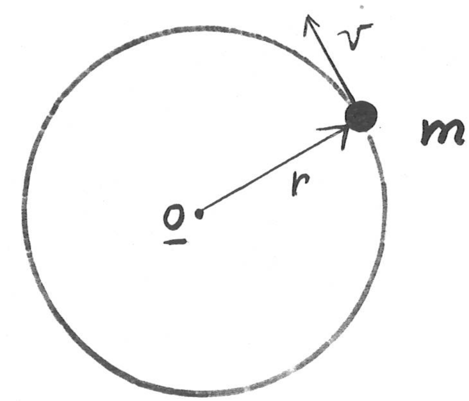
Physical Quantities

Linear	Angular
t	t
x	θ
v	ω (omega)
a_t	α (alpha)
m	I
F	τ (tau)
p	L

Angular Momentum

Chapter 11

Example: Ball on a String



Linear Momentum

$$\vec{p} = m\vec{v}$$

Angular Momentum

$$L_o = r\,p_t$$
$$= r\,m\,v_t$$

Important!

- Only the tangential component of velocity matters in Angular Momentum.
- L depends on the reference point O

For Circular Motion:

$$L_o = r m v$$

Let's write this in terms of purely angular coordinates.

$$v = r\omega$$

so

$$L_o = (m r^2)\,\omega$$

or

$$\boxed{L_o = I_o\,\omega}$$

This is the analog of $p = mv$

Why is angular momentum important?

Because, just like linear momentum, it can be **conserved**.

Linear Case:

If no external forces act, then the total linear momentum of a set of particles is constant.

$$0 = \vec{F}_{net} = \frac{d\vec{P}_{total}}{dt} = 0$$

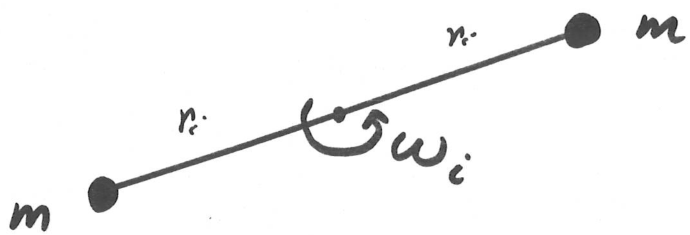
Rotational Case:

If no external torques act, then the total angular momentum of a set of particles is constant.

$$0 = \vec{\tau}_{net} = \frac{d\vec{L}}{dt} = 0$$

Example:

Suppose no external forces or torques act on a rotating dumbbell with $\omega_i = 4\pi$ rad/s



Initially, the masses are 1 meter apart, but while rotating the masses move in until they are $\frac{1}{3}$ meter apart. (Internal forces cause the masses to move.)

Question:

What is the final angular velocity? ω_f

Approach:

No external torques act so...

Angular Momentum is conserved.

$$L_i = L_f \qquad I = \sum m_i r_i^2$$

$$I_i \omega_i = \underset{\downarrow}{I_f}\, \underset{\uparrow}{\omega_f}$$

What happens to I? decreases

So what happens to ω? increases

$$\omega_f = \left(\frac{I_i}{I_f}\right)\omega_i$$

Moment of Inertia: $I = (2m)\,r^2$

so $\frac{I_i}{I_f} = 3^2 = 9$

(from 1 m apart to $\frac{1}{3}$ m apart)

$$\boxed{\omega_f = 9\,\omega_i = 36\pi\ \text{rad/s}}$$

Demo

While we're here, let's look at the initial and final Kinetic Energy.

Suppose $I_i = 9\ \text{kg}\cdot\text{m}^2$

then $I_f = 1\ \text{kg}\cdot\text{m}^2$

$$K_i = \tfrac{1}{2} I_i \omega_i^2 = \tfrac{1}{2}(9\ \text{kg}\cdot\text{m}^2)(4\pi\ \text{rad/s})^2$$

$$= 711\ \text{J}$$

$$K_f = \tfrac{1}{2} I_f \omega_f^2 = \tfrac{1}{2}(1\ \text{kg}\cdot\text{m}^2)(36\pi\ \text{rad/s})^2$$

$$= 6396\ \text{J} = 9\,K_i$$

$\Delta K = W_{net}$

How was work done?

How was work done?

I must exert a force to bring the masses in. This force points radially inward.

Is the work done positive or negative?

Positive. The displacement is also radially inward.

This force exerts no torque. Why not?

