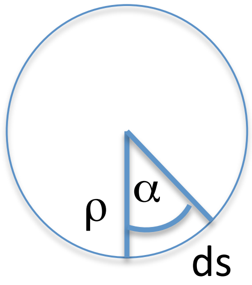
Define design trajectory (orbit)

- Length of dipole magnet and field define total bending angle of magnet:

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{B dl}{B\rho}$$

- Circular accelerator: total bending angle:= 2π

$$\alpha = 2\pi = \frac{\int B \ dl}{B\rho} = \frac{\int B \ dl}{\frac{p}{q}}$$



- **How many dipole magnets do we need in the LHC?**
  - Dipole length = 15 m
  - Field 8.3 T

$$\int B \ dl \approx N \ l \ B = 2\pi \frac{p}{q}$$

$$N = \frac{2\pi \ 7000 \ 10^9 eV}{8.3T \ 15m \ 3{\cdot}10^8 \frac{m}{s} e} = 1232$$

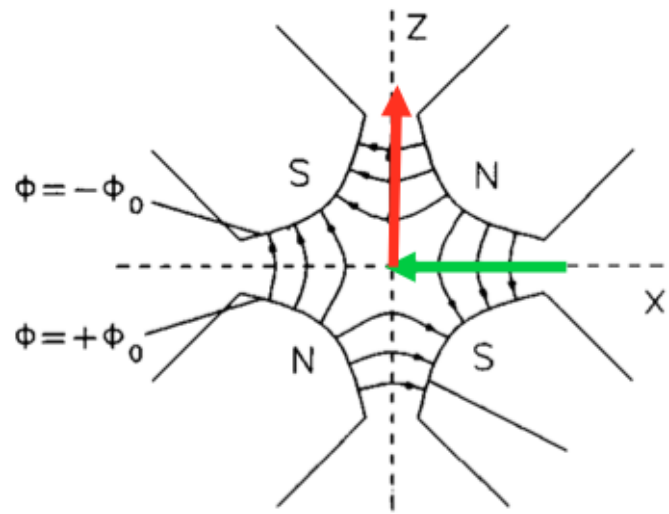
Focusing with Quadrupole Magnets

- Requirement: Lorentz force increases as a function of distance from design trajectory

