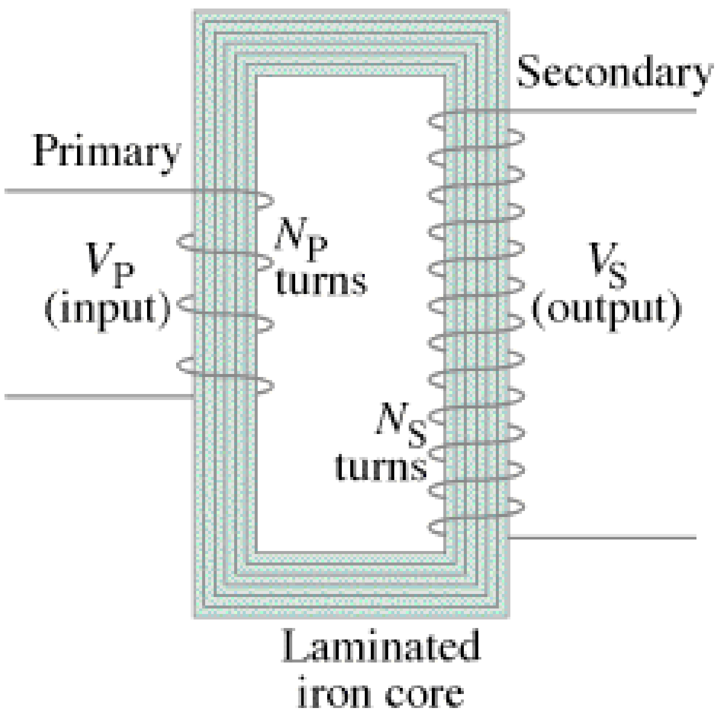
Transformers

➔Nothing comes for free, however!

- Increase in voltage comes at the cost of current.
- Output power cannot exceed input power!
- power in = power out
- (Losses usually account for 10-20%)

$$i_p V_p = i_s V_s$$

$$\boxed{\frac{i_s}{i_p} = \frac{V_p}{V_s} = \frac{N_p}{N_s}}$$



Transformers: Sample Problem

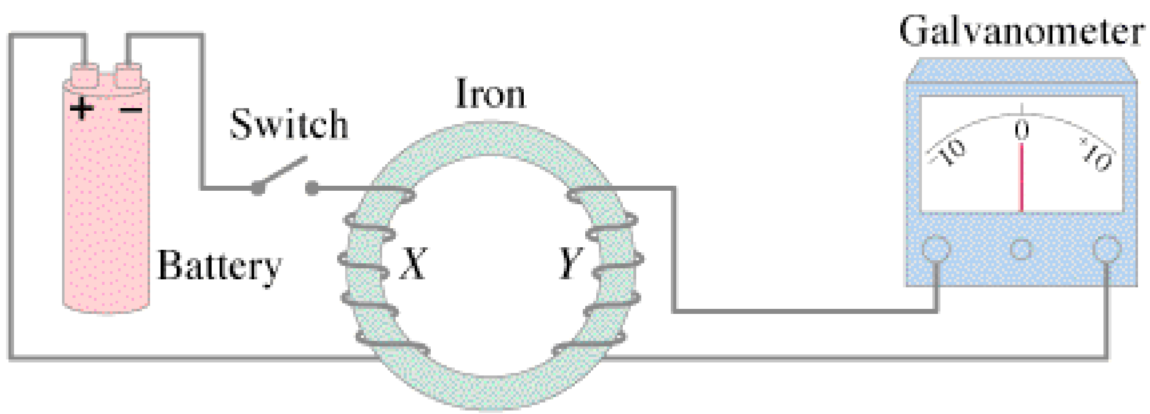
➔A transformer has 330 primary turns and 1240 secondary turns. The input voltage is 120 V and the output current is 15.0 A. What is the output voltage and input current?

$$V_s = V_p \frac{N_s}{N_p} = 120\left(\frac{1240}{330}\right) = 451\,\text{V}$$

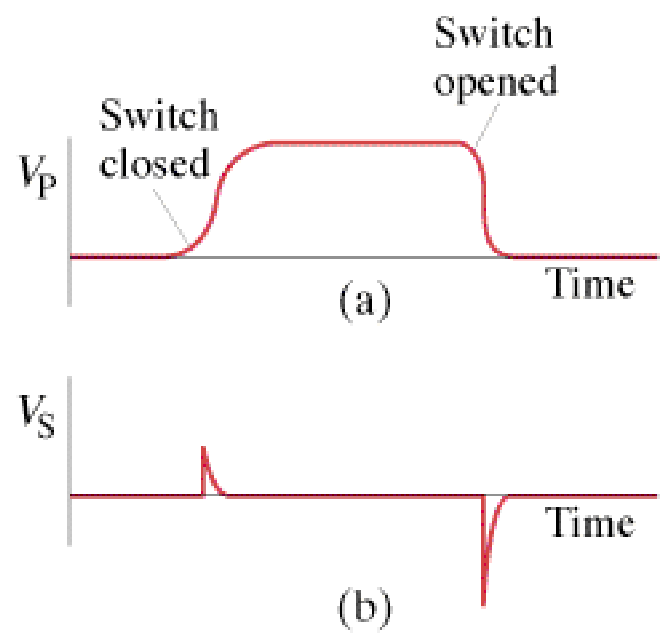
"Step-up" transformer

$$i_p V_p = i_s V_s \quad \Rightarrow \quad i_p = i_s \frac{V_s}{V_p} = 15\left(\frac{451}{120}\right) = 56.4\,\text{A}$$

