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Karel Van Acoleyen, and Jos Van Doorsselaere

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Captain Einstein: A VR experience of relativity

Karel Van Acoleyen and Jos Van Doorsselaere
Department of Physics and Astronomy, Ghent University, Krijgslaan 281, S9, 9000 Gent, Belgium

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Captain Einstein is a virtual reality (VR) movie that takes you on a boat trip in a world with a slow speed of light. This allows for a direct experience of the theory of special relativity, much in the same spirit as in the Mr. Tompkins adventure by George Gamow (1939). In this paper, we go through the different relativistic effects (e.g., length contraction, time dilation, Doppler shift, light aberration) that show up during the boat trip, and we explain how these effects were implemented in the 360° video production process. We also provide exercise questions that can be used-in combination with the VR movie-to gain insight and sharpen the intuition on the basic concepts of special relativity.
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I. INTRODUCTION

In the virtual reality (VR) movie Captain Einstein,[1] you experience a boat trip in the beautiful city center of Ghent in a world with a slow speed of light $c = 20\,\mathrm{km/h}$. Starting at a small initial velocity v, different accelerations bring you finally to the speed of light. During the various stages ($v/c \approx 40, 70, 85, 95\%, \ldots$), you can see how the different effects of special relativity emerge gradually. An obvious advantage with respect to the real world is that these effects now become more tangible as they are brought to a human scale. This was the original idea of Gamow that led to the intriguing short story *Mr. Tompkins* (see Fig. 1), which is set in such a dreamworld with a slow speed of light.[2] Later approaches in the same spirit include the nice animations of a relativistic bicycle trip through the city of Tübingen;[3,4] *Real Time relativity*,[5–7] an interesting first-person game that allows you to travel near the speed of light in different sci-fi settings; and *A Slower Speed of Light*,[8–10] an engaging first-person relativity game that takes place in a fantasy world. See also Ref. 11 for a thorough study on the visualization of both special relativity and general relativity.

The new aspect of our project lies in the 360° technology. First of all, it allowed us to record and use real images for creating our movie, as opposed to a 3D virtual model. This clearly helped for the realistic "feel" (but see also Sec. III for a discussion on the limitations of this approach). In addition, the 360° VR viewing experience puts the user "inside the theory," so to speak. Not only does this lead to a powerful immersive experience, but it also allows for a very natural way to examine the directional dependence of the relativistic effects, notably of the Doppler shift. See also Ref. 12 for a recent spin-off of,[8–10] involving a dome projection in a planetarium. In the broader context of general relativity, we should also mention the recent VR movies of black hole adventures.[13,14]

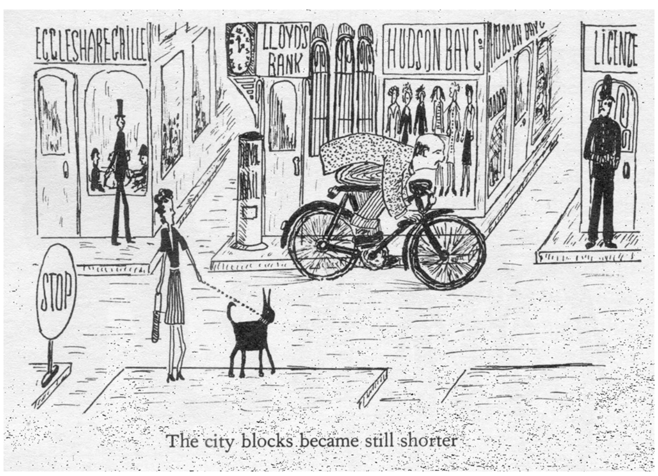


Fig. 1. Mr. Tompkins (Gamow, 1939) racing through the streets in a length contracted world.

Captain Einstein has so far been mainly used for science communication purposes. It has featured at festivals in Belgium and the Netherlands, drawing much interest and provoking enthusiastic reactions from people of different ages and backgrounds. However, the origin of the movie is actually more educational: it grew out of the practice of teaching the theory of relativity at Ghent University. The main purpose of this paper is to provide the scientific content behind the movie, which can be used directly by students or by lecturers in the context of an exercise session on special relativity. To this end, much of the discussion is put in the form of exercise questions. In fact, as will become clear in this paper, one can think of the whole Captain Einstein project as one big exercise in special relativity. We indicate a difficulty level (1 or 2) for the different exercises; the solutions are provided at the very end of the paper.

The paper is aimed at the level of someone with a rudimentary knowledge of special relativity, who has attended the first few lectures of an undergraduate level (special) relativity class or has read the first few chapters of some special relativity course (see, for instance, Ref. 19 for some excellent online notes). But even without this background, one should be able to follow a large part of the discussion. In Sec. V, we also give some suggestions for possible adaptations in the context of a first introduction to relativity at the secondary school level. At the end of the paper, we briefly comment on our experience with the movie so far, both in the context of science outreach and education.

Although it clearly would have been easier for Einstein to discover the theory of relativity in a world with a slow speed of light, deducing the laws of relativity from the direct observations still requires some work. Even to somebody with a prior training in relativity it is not entirely straightforward to relate our VR experience directly to the archetypical relativistic effects. The reason lies in the important difference between what is and what is seen: since the speed of light is finite, we do not see fixed-time snapshots, the further the object, the more we see it in the past. In more technical terms: at each instant, we see our past light cone rather than a particular spacelike hypersurface. This might seem to complicate the use of the movie for illustrating the basic effects of relativity, but as we hope to convey in this paper, thinking about the true observational consequences of, e.g., length contraction and time dilation can actually help to sharpen the understanding of these effects. We refer the reader to Refs. 15–17 for the original groundbreaking work on the visual consequences of relativity.

The science covered by our VR movie is basically the content of Einstein's 1905 paper[18] (apart from the Lorentz transformations of Maxwell's equations). In Sec. II, we study the role of length contraction, time dilation, and relativistic velocity addition in our movie. In Sec. III, we examine the actual observations in a world with a slow speed of light. This involves taking into account the relativistic light aberration. As we will show, the light aberration formula also goes to the heart of our 360° simulation. Finally, in Sec. IV, we consider the relativistic Doppler shift. To the best of our knowledge, this is the first time that the Doppler shift on realistic full light spectra of the sky has been simulated in this detail. This simulation produced the spectacular color effects in the movie with, for instance, the rainbowlike features that can be traced back to the different greenhouse absorption bands in the IR.

To set the conventions, we will consistently use primes (e.g., t', x', θ', λ', …) for coordinates, directions, wavelengths, etc. in the boat reference frame S'. This is your reference frame when you experience the VR boat trip. Non-primed coordinates, directions, etc., are reserved for the quay rest frame S.

II. MR. TOMPKINS

In 1939, George Gamow wrote the wonderful *Mr. Tompkins in wonderland*,[2] which is still one of the standard references in the popular science literature (George Gamow was not only successful as a writer; among his many fundamental contributions in physics, we can mention his work on the theoretical understanding of radioactive α-decay or his pioneering efforts in combining general relativity with nuclear physics in the context of the early universe, thereby laying the groundwork for the Big Bang theory). In the beginning of the book, the main character, Mr. Tompkins, wakes up in a dreamworld with a low speed of light ($c = 10\,\mathrm{mph}$), much like that in Captain Einstein. In this world, he is confronted with the effects of length contraction and time dilation. In Fig. 1, for instance, you see him racing through the streets, with the buildings (and people on the sidewalk) experiencing a length contraction.