- E.g. in the horizontal plane

$$F(x) = q \cdot v \cdot B(x)$$

- We want a magnetic field that



$$B_y = g \cdot x \qquad B_x = g \cdot y$$

➔ Quadrupole magnet

- **Gradient** of quadrupole

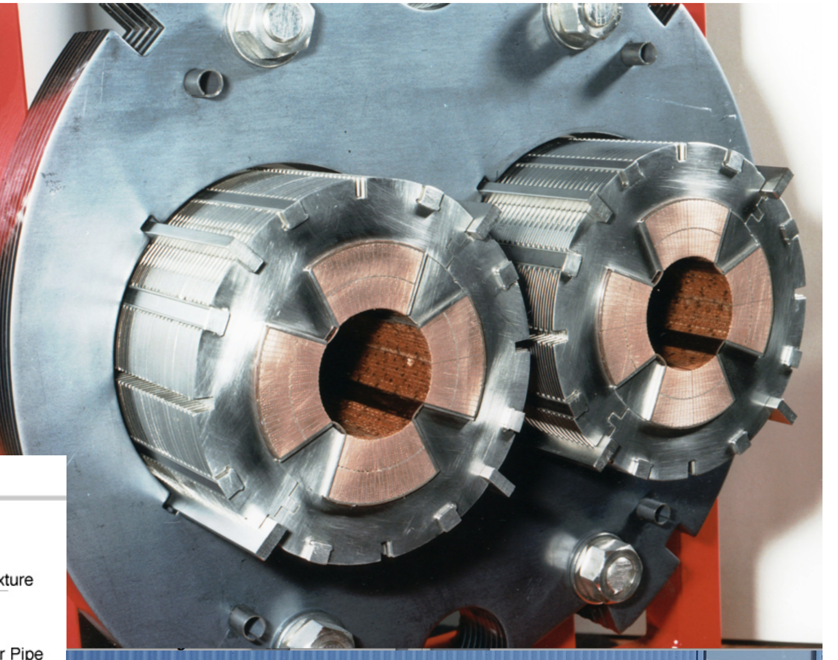
Normalized gradient, focusing strength

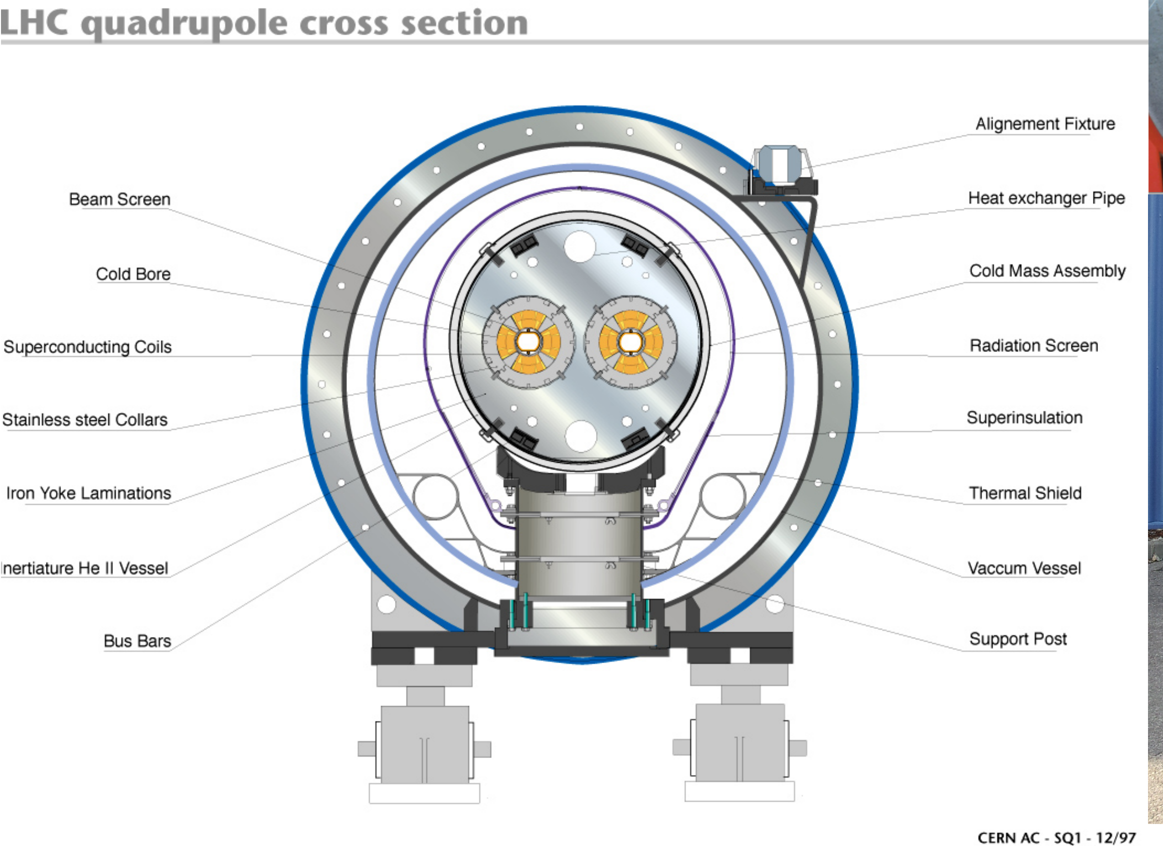
$$g = \frac{2\mu_0 n I}{r^2} [\frac{T}{m}]$$

$$k = \frac{g}{p/e} [m^{-2}]$$

The LHC main quad

- Length = 3.2 m
- Gradient = 223 T/m





Equation of Motion

- Taylor series expansion of B field:

$$B_y(x) = B_{y0} + \frac{\partial B_y}{\partial x} x + \frac{1}{2} \frac{\partial^2 B_y}{\partial x^2} x^2 + \frac{1}{3!} \frac{\partial^3 B_y}{\partial x^3} x^3 + \ldots$$

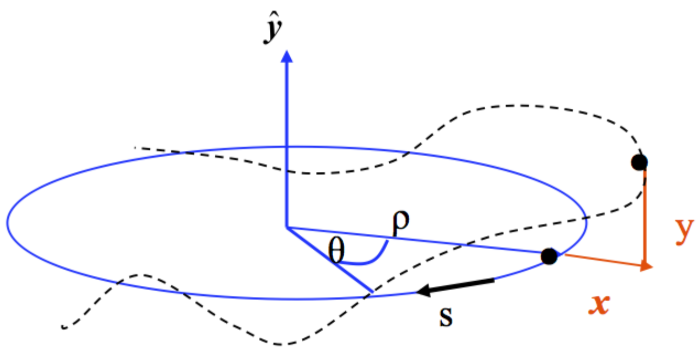
Normalize and keep only terms linear in x

$$\frac{B_y(x)}{p/e} = \frac{1}{\rho} + k\ x + \frac{1}{2} \cancel{m}\ x^2 + \frac{1}{3!} \cancel{n}\ x^3 + \ldots$$

