Chapter 7: Work & Energy

The work done by a force on an object is a special combination of that force and the displacement of the object.

If net work is done on an object, that object's kinetic energy will change.

The kinetic energy of an object is the energy due solely to its motion.

Kinds of Energy

- Electro-magnetic
- Chemical
- Nuclear
- Thermal
- Mechanical \{ Kinetic Potential

One of the iron-clad rules of physics: Energy is conserved - it cannot be created nor destroyed. Energy can, however, be converted from one form to another.
**Kinetic Energy**

The energy of an object due to its motion:

\[ K = \frac{1}{2} m v^2 \]

\( K \) is always positive or zero.

\( K \) is zero only when \( v = 0 \).

\( K \) is a scalar quantity:

- There is no direction associated with \( K \).
- \( K \) will not change when I change my coordinate system.

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**Ex.**

\[ \vec{v} = 3 \text{ m/s} \]

\[ m = 2 \text{ kg} \]

One coordinate choice:

\[ K = \frac{1}{2} m v^2 = \frac{1}{2}(2)(3)^2 = 9 \text{ J} \]

A different coordinate choice:

\[ \vec{v} = (-2.12, 2.12) \text{ m/s} \]

\[ K = \frac{1}{2} m v^2 = \frac{1}{2} m |\vec{v}|^2 = \frac{1}{2}(2)(3)^2 = 9 \text{ J} \]

\( |\vec{v}| = \sqrt{2.12^2 + 2.12^2} = 3 \)
**Dimensions & Units of Energy**

\[ K = \frac{1}{2}mv^2 \]

\[ [K] = \left[ \frac{1}{2} \right] [m] [v]^2 \]

\[ = 1 \cdot M \cdot \frac{L^2}{T^2} = \frac{ML^2}{T^2} \]

The S.I. unit of energy is the joule (J).

\[ 1 \text{ J} = 1 \text{ N} \cdot \text{m} = 1 \frac{\text{kg} \cdot \text{m}^2}{\text{s}^2} \]

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**Work**

**A special case**

The work done by a constant force \( \vec{F} \) on an object which undergoes a displacement \( \vec{s} \) is

\[ W = |\vec{F}| \cdot |\vec{s}| \cos \theta = \vec{F} \cdot \vec{s} \]

where \( \theta \) is the angle between the vectors \( \vec{F} \) and \( \vec{s} \) drawn tail to tail.
The work done on an object by a force can be zero.
\[ \vec{F} \text{ can be zero.} \]
\[ \vec{s} \text{ can be zero. Ex: holding a mass still.} \]
\[ \vec{F} \text{ and } \vec{s} \text{ can be perpendicular.} \]

The work done on an object by a force can be negative.
\[ \vec{F} \text{ is a } \textbf{scalar}. \]
Ex. The work done by friction on an object is negative.

\[
\begin{align*}
\vec{F}_k & \quad \vec{F} \\
\vec{N} & \quad \vec{S}
\end{align*}
\]

\[
\begin{align*}
\vec{F}_k & \quad \theta = 180^\circ \\
\vec{F} & \quad \vec{S}
\end{align*}
\]

\[
W_{f_k} = |\vec{F}_k| \cdot |\vec{S}| \cos 180^\circ \\
= -f_k \cdot s
\]

What about the work done on an object by static friction?

\[
W_{f_s} = 0 \quad \text{because} \quad s = 0
\]

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**Dot Product**

*(Scalar Product)*

The dot product of two vectors is a scalar. It is defined to be the length of one vector times the length of the second vector times the cosine of the angle between them, drawn tail to tail.

\[
W = \vec{F} \cdot \vec{S} = |\vec{F}| \cdot |\vec{S}| \cos \theta
\]
Other Definitions

\[ \text{Part of } \vec{B} \text{ along } \vec{A} = (|\vec{B}| \cos \theta) \]

The dot product of $\vec{A}$ and $\vec{B}$ is equal to the magnitude of $\vec{A}$ times the part of $\vec{B}$ that lies along $\vec{A}$. 

\[ \text{mag of } \vec{B} \]

The dot product of $\vec{A}$ and $\vec{B}$ is also equal to the magnitude of $\vec{B}$ times the part of $\vec{A}$ that lies along $\vec{B}$, \( (= |A| \cos \theta) \)
**Unit Vectors**

\[ \hat{e} \cdot \hat{e} = \left| \hat{e} \right| \left| \hat{e} \right| \cos \theta = 1 \cdot 1 \cdot \cos (0) = 1 \]
\[ \hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{j} \]
\[ \hat{k} \cdot \hat{k} = 1 \]

\[ \hat{e} \cdot \hat{j} = \left| \hat{e} \right| \left| \hat{j} \right| \cos \theta = 1 \cdot 1 \cdot \cos (90^\circ) = 0 \]
\[ \hat{j} \cdot \hat{j} = 1 \]
\[ \hat{k} \cdot \hat{j} = 0 \]

\[ \vec{A} = A_x \hat{e} + A_y \hat{j} + A_z \hat{k} \]
\[ \vec{B} = B_x \hat{e} + B_y \hat{j} + B_z \hat{k} \]
\[ \vec{A} \cdot \vec{A} = \left| \vec{A} \right| \left| \vec{A} \right| \cos (0) = \left| \vec{A} \right|^2 \]

