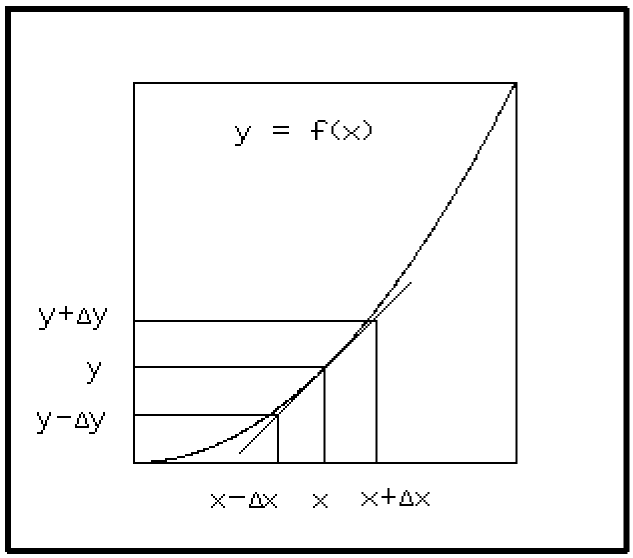
Error On The Mean

The purpose of this document is to derive an expression for the error on the average value of a set of measurements. The measurements are assumed to be random and that their histogram is Gaussian. Calculus is used to find an expression for the uncertainty in a function of a single variable given a measurement of the independent variable and its associated uncertainty. These ideas are combined to find an expression for the error on the average of a set of measurements.

Given a measurement $\{x \pm \Delta x\}$ and a function $y = f(x)$, how do we find Δy?



We approximate the function with a straight line, tangent to the curve at x, as shown in the figure above.

The formula for the slope of this tangent line is: $slope \;=\; \frac{\Delta y}{\Delta x}$. Solving for Δy we get
$\Delta y = \; slope \;\times\; \Delta x$

The slope of the tangent line is just the first derivative of f evaluated at x. $slope \;=\; f'(x)$. The slope can be negative but all uncertainties are intrinsically positive, so we take the absolute value.

Solving for Δy we get: $\boxed{\Delta y \;=\; \left|f'(x)\right| \times \Delta x}$

Given a set of n measurements of x_i with their associated uncertainties Δx_i, find the average value $\bar{x}$ and its uncertainty $\Delta \bar{x}$.

First we find the sum S and its error ΔS.

$S = x_1 + x_2 + x_3 + \;\ldots\; + x_n$ with $\Delta S = \sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 + \;\ldots\; + (\Delta x_n)^2}$

If all of the measurements were made with the same instruments e.g. rulers, then all of the uncertainties are the same, that is $\Delta x_i = \Delta x$ for all values of i.

Therefore,

$$\begin{aligned}\Delta S &= \sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 + \ ... \ + (\Delta x_n)^2} \\ &= \sqrt{(\Delta x)^2 + (\Delta x)^2 + (\Delta x)^2 + \ ... \ + (\Delta x)^2} \\ &= \sqrt{n(\Delta x)^2} \\ &= \sqrt{n}\,\Delta x\end{aligned}$$

If we write the average $\bar{x}$ as a function of S then $\bar{x} = \frac{1}{n} S$ and

$\Delta \bar{x} = \left| \frac{d}{dS}\left(\frac{1}{n}S\right) \right| \Delta S = \frac{1}{n}\Delta S$ but $\Delta S = \sqrt{n}\,\Delta x$

so $\Delta \bar{x} = \frac{\sqrt{n}}{n}\Delta x$ and we have $\boxed{\Delta \bar{x} = \frac{\Delta x}{\sqrt{n}}}$

References

- John R. Taylor, An Introduction To Error Analysis (2nd ed.), Ch 3: Propagation of Uncertainties, University Science Books, 1997.