Resonance

All objects have a natural angular frequency, $\omega_0$.

For a pendulum, $\omega_0 = \sqrt{\frac{g}{L}}$

For a mass on a spring, $\omega_0 = \sqrt{\frac{k}{m}}$

If a force is applied at a frequency close to $\omega_0$, the amplitude of oscillation will increase dramatically.

A force applied at a frequency far from $\omega_0$ will have little effect on the amplitude.

Chapter 14:
The Law of Universal Gravitation

A thought experiment:
Suppose that you climb a mountain and fire a projectile at different speeds horizontally.
How far does an object fall under the influence of gravity?

$$\frac{1}{2} g t^2$$

$t=0$ $t=1s$ $t=2s$ $t=3s$ ...
Isaac Newton calculated how far the Moon "fell" every second.

$$T = 27.3 \text{ days } \approx 1 \text{ month}$$

$$R = 240,000 \text{ miles}$$

$$v = \frac{2\pi R}{T} \approx 3000 \text{ ft/s}$$

$$a_n = \frac{v^2}{R} \approx 0.1 \text{ in/s}^2$$

How far does the Moon "fall" in 1 second?

$$d = \frac{1}{2} a_n t^2 = \frac{1}{20} \text{ or an inch}$$

How far does an apple fall in 1 sec

$$d = \frac{1}{2} gt^2 = 16 \text{ ft}$$

$$\frac{d_{\text{apple}}}{d_{\text{moon}}} \times 3600$$

Based on the relationship between the radius of a planet's orbit and the length of its year (its period), Kepler's Third Law, Newton assumed that the acceleration due to gravity, $g$, is inversely proportional to the square of the distance from the center of the Earth.
\[ R_{\text{moon}} = 240,000 \text{ miles} \]

\[ R_{\text{aple}} = \text{Radius of Earth} = 4000 \text{ miles} \]

\[ \frac{R_{\text{moon}}}{R_{\text{aple}}} = 60 \left( \frac{R_{\text{moon}}}{R_{\text{aple}}} \right)^2 = 3600 \]

The acceleration of any object toward the Earth \((g)\) is not a constant \(9.8 \text{ m/s}^2\) but is

\[ g(r) = \frac{k}{r^2} \]

where \(r\) is measured from the center of the Earth.

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**Universal Gravitation**

Every particle in the universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

\[ F = \frac{G m_1 m_2}{r^2} \]

**Direction:** along the line joining the centers of gravity of the two masses.

**Units of \(G\):**
\[ F = \frac{G m_1 m_2}{r^2} \]

\[ [F] = \frac{[G] [m_1] [m_2]}{[r]^2} \]

\[ [G] = \frac{[F] [r]^2}{[m_1] [m_2]} = \frac{(ML^2)(L)}{(M)^2} = \frac{L^3}{MT^2} \]

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**Units**

\[ \frac{m^3}{kg \cdot s^2} = \frac{N \cdot m^2}{kg \cdot s^2} \]

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**Experimentally measured value:**

\[ G = 6.672 \times 10^{-11} \ \frac{N \cdot m^2}{kg^2} \]

A very small number ⇒ gravity is weak!

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**Ex.** A block of mass \( m \) sits near the surface of the Earth. What is the force that the Earth exerts on the block?

\[ F = mg \]

\[ F = \frac{GM_E (m)}{r^2} = \left( \frac{GM_E}{R_E^2} \right) m \]

\[ g = \frac{GM_E}{R_E^2} = 9.8 \ \frac{m}{s^2} \]

\[ = \left( 6.672 \times 10^{-11} \ \frac{N \cdot m^2}{kg^2} \right) \left( 6 \times 10^2 \frac{kg}{m^2} \right) \]

\[ = \frac{6 \times 10^7 \text{ N}}{m^2} \]
Ex. What is the acceleration due to gravity at a height $h$ above the surface of the Earth?

$$n = (R_e + h)$$

$$g(n) = \frac{GM_E}{n^2} = \frac{GM_E}{(R_e + h)^2}$$

$$\leq 9.8 \text{ m/s}^2$$

What is the weight of an astronaut orbiting the Earth at an altitude $h$ of 2 Earth - radii?