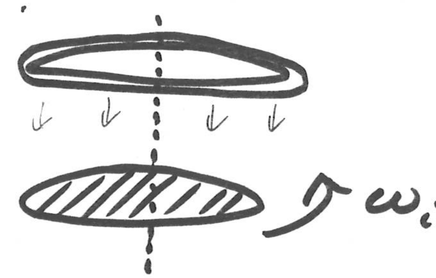
Another example:

"Perfectly inelastic angular collision"

A uniform disk of mass M and radius R spins with $\omega_i = 3\pi$ rad/s.

Onto this disk I drop a ring with mass M and radius R, which is not initially rotating.

If the two stick together what is the final angular velocity? ω_f



Approach?

Do any external torques act?

No external torques, so

L is conserved

$$L_i = L_f$$

$$I_i \omega_i = I_f \omega_f$$

$$I_i = \tfrac{1}{2} M R^2 \quad \text{(disk)}$$

$$I_f = \tfrac{1}{2} M R^2 + M R^2 \quad \text{(disk + ring)}$$

$$= \tfrac{3}{2} M R^2$$

$$\left(\tfrac{1}{2} \cancel{M}\cancel{R^2}\right)\left(3\pi \,\tfrac{\text{rad}}{s}\right) = \tfrac{3}{2} \cancel{M}\cancel{R^2}\, \omega_f$$

$$\boxed{\omega_f = \pi \,\tfrac{\text{rad}}{s}} = \tfrac{1}{3}\omega_i$$

$$K_f < K_i$$

Ex

I am going to hold weights at arm's length while rotating. When I drop the weights, what will happen to my angular speed?

increase

decrease

remain the same

why?

We have been casual in treating angular momentum so far.

$$\boxed{\vec{L} \text{ is a vector}}$$

$L = r m v_t$

Definition:

$$\vec{L}_o = \vec{r} \times \vec{p}$$

position vector

"cross product"

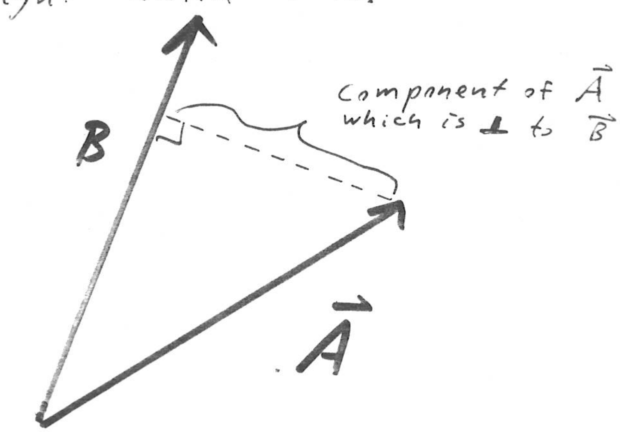
linear momentum vector

The cross product acts on 2 vectors to produce a third vector.

Cross Product

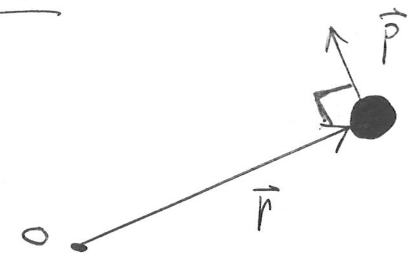
Physical Definition : $\vec{A} \times \vec{B}$

(#1) The magnitude of the component of vector $\vec{A}$ which lies perpendicular to vector $\vec{B}$ times the magnitude of vector $\vec{B}$. The direction of the cross product is perpendicular to <u>both</u> $\vec{A}$ and $\vec{B}$, as determined by the "right hand rule."



Direction of $\vec{A} \times \vec{B}$ is toward you.

Example:



$$\vec{L}_o = \vec{r} \times \vec{p}$$

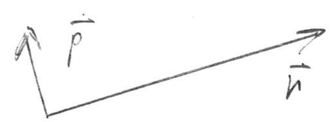
magnitude: $|\vec{L}_o| = |\vec{r}|\,|\vec{p}|$ in this case

why? Because $\vec{r} \perp \vec{p}$

direction: "right hand rule"

$\Rightarrow$ out of the screen, toward you.

It may help to redraw the vectors tail to tail.



$\vec{p} \times \vec{r}$?

Notice that

$$\vec{r} \times \vec{p} = -\vec{p} \times \vec{r}$$

This is true in general.

$$\boxed{\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}}$$

Definition (#2)

Magnitude of Cross Product

$$|\vec{A} \times \vec{B}| = |\vec{A}|\,|\vec{B}| \sin\theta$$

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta$$

where θ is the angle between $\vec{A}$ and $\vec{B}$.