$$\frac{B_y(x)}{p/e} \approx \frac{1}{\rho} + k\ x$$

Towards Equation of Motion

- Use different coordinate system: Frenet-Serret rotating frame

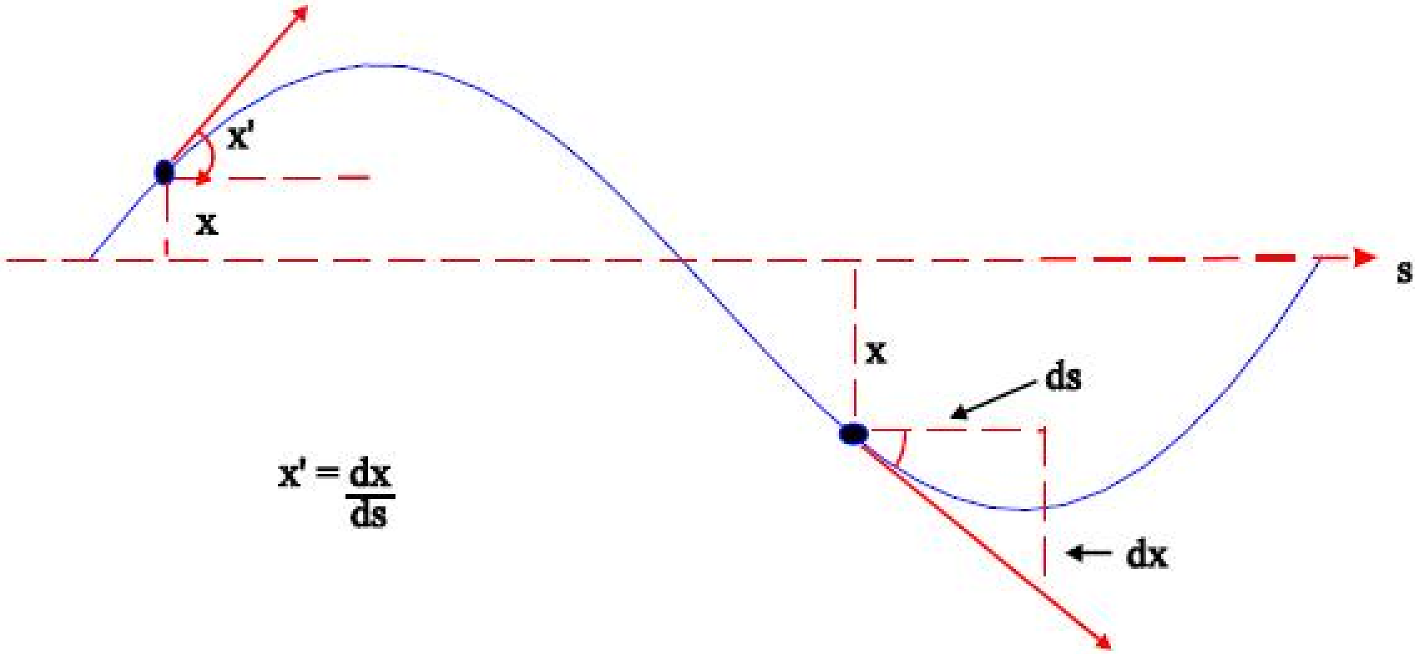


- The ideal particle stays on "design" trajectory. (x=0, y=0)
- And: x,y << r

- The design particle has momentum $p_0 = m_0 g v$.
-
- …. relative momentum offset of a particle

$$\delta = \frac{p-p_0}{p_0} = \frac{\Delta p}{p}$$

Equation of Motion



$$(x, x', y, y', z = s - \beta ct, \delta)$$

$$x' = \frac{dx}{ds} = \frac{p_x}{p_z} \qquad y' = \frac{dy}{ds} = \frac{p_y}{p_z}$$

- A particle is described with 6 coordinates

Equation of Motion

- All we have to do now is to write

$$F_r = m\ a_r = eB_y v$$

- in the Frenet-Serret frame,
- develop with x,y << r , and keeping only terms linear in x or y for magnetic field

after a bit of maths: the equations of motion

$$y'' + ky = 0$$
$$x'' + x(\frac{1}{\rho^2} - k) = 0$$

Assuming there are no vertical bends,
Quadrupole field changes sign between x and y

Solution of Equation of Motion

- Let's write it slightly differently:
- horizontal plane: $K = 1/\rho^2 - k$
- vertical plane: $K = k$

$$x'' + Kx = 0$$

Equation of the **harmonic oscillator** with spring constant K

- Solution can be found with ansatz
- Insert an: $x(s) = a_1 \cos(\omega s) + a_2 \sin(\omega s)$
- For K > 0: focusing $\omega = \sqrt{K}$
-

$$\boxed{x(s) = a_1 \cos(\sqrt{K} s) + a_2 \sin(\sqrt{K} s)}$$

Solution of Equation of Motion

- a_1 and a_2 through boundary conditions:

$$s = 0 \rightarrow \begin{cases} x(0) = x_0, & a_1 = x_0 \\ x'(0) = x'_0, & a_2 = \frac{x'_0}{\sqrt{K}} \end{cases}$$

- Horizontal focusing quadrupole, K > 0:

$$x(s) = x_0 \cos(\sqrt{K}s) + x'_0 \frac{1}{\sqrt{K}} \sin(\sqrt{K}s)$$

$$x'(s) = -x_0\sqrt{K} \sin(\sqrt{K}s) + x'_0 \cos(\sqrt{K}s)$$

- Use matrix formalism: TRANSFER MATRIX

$$\begin{pmatrix} x \\ x' \end{pmatrix} = M_{foc} \cdot \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\boxed{M_{foc} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}} \sin(\sqrt{K}s) \\ -\sqrt{K} \sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix}}$$