$$m = 100 \text{ kg}$$

$$W_{\text{surface}} = mg = 980 \text{ N}$$

$$n = 3R_e$$

$$F = W = \frac{GM_Em}{n^2} = \frac{GM_Em}{(3R_e)^2}$$

$$= \frac{1}{9} \left(\frac{GM_E}{R_e^2}\right)m = \frac{1}{9} mg_{\text{surf}}$$

$$= \frac{1}{9} (100\text{ kg})(9.8 \text{ m/s}^2) = 109 \text{ N}$$
Kepler's Laws

1) All planets move in elliptical orbits with the Sun at one of the focal points.

2) The radius vector drawn from the Sun to any planet sweeps out equal areas in equal time intervals.

3) The square of the orbital period of any planet is proportional to the cube of the semi-major axis of the elliptical orbit. \( T^2 \propto r^3 \)

These are all empirical laws; they describe the observations but do not explain them.

All three of Kepler's Laws can be derived from Newton's Law of Universal Gravitation:

\[
F = G \frac{M_1 M_2}{r^2}
\]
1) All planets move in elliptical orbits with the Sun at one focus.

2) The radius vector drawn from the Sun to any planet sweeps out equal areas in equal time intervals.

Both green-shaded areas represent one month.

Planets (and comets) move faster when they are closer to the Sun.
3) The square of the orbital period \((T)\) (the planet's "year") is proportional to the cube of the semi-major axis of the elliptical orbit.

We will consider the special case of a **circular orbit**.

\[
\Sigma F_n = m_p a_n
\]

\[
F = \frac{G M_s m_p}{r^2} = \frac{m_p v^2}{r}
\]

\[
\frac{v_0}{a} = \frac{2\pi r}{T}
\]

\[
\frac{G M_s}{r^2} = \left(\frac{2\pi r}{T}\right)^2 \frac{1}{r}
\]

\[
T^2 = \left(\frac{4\pi^2}{GM_s}\right) r^3
\]

This relation suggested to Newton that the force is inverse-square.
Ex. At what altitude must a satellite orbit to attain geosynchronous orbit?

Gravitational Potential Energy

If $g$ is not constant, then

$$U_g \neq mg y + \text{constant}$$

$$\vec{F} = \frac{G m_1 m_2}{r^2}, \quad \text{directed toward the origin (one of the masses)}$$

$\vec{dr}$ is an infinitesimal displacement vector directed away from the origin

$$\vec{F} \cdot \vec{dr} = -\frac{G m_1 m_2}{r^2} \, dr$$
\[ U(r) = - \int F \cdot dr = - \int \left( -\frac{Gm_1 m_2}{r^2} \right) dr \]

\[ = \int \frac{Gm_1 m_2}{r^2} dr \]

\[ = Gm_1 m_2 \int r^{-2} dr \]

\[ = Gm_1 m_2 \left( -\frac{1}{r} \right) + \text{constant} \]

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Ex Suppose that a small mass m "falls" to Earth from infinity far away. What is the change in the mass' potential energy?

\[ \Delta U = U_f - U_i = U(R_e) - U(\infty) \]

\[ = \frac{G M_e m}{R_e} - \left( -\frac{G M_e m}{\infty} \right) \]

\[ = \frac{G M_e m}{R_e} \]

What happened to this energy?

It becomes \(-\Delta K\).
Ex Suppose the mass has no initial velocity at \( r=0 \). What is its speed when it strikes the Earth?

\[
\Delta K + \Delta U = 0
\]

\[
k_f - k_i + \left(-\frac{GM_\text{E} m}{R_c}\right) = 0
\]

\[
\frac{1}{2} m v_f^2 = \frac{GM_\text{E} m}{R_c}
\]

\[
v_f = \sqrt{\frac{2GM_\text{E}}{R_c}} = 1.12 \times 10^4 \text{ m/s}
\]

**Escape velocity** = 25,000 mph

How fast must I throw a rock to get it out to \( r=\infty \) with zero speed?