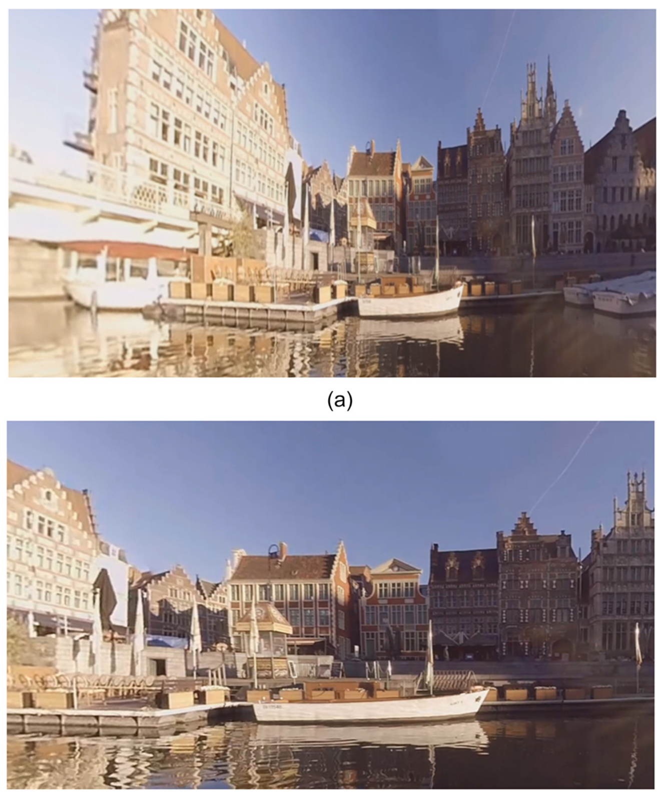
The time dilation and length contraction obviously also play an important role during Captain Einstein's boat trip. In Fig. 2, you see the analogous picture for a boat trip along the canals of a length-contracted Ghent at a velocity $v = 0.85c$. From the boat's perspective, it is of course the quay that is moving, which results in a length contraction $l' = l/\gamma(v)$ along the direction of movement for all the objects at rest in the quay frame. Here, we define the Lorentz factor $\gamma(v) \equiv 1/\sqrt{1 - v^2/c^2}$ and for $v = 0.85c$ we have $\gamma(v) \approx 2$, corresponding to the situation in the figure. From the Captain Einstein movie, you already know that this is not the actual view, but to be sure, the picture does represent the true geometry of the moving quay from the perspective of the boat observer O' at some fixed time t'. We reserve the discussion of the visual manifestation of length contraction for Sec. III, while the visual consequences of time dilation are discussed in Sec. IV. In this section, you are asked to compute the effect of length contraction and time dilation on the boat travel time, which, in turn, has its consequences for our simulation of speed.

Exercise 1 (level 1): Length contraction, time dilation, and frame rates

(a) Let us take two subsequent bridges over the canal that are separated by a distance l in the quay rest frame. Given the length contraction, if the boat is going at a constant speed v, what is the time $\Delta t'$ it takes the boat to drive from one bridge to the next?

(b) From the perspective of the quay observer, there is of course no length contraction for the distance between the two bridges. Still, he should be able to compute the boat travel time $\Delta t'$ and arrive at the same conclusion. Show that, by taking into account the time dilation, the quay observer indeed arrives at the same result for $\Delta t'$. This should clarify Maja's remark: "we get some speed for free, time is slowing down as we speed up."

(c) The original footage was shot on a boat going at a constant velocity $v_b (\approx 8\,\mathrm{km/h})$. To simulate other velocities, we simply applied a speed-up to the movie (see, for instance, this clip[20] for the same speed-up trick in the legendary 80's TV series "Knight Rider"). In a



(a)

(b)

Fig. 2. Above: point of view shot from the boat orthogonal to the velocity direction for a boat trip along a length contracted Graslei at speed $v = 0.85c$. Below: image in the quay rest frame ($v = 0$) for comparison.

non-relativistic world, we would have to speed up the movie by a factor v/v_b to simulate a certain velocity v. What is the speed-up that we had to apply for our relativistic movie? ■

Exercise 2 (level 2): "This g-force is crushing us," relativistic velocity addition and acceleration

The different accelerations during the boat trip bring you from one instantaneous inertial frame to the next: $S' = S(\vec{v})$, where $S(\vec{v})$ is the inertial frame that moves with a velocity $\vec{v}$ with respect to the quay frame $S = S(\vec{0})$. The movie shows you the visual effects of these accelerations; (un)fortunately we cannot let you experience the corresponding g-forces. But what is the physically experienced g-force g, corresponding to an acceleration $a = (dv/dt')$? (Here, t' is the time-coordinate for the boat observer O'! So, a is the acceleration that we measure on the boat by timing the change of speed on our speedometer.) For simplicity, you can assume a constant direction: $\vec{v}(t') = (v(t'), 0, 0)$.

At the end of the Captain Einstein movie, the boat is accelerating out of control toward the speed of light. We hear Maja shouting: "we have reached 19.8, …, 19.9 km/h, this g-force is crushing us." Assuming an acceleration a of about 0.1 km/h in one second ($a \approx 0.003g$) and a maximal physiologically sustainable g-force of 10 g, at what velocity would you pass out? ■

III. OBSERVING ON THE LIGHT CONE

The actual shapes of the buildings in the Captain Einstein movie (see, e.g., Fig. 3) are quite different from those seen in Fig. 2. Length contraction refers to the length (along the velocity direction) of moving objects taken at a fixed time t' in the considered reference frame. However, since the speed of light is finite, we do not see fixed-time snapshots. At each instant, we see our past light cone, rather than a particular spacelike hypersurface. This vision delay effect was neglected in the Mr. Tompkins book.

Exercise 3 (level 1): "Check out the towers!"

For the tower in Fig. 3, it is clear that the light traveling from the top of the tower toward our eye has taken a longer time than the light that emerged from the base of the tower. Use this to qualitatively explain the distortion in the picture, in particular, the direction in which the tower bends. ■

Exercise 4 (level 1): How to see the length contraction

At some point during the trip, Maja asks you casually to "see how the ancient houses are squeezed to half their width." She is, of course, referring to the length contraction, at 17 km/h, we have $\gamma(v) \approx 2$, which indeed gives a contraction by a factor of two. However, she was a bit cheeky there: to really see the length contraction, you have to look in a very specific direction.

Explain that by looking orthogonally to $\vec{v}$, one can indeed directly observe the length contraction, without distortions from the vision delay effect. So, if we take, for instance, a picture in the orthogonal direction, precisely at the moment when the moving object (for instance, the white fence in Fig. 4) appears in the center of the viewfinder, we can use the standard Euclidean geometry to compute its length l' from the opening angle α and the distance between the boat and the object (see also the left panel of Fig. 8).

Now also qualitatively argue from the vision delay effect why the house on the left in the picture does not appear to be contracted and why the white van on the right appears to be more contracted. Finally, in the same vein, you are asked to give a qualitative explanation of the so-called Terrell effect[16,17] on the fence: the apparent rotation of the left side of the fence (indicated with a red arrow in Fig. 4). ■

Let us now discuss how we actually created the relativistic images in our VR movie, in particular, how we simulated both the length contraction and the vision delay effect (the simulation of the Doppler effect on the colors is discussed in Sec. IV). At first sight, it seems that one needs the full 4D information to produce images that correctly take into account these effects; in particular, it seems one needs the 3D position for all the relevant objects at the appropriate times. However, as is apparent already in the original works

Fig. 3. "Check out the towers, how they are curved!" Point of view shot from the starboard side (righthand side) in the Captain Einstein movie.