Solution of Equation for Defocusing Quadrupole

- Solution of equation of motion with K < 0:

$$x'' + Kx = 0$$

- New ansatz is:

$$x(s) = a_1 \cosh(\omega s) + a_2 \sinh(\omega s)$$

- And the transfer matrix

$$M_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}s) \\ \sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}$$

Summary of Transfer Matrices

$K = 1/\rho^2 - k$ ....horizontal plane

$K = k$ ....vertical plane

- Uncoupled motion in x and y

- Focusing quadrupole, K >0:

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}s) \\ -\sqrt{K}\sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix}$$

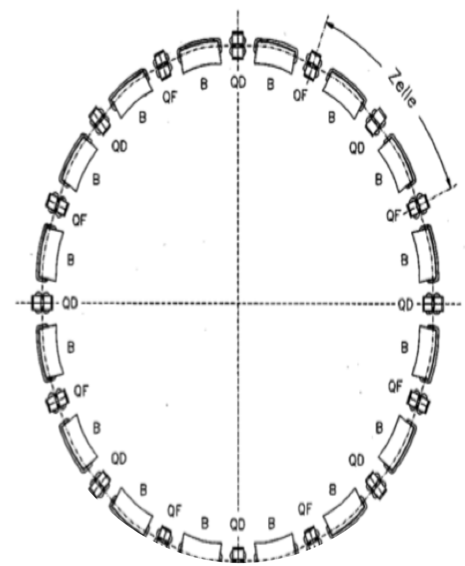
- Defocusing quadrupole, K< 0:

$$M_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}s) \\ \sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}$$

- Drift space: length $M_{drift} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$

Step 1 – when designing a synchrotron

- Design orbit with dipole magnets and alternating gradient lattice.



$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \quad M_{total} = M_{QF} \cdot M_D \cdot M_{Bend} \cdot M_D \cdot M_{QD} \cdot \ldots$$

The Hill's Equation

- We had…

$$x'' + Kx = 0$$

- Around the accelerator K will not be constant, but will depend on s

$$x''(s) + K(s)x(s) = 0$$

Hill's equation

- Where
  - restoring force ≠ const, K(s) depends on the position s
  - K(s+L) = K(s) periodic function, where L is the "lattice period"

- General solution of Hill's equation:

$$x(s) = \sqrt{\epsilon}\sqrt{\beta(s)}\cos(\psi(s) + \phi)$$

The Beta Function & Co

- Solution of Hill's Equation is a quasi harmonic oscillation **(betatron oscillation)**: amplitude and phase depend on the position s in the ring.

$$x(s) = \sqrt{\epsilon}\sqrt{\beta(s)}\cos(\psi(s) + \phi)$$

integration constants: determined by initial conditions

- The beta function is a periodic function determined by the focusing properties of the lattice: i.e. quadrupoles

$$\beta(s + L) = \beta(s)$$

- The "phase advance" of the oscillation between the point 0 and point s in the lattice.

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

The transport matrix revistited

- Definition: $\alpha(s) = -\frac{1}{2}\beta'(s)$ $\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$

$$x(s) = \sqrt{\epsilon}\sqrt{\beta(s)}\cos(\psi(s)+\phi)$$

$$x'(s) = -\frac{\sqrt{\epsilon}}{\sqrt{\beta(s)}}\alpha(s)\cos(\psi(s)+\phi) + \sin(\psi(s)+\phi)$$

Let's assume for s(0) = $s_{0,}$ y(0) = 0.

- Defines f from x_0 and x'_0, b_0 and $a_{0.}$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

Can compute the single particle trajectories between two locations if we know a, b at these positions!

$$M = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}(\cos\psi + \alpha_0 \sin\psi) & \sqrt{\beta\beta_0}\sin\psi \\ \frac{(\alpha_0-\alpha)\cos\psi-(1+\alpha\alpha_0)\sin\psi}{\sqrt{\beta\beta_0}} & \sqrt{\frac{\beta_0}{\beta}}(\cos\psi - \alpha\sin\psi)) \end{pmatrix}$$

The Tune

- The number of oscillations per turn is called “tune”

