Physical Quantities

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Angular Momentum

\[ \vec{L} = \vec{r} \times \vec{p} \]

Important!
- Only the tangential component of velocity matters in Angular Momentum.
- \( L \) depends on the reference point.

Example: Ball on a String

\[ m = \text{mass} \]
\[ r = \text{radius} \]
\[ \vec{v} = \text{velocity} \]
For Circular Motion:

\[ L = r m v \]

Let’s write this in terms of purely angular coordinates,

\[ v = r \omega \]

so

\[ L = (mr^2) \omega \]

or

\[ L = I \omega \]

This is the analog \( \mathbf{r} = m \mathbf{v} \)

Why is angular momentum important?

Because, just like linear momentum, it can be conserved.

Linear Cases

If no external forces act, then the total linear momentum of a set of particles is constant.

\[ 0 = \frac{d}{dt} P = \frac{d}{dt} \sum m \mathbf{v} = 0 \]

Rotational Case:

If no external torques act, then the total angular momentum of a set of particles is constant.

\[ 0 = \frac{d}{dt} L_{\text{net}} = \frac{d}{dt} \sum I \omega = 0 \]
Example:

Suppose no external forces or torques act on a rotating dumbbell with \( \omega_i = 4\pi \text{ rad/s} \)

\[ m \quad \bullet \quad \bullet \quad m \]

Initially, the masses are 1 meter apart, but while rotating the masses more in until they are \( \frac{1}{2} \) meter apart. (Internal forces cause the masses to move.)

Question:

What is the final angular velocity? \( \omega_f \)

Approach:

No external torques act so...

Angular Momentum is conserved.

\[ L_i = L_f \]

\[ I_i \omega_i = I_f \omega_f \]

What happens to \( I \)? decreases

so what happens to \( \omega \)? increases
\[ \omega_f = \left( \frac{I_i}{I_f} \right) \omega_i \]

Moment of Inertia: \( I = (2m) \pi^2 \)

so \( \frac{I_i}{I_f} = 3^2 = 9 \)

(from 1 m apart to \( \frac{1}{3} \) m apart)

\[ \omega_f = 9 \omega_i = 36 \pi \text{ rad/s} \]

While we're here, let's look at the initial and final Kinetic Energy.

Suppose \( I_i = 9 \text{ kg} \cdot \text{m}^2 \) then \( I_f = 1 \text{ kg} \cdot \text{m}^2 \)

\[ K_i = \frac{1}{2} I_i \omega_i^2 = \frac{1}{2} (9 \text{ kg} \cdot \text{m}^2)(4 \pi \text{ rad/s})^2 \]
\[ = 711 \text{ J} \]

\[ K_f = \frac{1}{2} I_f \omega_f^2 = \frac{1}{2} (1 \text{ kg} \cdot \text{m}^2)(36 \pi \text{ rad/s})^2 \]
\[ = 6396 \text{ J} = 9K_i \]

\( \Delta K = W_{\text{net}} \)

How was work done?
How was work done?

I must exert a force to bring the masses in. This force points radially inward.

Is the work done positive or negative?

Positive. The displacement is also radially inward.

This force exerts no torque. Why not?

Another example:

"Perfectly inelastic angular collision"

A uniform disk of mass $M$ and radius $R$ spins with $\omega_i = 3\pi$ rad/s.

Onto this disk I drop a ring with mass $M$ and radius $R$, which is not initially rotating.

If the two stick together, what is the final angular velocity? $\omega_f$

Approach?

Do any external torques act?
No external torques, so

\[ L \text{ is conserved} \]

\[ L_i = L_f \]

\[ I_i \omega_i = I_f \omega_f \]

\[ I_i = \frac{1}{2} MR^2 \quad \text{(disk)} \]

\[ I_f = \frac{1}{2} MR^2 + MR^2 \quad \text{(disk + ring)} \]

\[ = \frac{3}{2} MR^2 \]

\[ (\frac{1}{2} MR^2)(3 \pi \frac{\text{rad}}{s}) = \frac{3}{2} MR^2 \ \omega_f \]

\[ \omega_f = \pi \frac{\text{rad}}{s} = \frac{1}{3} \omega_i. \]

\[ \omega_f < \omega_i. \]

Ex:

I am going to hold weights at arm's length while rotating. When I drop the weights, what will happen to my angular speed?