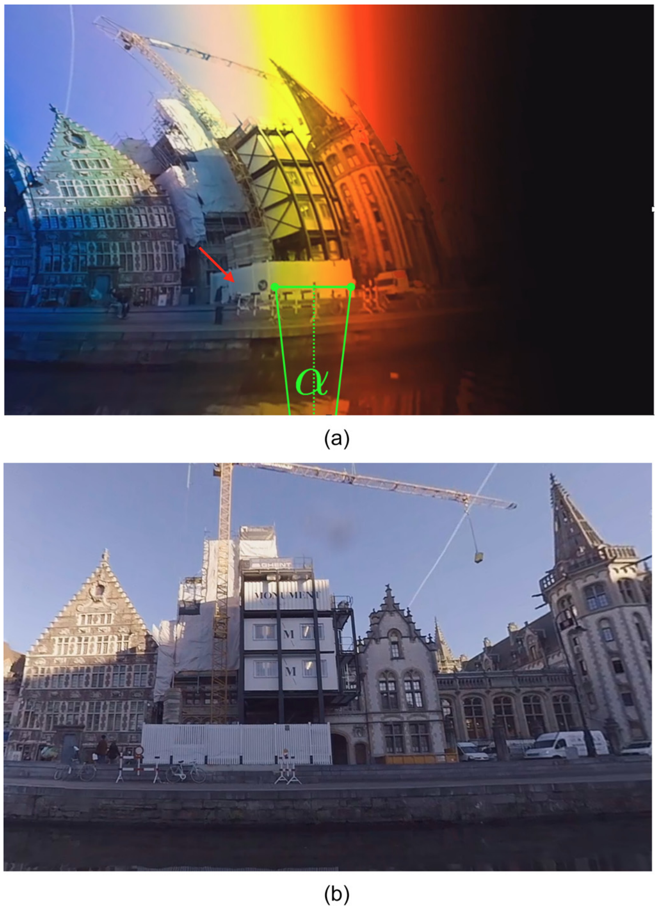


Fig. 4. Above: length contracted white fence as seen at starboard by Captain Einstein (the fence lies parallel to the velocity direction). The dotted line is the $\theta' = \pi/2$ direction orthogonal to the velocity ($\vec{v} \cdot \hat{n}' = v\cos\theta'$ with $\hat{n}'$, the unit vector in a particular viewing direction). The red arrow shows the Terrell effect on the fence. Below: view in the quay rest frame for comparison.

of Penrose and Terrell,[16,17] the 2D unit sphere of light directions $\hat{n}$ (see Fig. 5) for some appropriate observer O actually contains all the required information! The key insight is this: the very same photons that create an image for an observer O at some space-time point P are also responsible for the image that would be seen by another observer O' at the same space-time point P, but moving with a different velocity. To work out the image of a moving object, one can then start from the image for an observer O at rest in the object's restframe S (as in Fig. 5). In this restframe, there is no length contraction and vision delay effect, and the 2D unit sphere of light directions (and the resulting image) is therefore the same as in a non-relativistic world ($c \to \infty$). The object image as seen by a different observer O' that is not at rest in S, taking into account both the length contraction and vision delay effect, then entirely follows from the relativistic aberration formula, which transforms the light directions $\hat{n}$ for O to the light directions $\hat{n}'$ for O' (assuming in the case of an accelerated observer O' that the camera/eye is not affected by the g-force),

$$\cos\theta' = \frac{\cos\theta + \dfrac{v}{c}}{1 + \dfrac{v}{c}\cos\theta}. \tag{1}$$

Here, $\vec{v} \cdot \hat{n} = v\cos\theta$, $\vec{v} \cdot \hat{n}' = v\cos\theta'$, and the observer O' is moving with velocity $\vec{v}$ with respect to the object, or equivalently, the object is moving with velocity $-\vec{v}$ with respect to the observer O'. In Fig. 6, we show a visualization of this light aberration formula on the unit sphere.

The 360° image recording technology is perfectly suited for visualizing special relativity, precisely because it provides us exactly with the required information of the 2D unit sphere of light directions. As shown in Fig. 7(a), the images recorded by a 360° camera are typically stored in an equirectangular projection of the sphere. For our movie shoot, the camera was attached to a boat that was driving along the canals of Ghent. The boat was going at $v_b \approx 8\,\mathrm{km/h}$, which is, of course, a non-relativistic speed $v_b/c \approx 0$ in the real world. We can, therefore, effectively consider each recorded movie frame to be the 360° image for an observer O in the quay rest frame S of our virtual world with a slow speed of light. But notice that for this to hold, all the objects in the image have to be at rest in the quay frame. This is the reason why we had to stitch away our boat in the original images. This is also the reason why we shot the movie on a cold winter day; we did not want any other moving boats in the image. Unfortunately, the cold did not stop the traffic at the quay. So, the people on foot, the bikes, and the tram in the movie are not correctly depicted and can be considered as goofs that defy the laws of relativity (see exercise 8 for the correct visualization of the pedestrians).

Our simulation of the length contraction and vision delay effect then amounts to the application of the light aberration

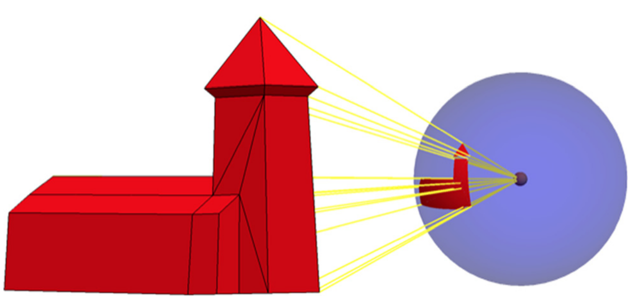


Fig. 5. The image of the red building encoded in the 2D sphere of light directions $\hat{n}$ for the observer at the center of the sphere. Here, we show the 2D sphere for the observer O at rest in the building's rest frame S.

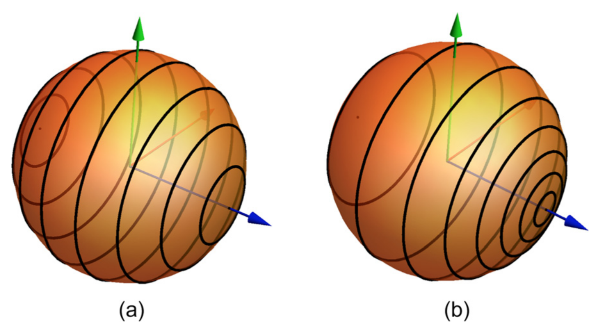


Fig. 6. Visualization of the light aberration formula (1). Left: the 2D unit-sphere of light directions $\hat{n}$ in the quay rest frame. The different black circles are parallels with respect to the (blue) x-axis that each correspond to a particular angle θ: $\hat{n} \cdot \hat{e}_x = \cos\theta$. Right: the corresponding unit-sphere of directions $\hat{n}'$ ($\hat{n}' \cdot \hat{e}_x = \cos\theta$), after a boost in the x-direction with $v/c = 0.7$. As can be seen from the transformed parallels, the field of view "collapses" into the velocity-direction.

transformation Eq. (1) on the original equirectangular input images (Fig. 7(a)), resulting in the 360° images for the boat observer O' (Fig. 7(b)). This transformation can be implemented fairly straightforward with a shader, an image transformation program common in computer graphics software (e.g., WebGL, Unity), written in a specific GPU compatible language. We have experimented with that, but in the end, for our final movie, we wrote a Matlab code that generates the relativistically transformed 360° images, frame by frame. At the risk of repetition, the non-trivial fact that this framewise approach is possible and correct is exactly the point of the Penrose–Terrell argument that we explained above. The main reason for using Matlab lies in our implementation of the Doppler shift (see Sec. IV); this required numerical interpolation on a large set of pre-generated RGB-to-RGB transformations, which we found more convenient to implement with Matlab. Finally, our procedure renders monoscopic images based on a monoscopic input. It would be interesting to extend this to stereoscopic images, which now give the information for two 2 D unit spheres of light directions, one for each eye. Notice that this is not a straightforward generalization since the two stereoscopic light spheres are centered at different space-time points. One, therefore, would have to take into account a length contraction effect for the interpupillary distance, which depends on the orientation of the field of view with respect to the velocity vector.