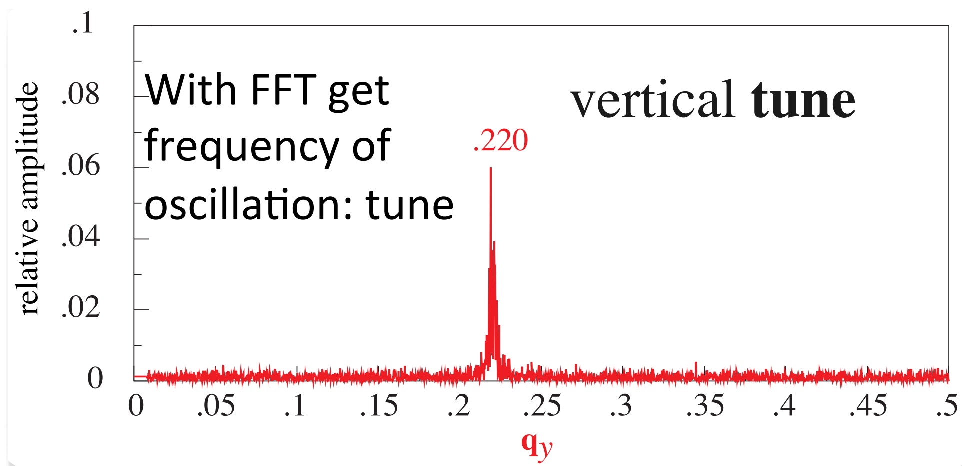
$$Q = \frac{\psi(L_{turn})}{2\pi} = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

- The tune is an important parameter for the stability of motion over many turns.
- It has to be chosen appropriately, measured and corrected.

Measure beam position at one location turn by turn

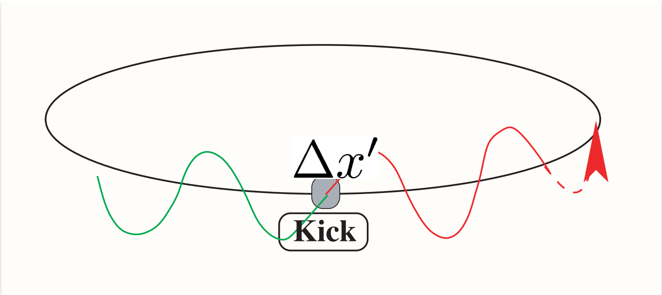
Beam position will change with

$$\propto \cos(2\pi Q i)$$



The Tune

- The choice of phase advance per cell or tune and hence the focusing properties of the lattice have important implications.



The perturbation at one location has an effect around the whole machine

$$x(s) = \frac{\Delta x'}{2} \cdot \sqrt{\beta(s_0)\beta(s)} \frac{\cos(\pi Q - \psi_{s_0 \to s})}{\sin(\pi Q)}$$

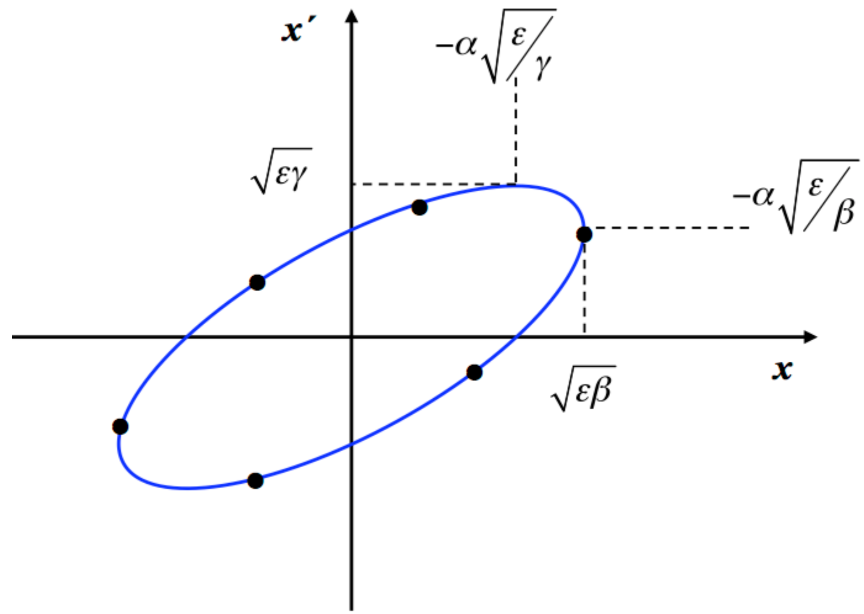
- Misalignment of quadrupoles or dipole field errors create orbit perturbations

➔ diverges for Q = N, where N is integer.

Phase-space ellipse

- The area of the ellipse is constant (Liouville):

$$A = \pi \cdot \epsilon$$



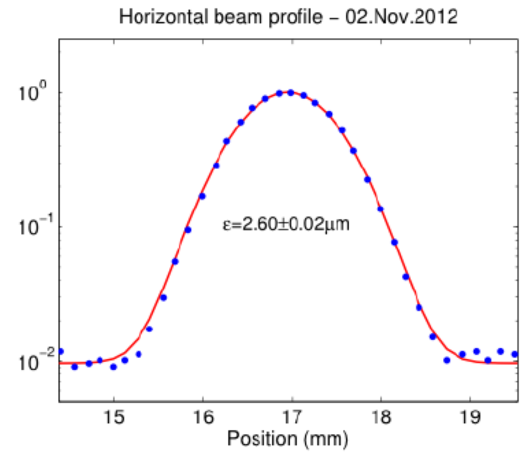
- The area of the ellipse is an intrinsic property of the beam and cannot be changed by the focusing properties.