**Cartesian Form**

\[ \vec{A} = (A_x, A_y, A_z) \]
\[ \vec{B} = (B_x, B_y, B_z) \]

then
\[ \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \]

The dot product is commutative:
\[ \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} \]
Real Definition of Work

The work done by a non-constant force on an object as it moves along any curved path is

\[ W = \int \mathbf{F} \cdot d\mathbf{s} \]

Whoa! What is \( d\mathbf{s} \)?

\( d\mathbf{s} \) is an infinitesimal vector that points along the path of the object.

How do I integrate \( \int \mathbf{F} \cdot d\mathbf{s} \)?

Use the definition of dot product.

\[ \mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k} \]

\[ d\mathbf{s} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k} \]

\[ \mathbf{F} \cdot d\mathbf{s} = F_x \, dx + F_y \, dy + F_z \, dz \]

\[ W = \int \mathbf{F} \cdot d\mathbf{s} \]

\[ = \left( \int_{x_i}^{x_f} F_x \, dx \right) + \left( \int_{y_i}^{y_f} F_y \, dy \right) + \left( \int_{z_i}^{z_f} F_z \, dz \right) \]
How much work is done by gravity as I move a mass of 1 kg up 1 m and across 2 m? (in a straight line)

\[ W = \int_{x_i}^{x_f} F_x \, dx + \int_{y_i}^{y_f} (F_y) \, dy \]

\[ \Delta S = dx \, dy \]

\[ F_x = -9.8N \]

\[ = \int_{0}^{1m} (-9.8) \, dy \]

\[ = -9.8y \bigg|_{0}^{1m} \]

\[ = -9.8(1) - (-9.8)(0) = -9.8 \, J \]

**Ideal Springs**

A spring that obeys Hooke's Law:

\[ F_x = -k \, x \]

\[ k \] is the displacement of the end of the spring from equilibrium.

\[ k \] is a constant of proportionality called the spring constant.

\[ k \text{ small } \Rightarrow \text{ easily stretched} \]

\[ k \text{ large } \Rightarrow \text{ very stiff} \]
The minus sign means that no matter how you choose coordinates the spring force will always point in a direction opposite to the displacement from equilibrium.

(Restoring Force)

Work done by a spring

\[ W = \int F \cdot ds \rightarrow \int F_x \, dx \]

\[ W_s = \int_x^{x_f} (-kx) \, dx = -k \frac{x_f^2}{2} - \left(-k \frac{x_i^2}{2}\right) \]

\[ = -\frac{1}{2} k x_f^2 + \frac{1}{2} k x_i^2 \]
Ex. What work is done by the spring when the spring is compressed 2m from equilibrium? \((k=10 \text{ Nm})\)

\[
W_s = \int (-kx) \, dx = -\frac{1}{2} kx^2
\]

\[
= -\frac{1}{2} (10 \text{ Nm}) (2)^2 = -20 \text{ J}
\]

\[
W_s \text{ is a scalar}
\]
Summary:

The work done by a spring on an object as the object is moved from equilibrium (stretched or compressed) is negative.

Can a spring do positive work on an object?

The work done by the spring is positive when the object is moved from a stretched or compressed state toward equilibrium.

\[
W_s = \int_{x_i}^{x_f} F_x \, dx = \int_{x_i}^{x_f} (-k_x) \, dx = -\frac{1}{2} k_x^2
\]

\[
-\frac{1}{2} \left( \frac{0.4 \text{ m}}{0.5 \text{ m}} \right) 0^2 - \left[ -\frac{1}{2} \cdot 10 \text{ N/m} (0.2)^2 \right] = 0 + 20 \text{ J}
\]

Start at \( x = 0 \) (equilibrium) then

\( W_s < 0 \)
Ex: How much work is done by our favorite spring ($k = \text{0.1 N/m}$) as the mass is moved from an initial position where the spring is compressed 2m to a final position where the spring is extended 2m?

\[ W = 0 \]

\[ W_{\text{total}} = \Delta K = K_f - K_i \]

\[ W_f = \int \vec{F} \cdot d\vec{s} \]

\[ K = \frac{1}{2} m v^2 \]

Environment \rightarrow particle motion

Just like Newton's Second Law

\[ \sum \vec{F} = m \vec{a} \]
Ex (Atwood's machine) What is the speed of the 7 kg mass when it has fallen from rest a distance of 1 m?