Exercise 5 (level 2): From 2D to 3D, from light aberration to length contraction

So to create our movie, we used a local transformation (1) for the 2D unit-sphere of light directions hitting the observer at a certain time. One can verify, in general, that the light aberration formula indeed reflects two effects in the 4D world: the vision delay effect and the length contraction. In this exercise, we focus on the particular situation of the

(a)

(b)

Fig. 7. Above (a): frame of the original footage of the full sphere of directions $\hat{n}$ in equirectangular projection: $\hat{n} = (\sin(\pi - v)\cos(u), \sin(\pi - v)\sin(u), \cos(\pi - v))$, with $u \in (-\pi, \pi]$ the horizontal coordinate and $v \in [0, \pi]$ the vertical coordinate in the picture. The boat is going in the x-direction: $\vec{v} = (v,\ 0,\ 0)$. Below (b): corresponding frame after applying the light aberration formula (1) for $v = 0.7c$. Side question: can you understand what happened with the plane trail in the picture?.

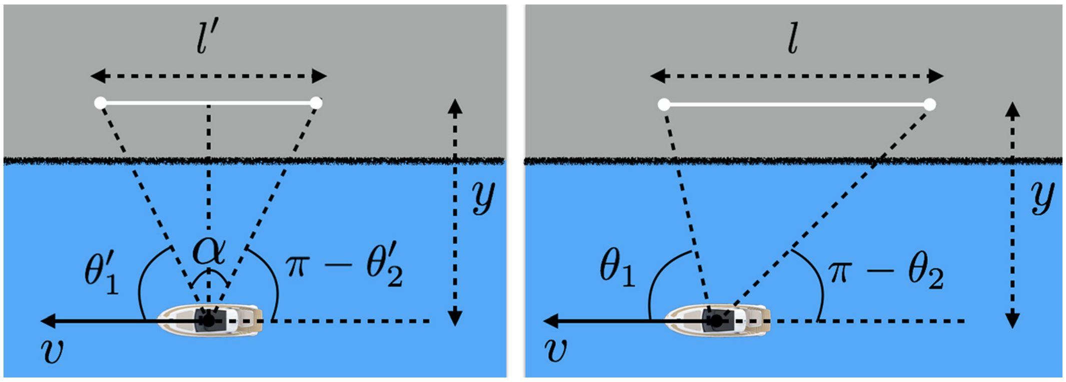


Fig. 8. Left: fence as seen by the boat observer, with the endpoints at angles $\theta'_1 = \pi/2 - \alpha/2$, $\theta'_2 = \pi/2 + \alpha/2$ with respect to the velocity vector. Right: the same fence as it would be seen in the quay rest frame (also the fence rest frame) for an observer at the same position as the boat observer.

previous exercise 4 and demonstrate that for an orthogonal viewing direction $\theta' = \pi/2$, the aberration formula is indeed geometrically equivalent to the length contraction. As shown in exercise 4, for this particular viewing angle, we can use the Euclidean geometry to relate the opening angle α of the two endpoints to a length l' of the moving fence: $l' = 2y\tan(\alpha/2)$, where we take the object lying parallel to $\vec{v}$ and y is the distance from the observer to the center of the object (see Fig. 8). Now, use the aberration formula to show that the length l in the object's rest frame (i.e., the quay rest frame S) is indeed longer by a factor $\gamma(v)$. ■

IV. DOPPLER EXTRAVAGANZA

Let us now discuss color. The relativistic Doppler formula reads (with again θ' the direction in S' with respect to the velocity direction)

$$\lambda' = \lambda \frac{1 - \frac{v}{c}\cos\theta'}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \frac{\lambda}{D}. \tag{2}$$

We first illustrate the Doppler formula by looking at its effect on monochromatic light. In Fig. 9(a), you see the effect on blue light of $\lambda = 475\,\mathrm{nm}$, for $v/c = 40\%$ (85%). At these relativistic speeds, the Doppler shifted wavelengths span the full visible spectrum $\approx [390\,\mathrm{nm}, 700\,\mathrm{nm}]$,[21] and even go beyond the visible spectrum, into the UV: $\lambda' < 390\,\mathrm{nm}$ for directions $\theta' < 52°$ ($\theta' < 48°$) and into the IR: $\lambda' > 700\,\mathrm{nm}$ for directions $\theta' > 151°$ ($\theta > 75°$), resulting in black patches of invisible light both behind and in front of the boat.

The considered values of v/c for the Doppler shift on blue light in Fig. 9(a) correspond to the velocity at the beginning of our movie ($8\,\mathrm{km/h} = 0.4c$) and to the velocity after the second acceleration ($17\,\mathrm{km/h} = 0.85c$). Yet, the colors at the sky in our movie do not look at all like in the figure above! The reason lies in the difference between the pure blue light and the incredibly rich blue sky that consists of a full spectrum of different wavelengths, starting in the UV and ending deep in the IR part of the light spectrum (see Fig. 10). The Doppler effect on such a realistic blue sky spectrum produces the color pallet of Fig. 9(b); see also Fig. 11 for a point of view shot in our movie. At the end of this section, we will expand a bit more on our simulations of the Doppler shift, but now you are first asked to explain the qualitative features of the colors in the movie and think about the manifestations of time dilation.

Exercise 6 (level 1): Captain Einstein and the Doppler shift: Seeing the invisible

In this exercise, we focus on the $v = 0.85c$ case (Fig. 9(b), right panel). At this relativistic speed, the Doppler shift allows you to see parts of the blue sky spectrum (Fig. 10) that normally lie beyond our visible window. Can you explain the yellow and red bands at orthogonal viewing directions $\theta' \approx 90°$? In the movie, Maja reassures us: "don't worry about the dark behind you, it's not a black hole." Can

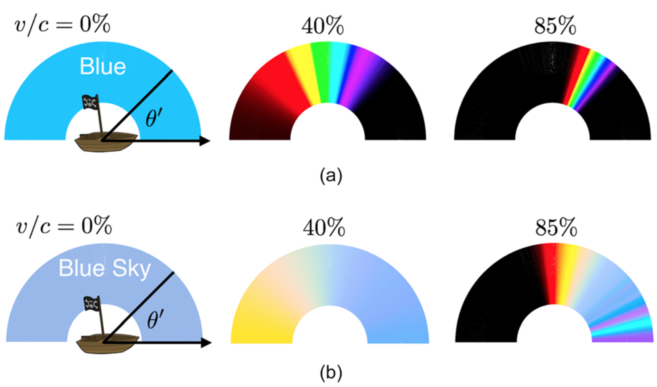


Fig. 9. (a) Doppler shifted monochromatic blue, $\lambda = 475\,\mathrm{nm}$, at 40% and 85% of lightspeed. (b) Doppler shift for the diffuse blue sky spectrum that was used as input for the Captain Einstein movie (see Fig. 11 for a point of view shot).