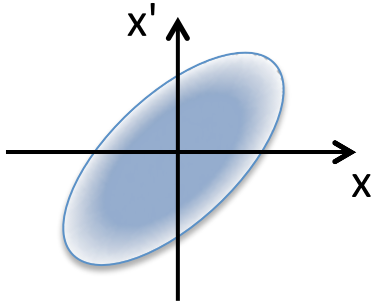
Emittance of an ensemble of particles

- Typically particles in accelerator have Gaussian particle distribution in position and angle.

$$\rho(x) = \frac{N}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{x^2}{2\sigma_x^2}}$$

Transverse profile measurement and Gauss fit





Define beam emittance e as ellipse with area in phase-space that contains 68.3 % of all particles. Such that

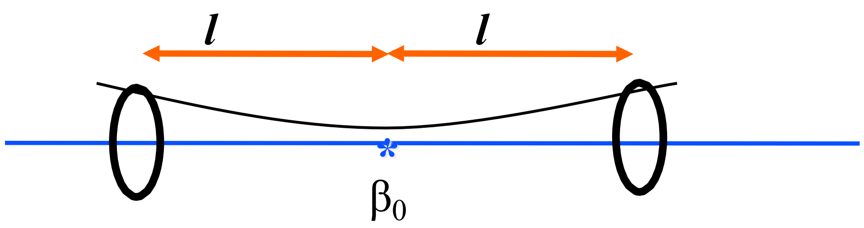
$$\sigma_x = \sqrt{\varepsilon \beta_x}$$

Particles (protons) follow an trajectory that oscillates around the ideal circular path.
The oscillation is independent for vertical and radial components.
The oscillation has same properties but each particle has an independent trajectory govern by its initial vector and the derivatives.
The bunch is defined by an envelope of all trajectories.
At any point along the ring, the distribution of the density of the trajectories is gaussian and we assume that the beam pipe must be large enough to allow for 10σ of this density distribution to pass.

In order to increase the probability of particle collision we want to maximize the particle density at the beam intersection point. This is done by "low beta inserts" – a special set of magnets with strong focusing that narrows the beam envelope. Close to the interaction point the beam must get close to each other into one beam pipe. We do not want the interactions to take place away from the center of the detector so beams are blown up when travelling along the same path and brought to high density (small transverse cross section) at the interaction point.

Minibeta insertion

Minibeta insertion is a symmetric drift space with a beta waist in the center of the insertion!



The insert must have a high level of magnetic symmetry so that the trajectories of particles at the exit will match the regular lattice. Typically done with a set of three magnets at each side of the intersection.

Beta describes the radius of curvature of the beam envelope at the interaction point.

LHC design – not reached so far

β = 55 cm to give the transverse dimension of the beam σ = 16 μm.

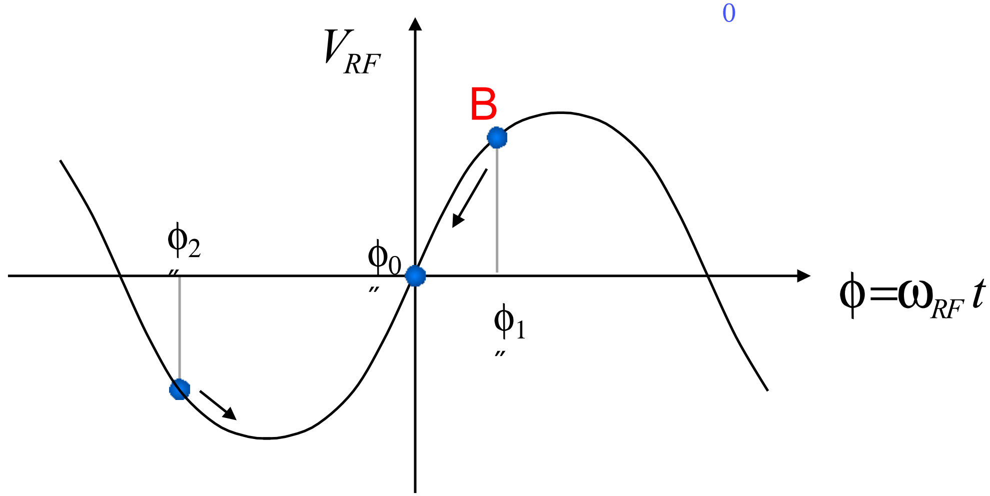
Synchrotron Oscillations

Like in the transverse plane the particles are performing an oscillation in longitudinal space.