\[
\begin{align*}
T &= W_1 = m_1 a \\
T &= W_2 = m_2 (\varepsilon a)
\end{align*}
\]

\[
a = \frac{m_0 - m_1}{m_1 + m_2} g
\]

\[
\frac{v_f^2}{2} = v_0^2 + 2a \Delta y
\]

\[
= 0 + 2 \left( \frac{7 - 2}{7 + 2} \right) g \cdot \frac{4}{5} (-1 m)
\]

\[
v_f = 1.8 m/s
\]

Now with the work-energy theorem:

\[
W_{net} = \Delta K = K_f - K_i
\]

It is critical that we consider the entire system (both masses):

\[
W_{net} = \Delta K = K_f - K_i
\]

\[
\begin{align*}
\text{What is the work done by the force of gravity on } m_1? \\
\end{align*}
\]

\[
W_{g1} = \int F_y \, dy = \int (-m_1 g) \, dy
\]

\[
= -m_1 g y \bigg|_0^1 = -(7 kg)(9.8 m/s^2)(1 m)
\]

\[
= -72.5 J
\]

\[
\frac{m_1}{m_2} = \frac{3}{5} \cos 150^\circ
\]
What is the work done by the force of gravity on \( m_2 \)?

\[ W_{g2} = \int_{x^1}^{x^0} F_y \, dy = \int_{x^1}^{x^0} (-m_2 g) \, dy \]

\[ = -m_2 g (y^0 - y^1) = -((7.8)(8.8)/(-1m)) \]

\[ = 66.6 \text{ J} \]

\( W_T = \text{internal force} \)

What is the work done by the tension in the rope on \( m_1 \) and \( m_2 \)?

\[ W_T = \int_{m_1}^{m_2} T \, dy = 0 \]

\[ W_{net} = \Delta K = K_f - K_i = \frac{k_f}{m_1} + \frac{k_f}{m_2} \]

\[ -49 \text{ J} + 68.6 \text{ J} = \frac{1}{2} m_1 v_f^2 + \frac{1}{2} m_2 v_f^2 \]

\[ - (\frac{1}{2} m_1 \frac{86^2}{m} + \frac{1}{2} m_2 \frac{18^2}{m}) \]

\[ 19.6 \text{ J} = \frac{1}{2} (5 kg) v_f^2 + \frac{1}{2} (7 kg) v_f^2 \]

\[ = \frac{1}{2} (12 kg) v_f^2 \]

\[ v_f = \sqrt{\frac{(19.6 \text{ J}) 2}{12 \text{ kg}}} = 1.81 \text{ m/s} \]
Rule of Thumb

When solving for acceleration or tension, it is generally easier to use \( \pm F = ma \) (Newton II).

When solving for displacement or velocity, it is generally easier to use \( W_{net} = AK \) (work-energy theorem).

But the physics is the same. Any problem can be solved by either method.

Power

Power is the rate at which work is done.

\[
P_{\text{Avg}} = \bar{P} = \frac{\Delta W}{\Delta t} = \frac{W_f - W_i}{t_f - t_i}
\]

\[
P_{\text{Inst}} = P = \frac{dW}{dt} = F \cdot \mathbf{v}
\]

Dimension: \( [P] = \frac{ML}{T^2} \cdot \frac{L}{T} = \frac{ML^2}{T^3} \)

Unit: 1 watt = 1 W = 1 \( \frac{Kg \cdot m^2}{s^3} \)

Vector or Scalar?
Ex: I want to cook a steak by dragging it behind my car.

\[ P = \overrightarrow{F} \cdot \overrightarrow{v} = \overrightarrow{F}_{k}\overrightarrow{v} = -m_k \alpha v_1 \]

\[ f_k = m_k N = m_k mg \]

\[ P = -m_k mg v \]

\[ W_{\text{friction}} = \int P \, dt = -m_k mg v \Delta t \]

Heat energy = \( \overline{W_{\text{friction}}} = m_k mg v \Delta t \)

\[ \approx (0.7)(1 \text{ kg})(9.8 \text{ m/s}^2)(27 \text{ s})(3600 \text{ s}) \]

\[ = 667,000 \text{ J} = 159 \text{ Food Cal.} \]

Ex: A baseball pitcher throws a ball, accelerating it from rest to 90 mph. Is the work done on the ball by his arm's force positive or negative?

\[ W_{\text{friction}} = \Delta K = K_f - K_i \]

\[ K_f = \frac{1}{2} m v^2 \]

\[ K_i = 0 \]

\[ W_{\text{friction}} = \Delta K = \frac{1}{2} m v^2 - 0 = \text{positive} \]

The catcher catches the ball. Does the force of the catcher's arm do positive or negative work?

\[ \Delta K = K_f - K_i \]

\[ K_i = \frac{1}{2} m v_i^2 \]

\[ W_{\text{catch}} = 0 - \frac{1}{2} m v_f^2 \text{ negative} \]