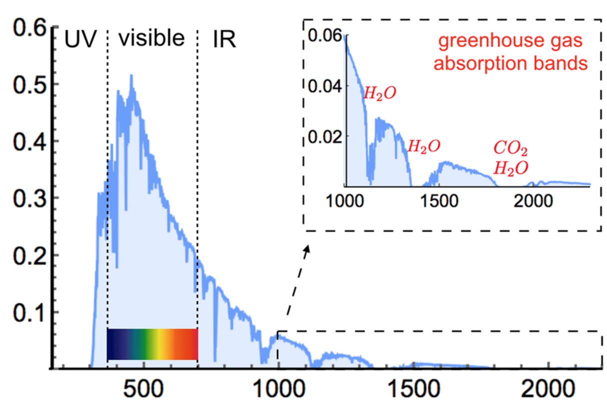


Fig. 10. The diffuse sky radiation (data from Ref. 22). Horizontal axis: wavelengths in units of nm; vertical axis: spectral power distribution $I(\lambda)$ in units of $W/(\mathrm{m}^2 \cdot \mathrm{nm})$. Inset: part of the IR tail with the greenhouse gas absorption bands.

Fig. 11. Point of view shot from Captain Einstein at $v = 0.85c$, taken at an angle $\theta' \approx 30^\circ$ with respect to the velocity direction.

you explain the true reason for this darkness? Finally, can you see why the greenhouse gas absorption bands are responsible for the distinct blue and purple bands in front of the boat? (Don't worry here about the overall brightness in front of the boat, as we explain below, the searchlight effect boosts the light intensity in the frontal directions.) ■

Exercise 7 (level 1): How to see the time dilation: Transverse Doppler effect

The relativity of time, and, in particular, the time dilation, is probably the most spectacular consequence of the theory of relativity. In Sec. II, we discussed the effect of time dilation on the experience and simulation of speed in our movie. Now, we discuss other manifestations. The most direct verification of time dilation results from comparing two clocks after different trips.[23] The fact that our captain Maja Einstein is still alive alludes to this type of twin paradox experiment. In our movie, she is thirty-something, while her brother died in 1955. But this is, of course, a bit of a gimmick. However, you can also directly see the time dilation during the boat trip. For this, you simply have to look at the sky. Indeed, the relativistic Doppler effect is, in fact, the combination of two phenomena: the (de)compression of the light waves due to the radial motion of the source with respect to the observer (classical Doppler effect, numerator of Eq. (2)) plus the time dilation and, therefore, frequency shift due to the relative velocity of the source (transverse Doppler effect, denominator of Eq. (2)). At angles $\theta' = \pi/2$, we see that part of the sky for which the radial velocity is zero. The Doppler shift in those directions is therefore purely the result of time dilation: the "sky clock" that is ticking at a slower rate, resulting in an observed light frequency that is smaller by a factor $\sqrt{1 - v^2/c^2}$. For this exercise, we simply ask you to put on the VR headset and verify this transverse Doppler effect during the Captain Einstein boat trip. ■

Exercise 8 (level 2): Doppler ganger: From fast to slow motion

During our boat trip, you can see several pedestrians that are speed walking in "silent movie style." As we explained in Sec. III, our procedure cannot correctly depict the objects (like pedestrians) that are in motion with respect to the quay. In this exercise, we examine what the observed walking pace should have been. We consider somebody walking along the quay with a velocity $\vec{u}$ (see Fig. 12) in the quay frame:

(a) What is the velocity $u' = -dx'/dt'$ of our pedestrian friend in the boat frame?

(b) In her eigenframe, the pedestrian is walking at a pace f_0 of one step per second. What is the walking pace f' that we would observe on the boat as a function of our viewing direction θ'? For what angles θ' would we observe the pedestrian in slow (fast) motion: $f' < f_0$ ($f' > f_0$). ■

We now discuss in more detail how we actually simulated the Doppler shift. The first thing to notice is that human color sensation involves a huge compression of the full spectrum of wavelengths into excitation rates of only three types of cone cells, sensitive to three different regions of the visible window. So radically different spectral distributions can give rise to the same color. This, of course, also lies at the basis of color reproduction, with the three types of RGB pixels in digital imagery or the four types of CMYK dots in printing. The RGB values corresponding to a particular spectral power distribution (SPD) $I(\lambda)$ are obtained from the related

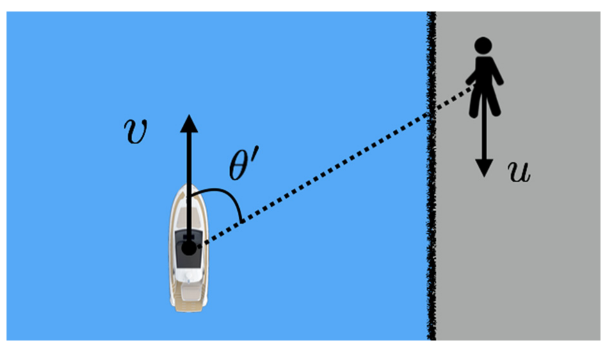


Fig. 12. Pedestrian walking with velocity $\vec{u} = -u\hat{e}_x$ in the quay frame, viewed at an angle θ' in the boat frame. The boat has velocity $\vec{v} = v\hat{e}_x$ in the quay frame.

tri-stimulus XYZ values, which, in turn, are computed by integrating the SPD weighted by the CIE matching functions (see Ref. 24 for more details),

$$\{X, Y, Z\} = K \int d\lambda\ \{x(\lambda), y(\lambda), z(\lambda)\}\, I(\lambda). \tag{3}$$

Here, K is a constant that, in practice, is fixed by calibrating the luminosity on some reference light source.[24] In our case, this was done implicitly by fixing our RGB-to-RGB map to the identity in the absence of any Doppler shift, $D = 1$ (see the discussion below). In Fig. 13, we show the precise form of the matching functions.

This compression, with the RGB (or XYZ) values only containing very limited information on the actual underlying spectrum, presented a challenge for our Doppler simulations based on the RGB pixel values of the input images (e.g., Fig. 7(a)). Our strategy for implementing the Doppler shift as an RGB-to-RGB map consists of two steps: first, we infer a plausible SPD for each pixel from the input RGB values. Second, we use this SPD to compute the Doppler-shifted RGB value.

Given the correct input spectrum, the second step is essentially exact. The (direction-dependent) SPD $I'(\lambda')$ in the boat frame S' follows from the input SPD $I(\lambda)$ (see, e.g., Fig. 10 for the blue sky spectrum) in the quay/sky rest frame S (with $\lambda' = \lambda/D$, see Eq. (2)):

$$I'(\lambda')d\lambda' = D^4 I(\lambda)d\lambda. \tag{4}$$

From Eq. (3), we easily compute the Doppler shifted XYZ and related RGB values, corresponding to a certain SPD.

Notice the D^4-factor in Eq. (4), this is the so-called searchlight effect:[25] the light from the blueshifted directions $(D > 1)$ gets brighter, while the light from the redshifted directions $(D < 1)$ is dimmed. As discussed above, the colors in front of our boat arise from blueshifted parts of the IR spectrum of the blue sky. Here, the full searchlight effect counteracts and even overcompensates the diminished power in the IR tail, resulting in very bright light already for "moderate velocities" $v \gtrsim 0.7c$. Together with the darkening of the redshifted regions, this produces images like in Fig. 14, with oversaturated bright bands in front of the boat on a rather dull dark background. To clearly show all the color nuances of the Doppler shift, we chose to alter this aspect of relativity, by dimming down the searchlight effect for our actual movie, effectively working with a different D-dependent pre-factor in (4).