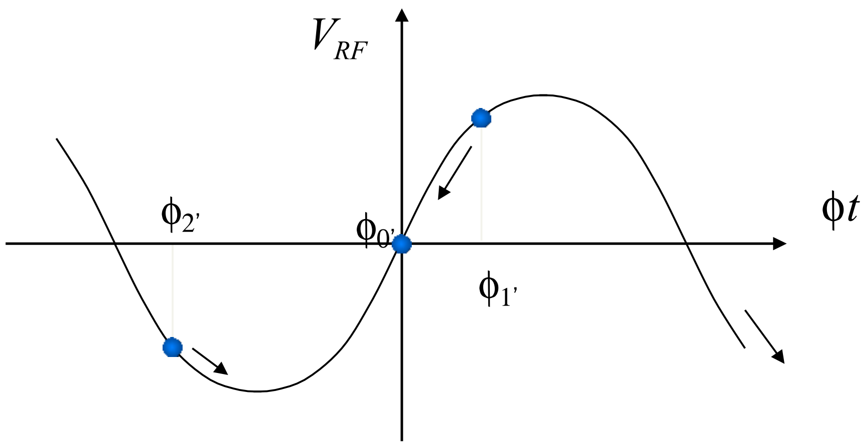
Particles keep oscillating around the stable synchronous particle varying phase and dp/p.

Assume: no acceleration, B = const, below transition $\eta > 0$

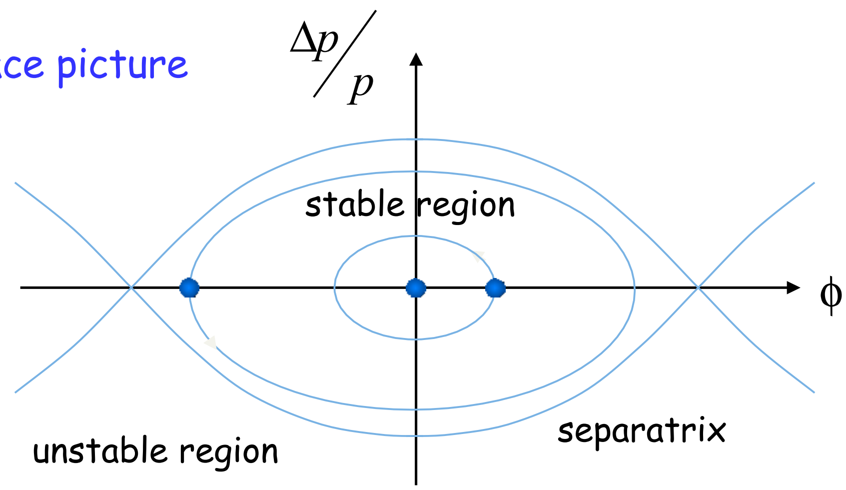
Stable phase = 0. B will oscillate around ϕ_0.



Synchrotron Oscillations – No acceleration

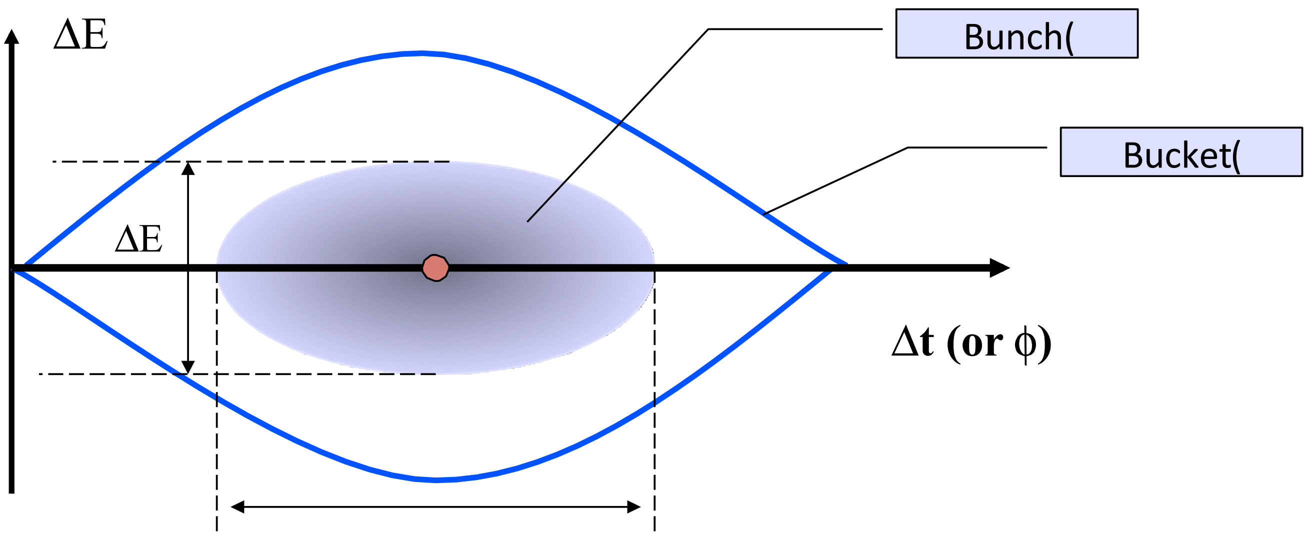


Phase space picture



Bucket & Bunch

The bunches of the beam fill usually a part of the bucket area.