- increase
- decrease
- remain the same

Why?
We have been casual in treating angular momentum so far. 

\[ L \text{ is a vector} \]

Definition:

\[ L = \mu \omega \]

The cross product acts on 2 vectors to produce a third vector.

**Cross Product**

**Physical Definition:** \( \vec{A} \times \vec{B} \)

1. The magnitude of the component of vector \( \vec{A} \) which lies perpendicular to vector \( \vec{B} \) times the magnitude of vector \( \vec{B} \). The direction of the cross product is perpendicular to both \( \vec{A} \) and \( \vec{B} \), as determined by the "right hand rule."

Direction of \( \vec{A} \times \vec{B} \) is toward you.
Example:

\[ \vec{L} = \vec{r} \times \vec{p} \]

magnitude: \( |\vec{L}| = |\vec{r}| |\vec{p}| \) in this case

Why? Because \( \vec{r} \perp \vec{p} \)

direction: "right hand rule"

⇒ out of the screen, toward you.

It may help to redraw the vectors tail to tail.

\[ \vec{p} \quad \vec{r} \]

\[ \vec{p} \times \vec{r} \]

Notice that

\[ \vec{r} \times \vec{p} = -\vec{p} \times \vec{r} \]

This is true in general.

\[ \vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \]

Definition (#2)

Magnitude of Cross Product

\[ |\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta \]

\[ \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta \]

Where \( \theta \) is the angle between \( \vec{A} \) and \( \vec{B} \).

Direction: From the right hand rule.

What is \( \vec{r} \times \vec{r} \)?
**Vector quantities:**
\[ \dot{\theta}, \vec{\omega}, \vec{a}, \vec{v}, \vec{L} \]

**Scalar quantities:**
\[ t, I \text{ (moment of inertia)} \]

**Example:**
An airplane flies directly over an airport with a horizontal speed of 150 m/s at a height of 300 m. Its mass is 5000 kg. The plane flies south.

1) What is the angular momentum with the airport as the origin?

2) If the plane continues to fly at the same height and velocity, what is its angular momentum about the same origin 10 seconds later?
West is into screen  
East is out of screen

\[ \vec{L}_0 = \vec{r} \times \vec{v} = \vec{r} \times (m \vec{v}) \]

\[ |\vec{L}_0| = |\vec{r}| / |m \vec{v}| \sin 90^\circ \]
\[ = (300 \text{ m}) (5000 \text{ kg})(150 \text{ m/s}) \]
\[ = 225 \times 10^6 \text{ kg m}^2 / \text{s} \]

Direction?
out of screen \Rightarrow \text{East}
\( \vec{L}_0 \) is exactly the same!

**Why?**

\[ \vec{R} = v \cdot t = 1500 \text{ m} \]
\[ R = R_2 = 300 \text{ m} \]
\[ R = R \sin \theta \]

\[ \vec{L}_0 = \vec{R} \times \vec{P} \]

Only perpendicular components matter.

\[ \vec{R} = 1500 \text{ m} \text{ south} + 300 \text{ m} \text{ up} \]
\[ \vec{P} = m \vec{v} \text{ south} = 750 \times 10^3 \text{ kg} \cdot \text{m/s} \text{ south} \]

South \times \text{south} = ? \quad \text{C}

Up \times \text{south} = ? \quad \text{East}

\[ \vec{L}_0 = 225 \times 10^6 \text{ kg} \cdot \text{m}^2/\text{s} \text{ East} \]

This makes sense...

\[ \vec{P} = m \vec{v} \] is a constant,

so there is no net external force acting on the plane,

so there is no net external torque acting on the plane,

so the plane's angular momentum about the airport must be a constant.
Ex: If I sit on a frictionless chain at rest, holding a spinning bicycle wheel and flip the wheel 180°, what will happen? The wheel is initially spinning ccw as seen from above.

a) Nothing
b) Spin clockwise
c) Spin counterclockwise
d) Get dizzy and fall off chain

Why? \( \vec{L}_i \) is up.