Transformers

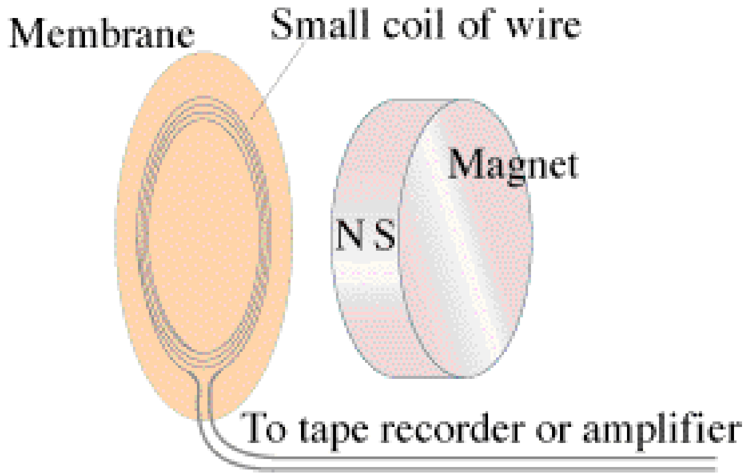


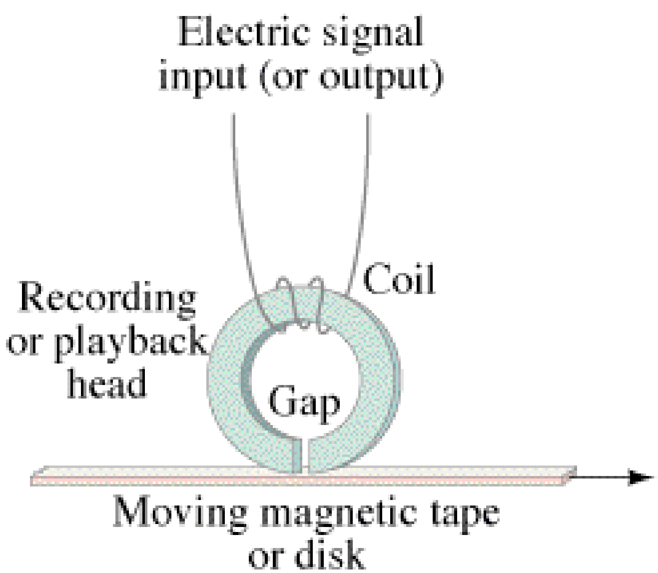
- This is how first experiment by Faraday was done
- He only got a deflection of the galvanometer when the switch is opened or closed
- Steady current does not make induced emf.



Applications

Microphone





Tape recorder

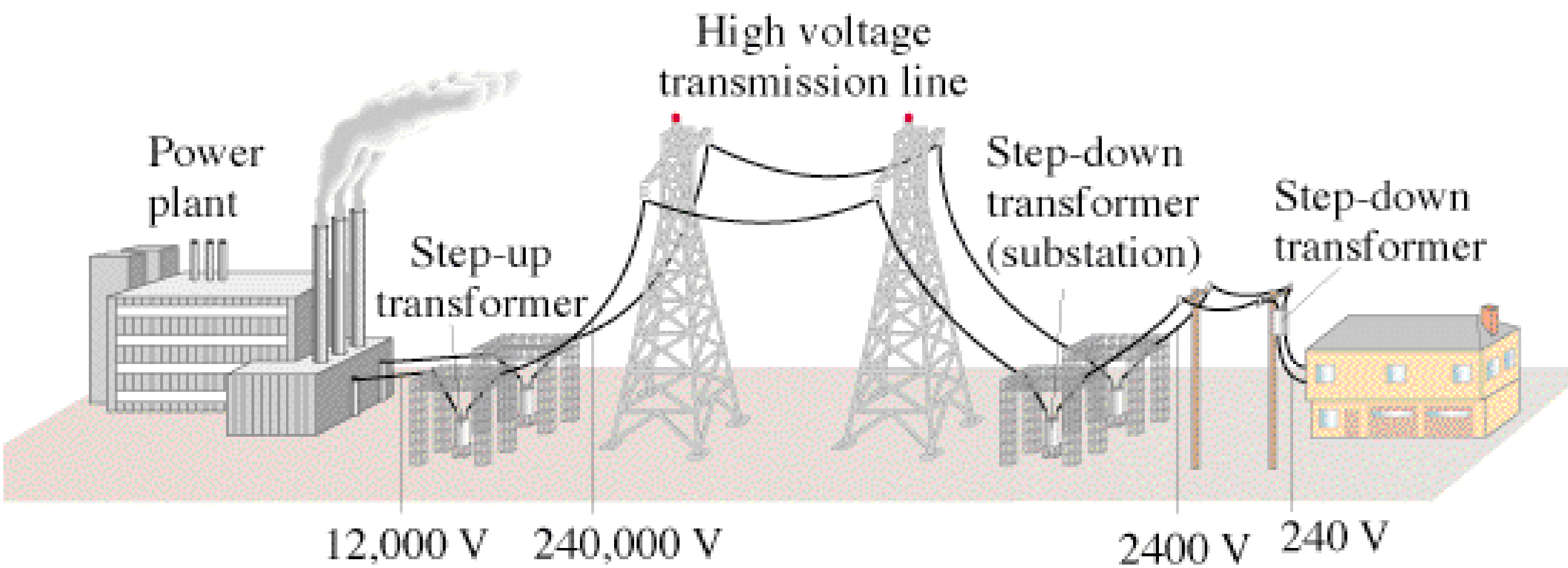
ConcepTest: Power lines

➔At large distances, the resistance of power lines becomes significant. To transmit maximum power, is it better to transmit (high V, low i) or (high i, low V)?

- (1) high V, low i
- (2) low V, high i
- (3) makes no difference

Power loss is i^2R

Electric Power Transmission



i^2R: 20x smaller current $\Rightarrow$ 400x smaller power loss