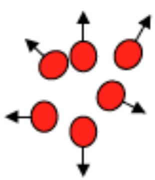


LHC – bucket spacing 25 ns -> 40 MHz of bucket-bucket crossing
Not all buckets are filled up with particles
Not all bunches have the same number of particles

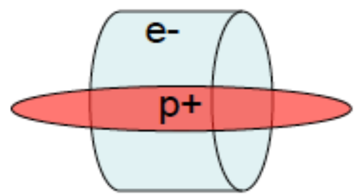
Collective Effects

Three categories: can cause beam instabilities, emittance blow-up, beam loss....!

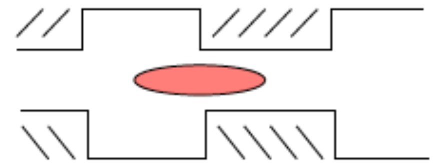
Beam-self: beam interacts with itself through space charge.



Beam-beam: colliding beams in colliders or ambient electron clouds (e-p instability).



Beam-environment: beam interacts with machine (impedance-related instabilities).

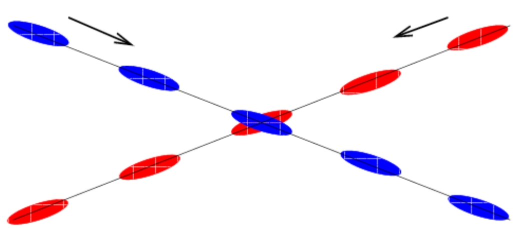


Tune'spread'' $\propto \frac{1}{\beta^2\gamma^3}$

'

THE'limita-on'in'low' energy'machines'

Colliding'beams.'Tune' spread/shi['due'to'headJ on'collisions'and'long' range'collisions.''



Beam'induces'field'in' accelerator'environment.' Wake'fields.' Wake'fields'can'act'back'on' trailing'beam.'

'

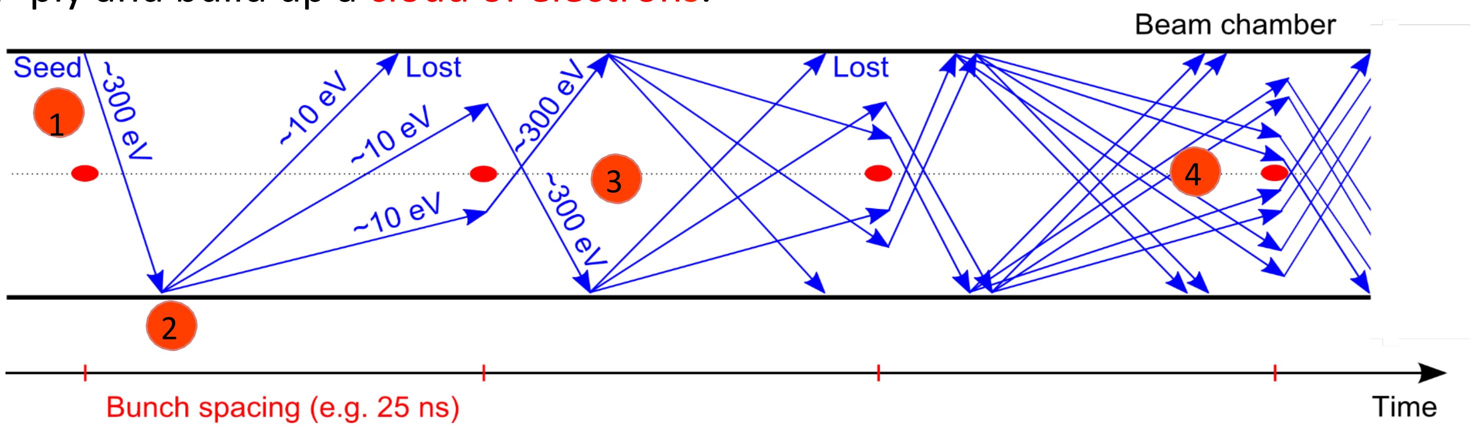
Fourier'transform'of'Wake' field'is'impedance.'

'

Can'lead'to'component' hea-ng'and/or'instability.'

Electron cloud – One of the LHC Challenges

In'high intensity accelerators with positively charged beams and closely spaced bunches electrons'liberated'from'vacuum'chamber'surface'can' mul- ply'and'build'up'a'cloud'of'electrons.



1 Seed electrons accelerated by beam

2 Produce secondary electrons when hiting wall

3 Secondary electrons accelerated produce more electrons on impact

4 May lead to exponential growth of electron density (multipacting)

- Trailing bunches of train interact with dense e-cloud'
  - Transverse instabilities
  - Transverse emtance blow-up
  - Particle losses
- Other unwanted effects:
  - Heat load on the beam chamber
  - Vacuum degradation