No external torques. \( \Rightarrow \vec{L}_i = \vec{L}_f \)

Correct Ans = Spin counterclockwise

Why?

Angular momentum is conserved,

\[ \vec{L}_i = \vec{L}_f \]

as a vector.

\[ \vec{L}_i = \vec{L}_i^\text{wheel} + \vec{L}_i^\text{me} \]

\( \uparrow \text{up} \quad 0 \)

After the flip

\[ \vec{L}_f = \vec{L}_f^\text{wheel} + \vec{L}_f^\text{me} \]

\( \downarrow \text{down} \quad \uparrow \text{up} \Rightarrow \text{ccw} \)
Compare $|\vec{L}_{me}|$ to $|\vec{L}_{wheel}|$.

Remember that we can add vectors graphically.

Before flip $\vec{L}_i = \vec{L}_{wheel}$

After flip $\vec{L}_f = \vec{L}_{wheel} + \vec{L}_{me}$

We know $\vec{r} = I \vec{\alpha}$ 

A torque provides an angular acceleration which changes the angular velocity. This can change the angular momentum.

But $\vec{r}, \vec{L}, \vec{\alpha}, \vec{\omega}, d\phi$ are all vector quantities. Direction is important!

\begin{align*}
\vec{L} &= I \vec{\omega} \\
\vec{L} &= \vec{r} \times \vec{p}
\end{align*}
Torque is a vector:
\[ \vec{\tau} = I \vec{\omega} \quad (\vec{p} = m \vec{v}) \]
\[ \vec{\tau} = \vec{r} \times \vec{F} \]

The Cross Product again!

Let's find a relation between angular momentum \( \vec{L} \) and torque \( \vec{\tau} \).

Start with: \( \vec{\tau} = \vec{r} \times \vec{F} \)

and substitute: \( \vec{F} = \frac{d\vec{p}}{dt} = m\vec{a} \)

Remember this while we explore angular momentum.
\[ \vec{L} = \vec{r} \times \vec{p} \]

Take the time derivative of both sides.

\[ \frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \]

\[ \frac{d\vec{r}}{dt} = \vec{v} \quad \text{and} \quad \vec{p} = m\vec{v} \]

\[ \text{this gives} \quad \vec{v} \times m\vec{v} = 0 \]

So

\[ \frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} \]

\[ \vec{\tau} = \frac{d\vec{L}}{dt} \]
Simple notation:
\( \hat{\mathbf{c}} \) and \( \hat{\mathbf{L}} \) are in the same direction. E.g. bicycle wheel speeding up

But the equation
\[ \hat{\mathbf{c}} = \frac{d\hat{\mathbf{L}}}{dt} \]
does not say that \( \hat{\mathbf{c}} \) and \( \hat{\mathbf{L}} \) are in the same direction. It says that torque and the change in angular momentum are in the same direction.

\( \hat{\mathbf{c}} \) in same direct \( \hat{\mathbf{L}} \)

Bicycle wheel speeding up:

Now apply a torque

\( \hat{\mathbf{L}}_i \)

The force \( \hat{\mathbf{F}} \) points into the screen

\( \hat{\mathbf{c}} \)

\( \Delta \hat{\mathbf{L}} = \hat{\mathbf{L}}_f - \hat{\mathbf{L}}_i \) points up

\( \uparrow \Delta \hat{\mathbf{L}} \) in the same direction as the torque,
Suppose \( \vec{L} \) and \( \vec{L} \) are not in the same direction.

**Demo**

\( \vec{L} \) changes, not in magnitude but in direction. There must be a torque acting. What is the source of the torque?

Gravity

What the heck is going on here:

- [Image: Diagram showing forces and vectors]
  - Top view
  - The force \( \vec{m}g \) is into the screen
  - Torque is always perpendicular to the angular momentum in this case

Direction? \( \vec{L} \) Out
Result:

Precession

The angular momentum vector $\vec{L}$ rotates in a circle.

Why can't the spinning wheel fall down?

To fall down, $\Delta \vec{L}$ must be down. Since $\vec{\alpha} = \frac{d\vec{L}}{dt}$, the torque must be down. What is the direction of the force if the torque is down?

$\vec{F}$ must be out of the screen. There is no force in this direction.