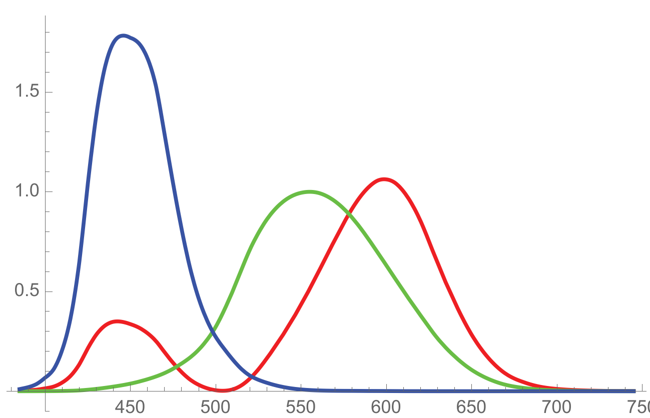


Fig. 13. CIE tri-stimulus matching functions, $x(\lambda)$ (red), $y(\lambda)$ (green), $z(\lambda)$ (blue). (Horizontal axis is in units of nm, the (x, y, z) functions are dimensionless.)

The first step of our simulation then consists of the determination of the input spectra SPD from the input RGB values. We have considered three reference SPD's: in addition to the blue sky spectrum that we discussed above, we also considered light reflected from water and from brick, by taking into account the proper reflection spectra (data obtained from Ref. 26). The inferred SPD's then interpolate between the three different cases (sky, brick, water) based on the input RGB-value and the position of the pixel in the original image.

Some comments are in order here. While our resulting RGB-to-RGB map should be pretty accurate for the quasi uniform blue sky (barring the aforementioned altered searchlight effect), the color changes of the other "materials" in the images are at best a qualitative approximation. We list a few obvious shortcomings: clearly, not all buildings or other objects in the movie are made of red brick. Furthermore, certain regions are illuminated by direct sunlight, whereas we assumed incident diffuse sky radiation everywhere. Also, lacking the data, we simply took a flat water reflection spectrum for wavelengths $\lambda > 2250\,\text{nm}$, which starts to have its effect for velocities $v \gtrsim 0.8c$. Finally, we have also ignored thermal radiation from the different objects. Taking $\lambda_{\text{thermal}} \approx 10\,000\,\text{nm}$, we can estimate that this would only have a visual effect at ultra-relativistic velocities.

Exercise 9 (level 2): The searchlight effect

As a last exercise, we ask you to verify the searchlight effect (4). To this end, show that an incoming photon flux $F = (dN/dtd\Omega)$ (with N the photon number) in a particular direction on the unit sphere of directions in the reference frame S transforms as $F' = D^3 F$ to an incoming flux F' in the transformed direction of the unit sphere in the reference frame S' (see Fig. 6). The extra factor D in (4) then follows from the energy Doppler shift: $h\nu' = D\, h\nu$. ■

V. SECONDARY SCHOOL LEVEL

The discussion and exercise questions in the previous sections are mainly at the level of undergraduate university physics students, but our movie can also be used as part of a first introduction to relativity in secondary school/high school. Here, we suggest some adaptations appropriate for this educational level, with some alternative questions. Our website[1] also gives some scientific background at a lower level, more or less in line with the discussion below.

A. Time and space

The empirical fact at the basis of special relativity, flying against our common notion of space and time, is, of course, the universality of the speed of light (in vacuum) $c = 299\,792\,485\,\text{m/s}$. Special relativity tells us exactly how our Galilean notions of space and time should be corrected in light of this fact. It turns out that a moving clock ticks at a slower rate (time dilation) and a moving object gets contracted in the direction of movement (length contraction). The time dilation, for instance, can be easily demonstrated (at the qualitative or quantitative level) with the light-clock thought experiment.[27]

Fig. 14. Point of view shot at $v = 0.85c$ taking into account the full searchlight effect. (Compare with the corresponding image of the actual movie in Fig. 11.)

Question 1. In the world of Captain Einstein, you are traveling on a boat at relativistic speeds and your clock (biological or actual clock aboard the boat) is ticking at a slower pace because of the time dilation. Here, you are asked to think about the actual implication of such a slowdown of time: does this mean that the houses on the quay should pass by slower or should they pass by faster? In addition, related to this: what is the effect on the travel time? Going from one bridge to the next, is the travel time as measured by your clock larger or smaller because of this time dilation. After some thought, you can verify your answer on the movie clip (see Step 4: Time) in the science section of our website.

Question 2. In the previous question, we have discussed the time dilation aboard the boat, due to its speed with respect to the quay. You should have concluded that this implied a smaller travel time, going from one bridge to the next. However, speed is relative: for the boat observer (you in the movie), the boat actually has no velocity, and it is the quay that is moving! So the clocks at the quay are the ones that should be ticking at a slower pace. How can you explain your smaller travel time from this other perspective?

The two questions above go to the core of special relativity, so it is quite normal if they leave the student a bit puzzled. In fact, they are conceptually equivalent to exercise 1 in the main text (Sec. II). Equipped with the exact formula for time dilation and length contraction, one can also solve that exercise. Of course, it is also fun to play around with the time dilation formula in the context of interstellar travel in the real world.[28]

B. Vision delay

Question 3. Since the speed of light is finite, we literally always look back in time. If you look at the moon, how much in the past are you actually seeing it? What do you get for the sun and the next star? Also, find out the astronomical object that we have seen the furthest in the past.

Question 4. In the world of Captain Einstein with its slow speed of light, this vision delay effect is responsible for the severe distortions of the buildings. Can you, for instance, explain the curvature of the towers by solving exercise 3 in Sec. III?

C. Doppler shift

The relativistic Doppler shift of light is responsible for the spectacular color effects in the movie. We refer to the Doppler extravaganza section of the website for a discussion of the different effects. An additional phenomenon that can be touched upon in this context is the greenhouse effect as manifested by the greenhouse absorption bands in the sky spectrum.

VI. EXPERIENCE

In the context of science communication and outreach, Captain Einstein has so far been "tested" on more than 3000 people at different festivals in Belgium and the Netherlands. Our festival setup typically involves a real canoe or kayak (on land), in which the passengers, equipped with a VR set, experience the virtual boat trip. The basic premise of a world with a slow speed of light in combination with the VR experience itself attracts a wide range of people of different backgrounds—young and old, with or without a particular interest in science. The response after viewing the movie is, in general, very positive—many interesting questions come up, often leading to animated discussions. This helped shape our website[1] that accompanies the recent online launch of the movie.

The movie has also been used as an educational tool, for secondary school students, and in training sessions for science teachers. At Ghent University, we use the movie in the context of the relativity course (third year bachelor physics and astronomy). All students (≈50) view the movie on a smartphone VR viewer, either at home or in class. This then serves as the basis both for a discussion and exercise session in class and a home assignment. During the session in class, for illustrating several aspects of the movie, several VR headsets are passed to the students. In addition, we project a point-of-view image on a central screen. It is convenient to play the movie on a laptop or computer directly from youtube in a 360° viewing mode, allowing one to "look around" and pause at appropriate times.

From the educational perspective, the immersive 360° experience, of course, gives the students a new handle on relativity, notably on the directional dependence of the effects. However, in our estimation, the main advantage of the movie is actually simply that it gives a fun and concrete setting for studying the basic concepts of special relativity. We refer to Ref. 7 for a study on the positive learning impact of the related Real Time Relativity game.

VII. CONCLUSIONS

In this paper, we have presented the Captain Einstein project. We discussed how the different effects of special relativity manifest themselves in the movie. Furthermore, we have explained how the relativistic light aberration formula (1) lies at the heart of our simulation, as it maps the recorded 360° images to the corresponding images for a camera moving at relativistic speeds. We also discussed the limitations of this procedure, namely, that all objects in the initial reference frame should be at rest. Finally, we described our approach for the semi-realistic simulations of the Doppler shift, assuming physically relevant light spectra behind the RGB pixels in the recorded images.

As a science communication and outreach tool, we evaluate Captain Einstein very positively from our experiences so far. Both the concept and the VR experience itself attract a broad public, and the movie triggers many questions both on relativity and the nature of light and color. Our website[1] was developed to cover some first answers to these questions.

We have also experimented with Captain Einstein as an educational tool, e.g., in the context of a relativity course at the undergraduate university level. Also here our evaluation is positive; it allows for a direct experience of some of the basic special relativistic effects and serves well as the basis for an engaging discussion and exercise session. As a final comment, based on our experience, we believe VR applications can indeed present a powerful tool for education, on top of the "old school" methods: direct teacher student interaction, blackboards, and books (digital or not).

VIII. EXERCISE SOLUTIONS

Exercise 1

(a) The length l' between the two bridges as measured in the boat frame is contracted: $l' = l/\gamma(v)$. This gives us

$$\Delta t' = \frac{l}{v\gamma(v)}. \tag{5}$$

So, we have that for $v \to c$, the boat crosses the distance between the two bridges in a time interval $\Delta t' \to 0$. Even in a world with maximal velocity $c = 20\,\mathrm{km/h}$, we can, in principle, get everywhere as fast as we want, simply because all distances are length contracted to zero!

(b) From the perspective of the quay rest frame, the distance between the bridges is, of course, independent of the boat velocity v, so we have $\Delta t = l/v$ for the measured boat travel time. But if we now take into account the time dilation ("…time is slowing down, as we speed up") for the moving clock on board the boat, we recover the result of (a): $\Delta t' = \Delta t/\gamma(v)$.

(c) In the real world $v_b/c \approx 0$, which means that we can interpret the frames in the original footage as a series of pictures at different positions in the quay rest frame. The speed-up of the movie should correspond to, e.g., the number of houses (or bridges) that pass by per unit time, not the distance traveled per unit time. In light of (a) (or (b)), it is clear then that we should speed up the movie by a factor $\gamma(v)v/v_b$.

Exercise 2

Let us compute the g-force corresponding to an acceleration $a = (dv/dt')$ at $t' = t'_0$. At that time, the infinitesimal velocity change reads $v(t'_0 + dt') = v_0 + a_0\,dt'$, with $v_0 \equiv v(t'_0)$ and $a_0 \equiv (dv/dt')|_{t'=t_0}$. To obtain the corresponding g-force, we boost to the instantaneous inertial boat rest frame $S(v_0)$. In this inertial frame, we can use Newtonian physics to equate the acceleration to the experienced g-force. The velocity addition formula, going from $S = S(0)$ to $\bar{S} = S(v_0)$, reads

$$\bar{v} = \frac{v - v_0}{1 - \frac{v_0 v}{c^2}}. \tag{6}$$

Plugging in $v(t'_0 + dt') = v_0 + a_0\,dt'$ then gives us

$$\bar{v}(t'_0 + dt') = \frac{a_0 dt'}{1 - \frac{v_0^2}{c^2}}, \tag{7}$$

which finally allows us to read off the g-force g, corresponding to an acceleration $a = (dv/dt')$ at velocity v,

$$g = \frac{a}{1 - v^2/c^2} = \gamma^2(v)a. \tag{8}$$

For a constant acceleration $a = 0.003g$, assuming that we pass out at $10g$, we then find a critical velocity $v_{crit} = 19.997\,\mathrm{km/h}$.

Exercise 3

The top of the tower was positioned more to the left than the bottom of the tower when the respective photons left, leading to the distorted picture in Fig. 3.

Exercise 4

When the center of the fence appears in the center of our viewfinder, aiming at an angle $\theta' = \pi/2$, and assuming that the fence lies parallel to the velocity direction, we know that the traveled distance of the photons from both endpoints to our camera is the same. The picture, therefore, shows both endpoints at the same time t' in our reference frame S'. In addition, we can therefore indeed use the standard Euclidean geometry in our 3D-space to relate the length l' of the fence, to the opening angle α and the distance y to the boat: $l' = 2y\tan(\alpha/2)$.

For the house at the left, the photons coming from the left endpoint traveled over a longer distance than those coming from the right endpoint. The left endpoint is therefore shown at an earlier time in the past than the right endpoint: $t'_L < t'_R$. So at t'_L, the house was positioned more to the left than at t'_R. This elongates the apparent shape in the picture, counteracting the length contraction. For the van at the right, we now have that

$t'_L > t'_R$, producing an extra apparent contraction in addition to the length contraction. Similarly, one can understand the Terrell effect on the fence from the different times at which the photons left the side part of the fence (see Fig. 15).

Exercise 5

First of all, notice that in the quay rest frame, we do not have to worry about the vision delay effect as the endpoints of the fence do not move. From Fig. 8, we then find

$$l = y(\cot\theta_1 - \cot\theta_2). \tag{9}$$

We can then use the aberration formula (1) to get an expression in terms of the angles θ' in the boat rest frame. First, we can easily invert (1), by substituting $v \to -v$

$$\cos\theta = \frac{\cos\theta' - \frac{v}{c}}{1 - \frac{v}{c}\cos\theta'}, \tag{10}$$

after which we can also solve for $\sin\theta$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \frac{\sin\theta'\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}\cos\theta'}, \tag{11}$$

arriving at

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \gamma(v)\frac{\cos\theta' - \frac{v}{c}}{\sin\theta'}. \tag{12}$$

With $\theta'_1 = \pi/2 - \alpha/2$ and $\theta'_2 = \pi/2 + \alpha/2$, Eq. (9) then gives us

$$l = 2y\gamma(v)\tan\left(\frac{\alpha}{2}\right) = \gamma(v)l', \tag{13}$$

which is precisely the length contraction formula.

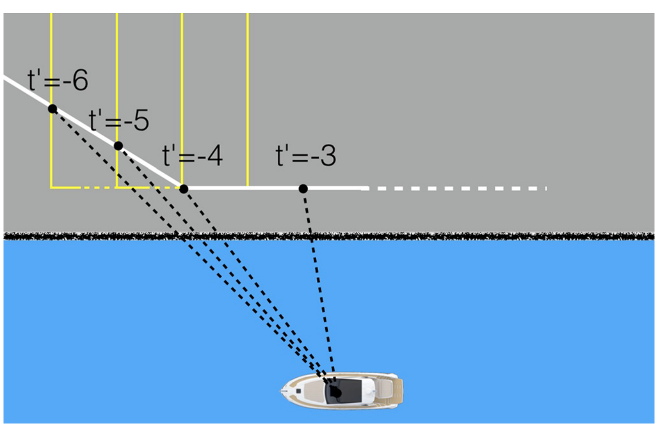


Fig. 15. The apparent rotation of the left side of the white fence (Terrell effect) explained from the vision-delay effect. The figure shows the horizontal plane for the boat observer. The yellow lines are the different fixed-time snapshots (in the boat rest frame S') of the left part of the fence. The black dots indicate where particular photons have left the fence at different times t', to enter the camera at $t' = 0$. The white lines show the resulting image for the fence as observed on the boat at $t' = 0$.

Exercise 6

To understand the origin of the different colors in Fig. 9(b) or Fig. 11, you have to look at the particular wavelength intervals that are Doppler shifted (by Eq. (2)) into the visible window for the different viewing directions θ' (see Fig. 16). In this way, you can understand that the yellow and red colors for orthogonal viewing angles $\theta' \approx 90°$ originate from the UV band (300 nm, 400 nm) in the blue sky spectrum, which for increasing angles $\theta' \gtrsim 80°$ gets redshifted further and further into the low frequency part of the visible window (580 nm, 700 nm). This up to the point that the UV band is shifted completely beyond the visible window, resulting in a darkness for directions $\theta' \gtrsim 100°$. So the reason for the dark behind you is simply the absence of UV radiation in the blue sky spectrum for wavelengths $\lambda \lesssim 300$ nm.

Similarly, the distinct blue and purple bands for small angles θ' arise from the IR part of the spectrum that is now blueshifted ($D > 1$) into the visible window. Were it not for the greenhouse absorption bands, the IR tail would be smooth, resulting in a rather uniform color, independent of the precise viewing direction θ'. However, the absorption bands introduce sharp features, producing distinct colors depending on the particular position of the Doppler shifted bands in the visible spectrum. (See the end of Sec. III for more details on the relation between spectra and colors.)

Exercise 7

The transverse Doppler effect can be simply observed by looking straight up during the boat trip (see Fig. 17). Notice that the $v/c = 0.4$ image does not differ much from the $v = 0$

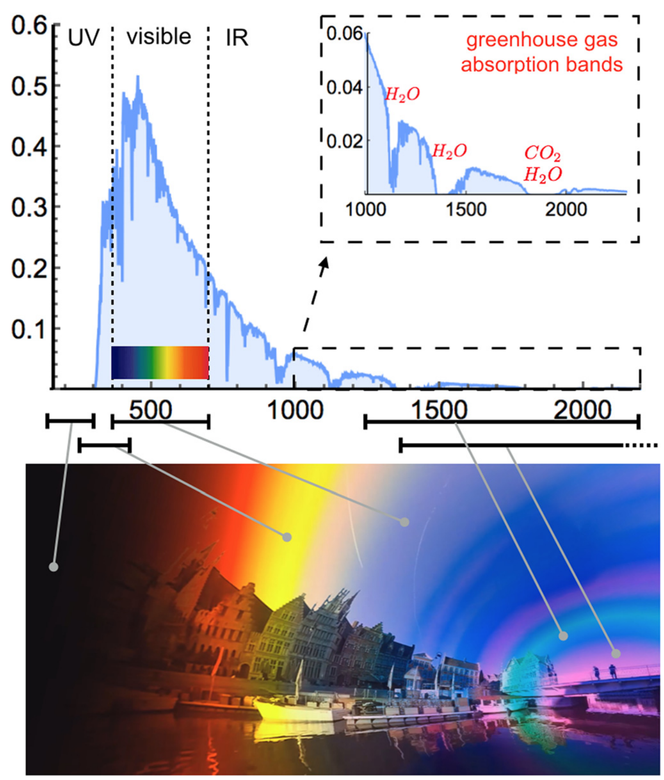


Fig. 16. For directions $\theta' = \{105°, 80°, 56°, 12°, 3°\}$ (and for $v = 0.85c$), the different intervals that are Doppler shifted into the visible window (390 nm, 700 nm), producing the different color effects on our relativistic boat trip. (Top panel: see Fig. 10 for the definition of the axes and inset.)

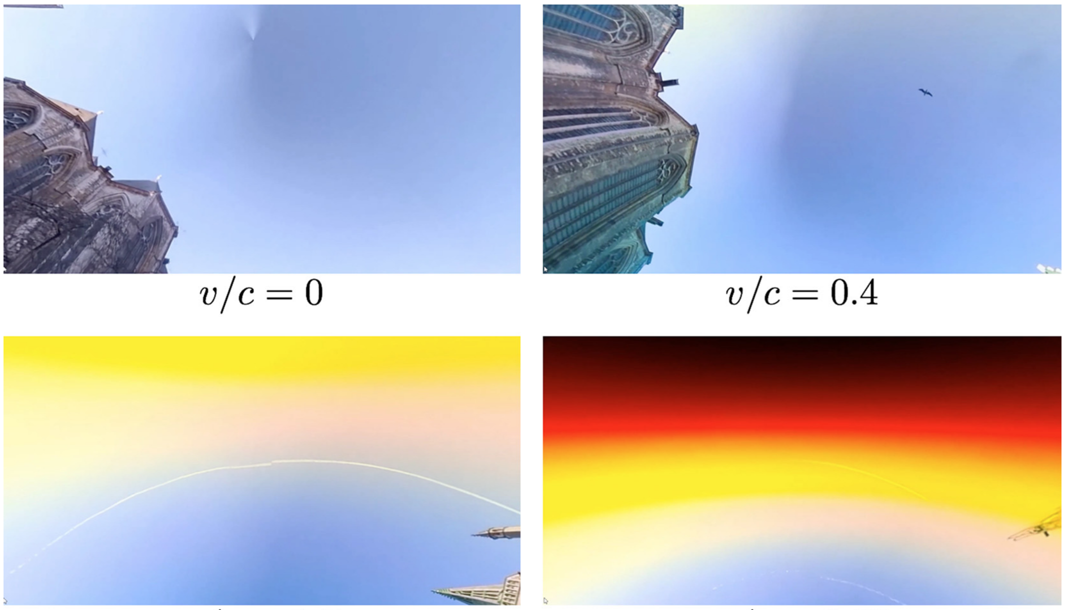


Fig. 17. The transverse Doppler effect: point of view shots at different speeds, always looking straight up.

view: the transverse Doppler effect is a truly relativistic order v^2/c^2 effect, manifesting itself rather late during the boat trip. (And it is equivalent to the time dilation, as we discussed in the exercise question.)

Exercise 8

(a) The pedestrian's velocity u' in the boat frame follows immediately from the velocity addition formula

$$u' = \frac{u+v}{1+\frac{vu}{c^2}}. \tag{14}$$

(b) The Doppler formula applies for any periodic signal that is transmitted with light speed (identifying $\lambda = c/f$ with f the signal frequency). By using the Doppler formula to go from the pedestrian's reference frame S_0 to the boat frame S', we then immediately find

$$f' = f_0 \frac{\sqrt{1-\frac{u'^2}{c^2}}}{1-\frac{u'}{c}\cos\theta'}, \tag{15}$$

for the visually observed walking pace f'. Solving $f' = f_0$, we then find a critical angle θ_c

$$\theta_c = \arccos\left(\frac{c}{u'} - \sqrt{\frac{c^2}{u'^2}-1}\right) < \pi/2. \tag{16}$$

For viewing directions $\theta' < \theta_c$, we would see the properly visualized pedestrian in fast motion, while for $\theta' > \theta_c$, we would see her in slow motion. Notice that in orthogonal directions $\theta' = \pi/2$, we would see a walking pace f' slowed down by a factor $\sqrt{1-u'^2/c^2}$. This is, of course, again the transverse Doppler effect, a visual manifestation of the time dilation, in this case showing us directly the slowdown of the pedestrian's eigentime with respect to the boat frame time t.

Exercise 9

Similar to the previous exercise, we can apply the Doppler formula also on the photon flux "frequency" $f = (dN/dt)$

$$\frac{dN'}{dt'} = D\frac{dN}{dt}. \tag{17}$$

While for the surface element $d\Omega = d\cos\theta d\varphi$, we have from the inverted aberration formula (1) (replacing $v \to -v$, $\theta \leftrightarrow \theta'$)

$$d\Omega = d\cos\theta d\varphi = \frac{1-v^2/c^2}{\left(1-\frac{v}{c}\cos\theta'\right)^2} d\cos\theta' d\varphi = D^2 d\Omega'. \tag{18}$$

Together, this amounts to

$$F' = \frac{dN'}{dt'd\Omega'} = D^3\frac{dN}{dtd\Omega} = D^3 F. \tag{19}$$

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