Direction: from the right hand rule.

What is $\vec{r} \times \vec{r}$?

Vector quantities:

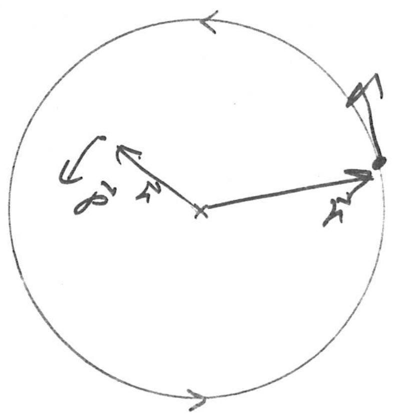
$d\vec{\theta}$, $\vec{\omega}$, $\vec{\alpha}$, $\vec{\tau}$, $\vec{L}$

Scalar quantities:

t, I (moment of inertia)

Ex. What is the direction of $\vec{L}$ for a spinning wheel?

$$\vec{L} = \vec{r} \times \vec{p}$$



$$\vec{L} = I\vec{\omega}$$

Direction of $\vec{\omega}$?

out

Example:

An airplane flies directly over an airport with a horizontal speed of 150 m/s at a height of 300 m. Its mass is 5000 kg. The plane flies south.

1) What is the angular momentum with the airport as the origin?

2) If the plane continues to fly at the same height and velocity, what is its angular momentum about the same origin 10 seconds later?

South  North

West is into screen
East is out of screen

$$\vec{L}_o = \vec{r} \times \vec{p} = \vec{r} \times (m\vec{v})$$

$\vec{r}$ and $\vec{v}$ are perpendicular

$$|\vec{L}_o| = |\vec{r}|\,|m\vec{v}| \sin 90^\circ$$
$$= (300\,m)(5000\,kg)(150\,m/s)$$
$$= \boxed{225 \times 10^6 \ kg \cdot m^2/s}$$

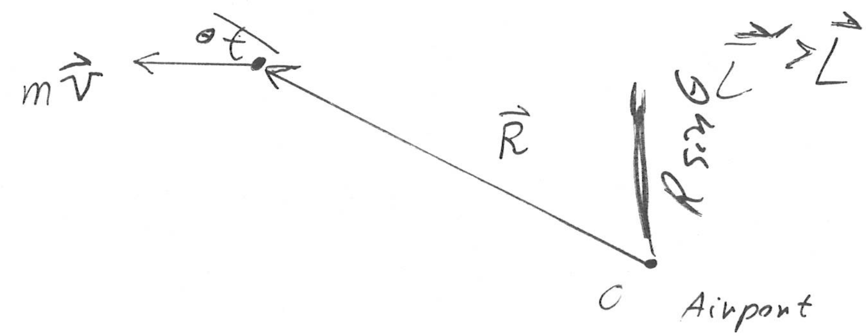
Direction?

Out of screen $\Rightarrow$ $\boxed{\text{East}}$

Part 2:

The plane continues to fly for 10 seconds.

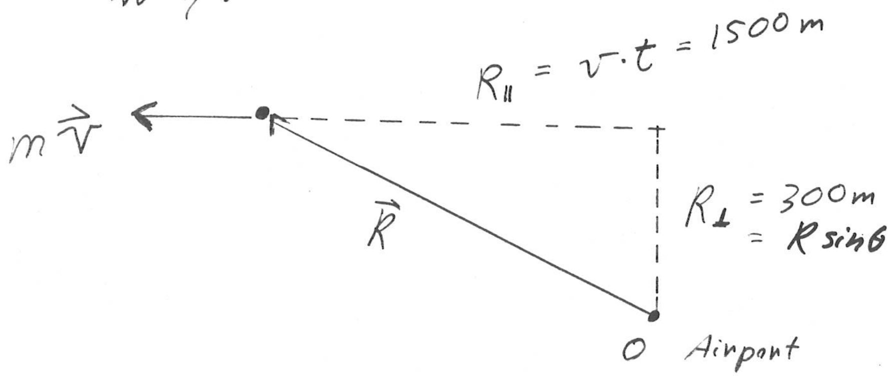
What is $\vec{L}_o$ now?



$$|\vec{L}_o'| = |\vec{R} \times m\vec{v}|$$
$$= |\vec{R}|\,|m\vec{v}| \sin\theta$$

$\vec{L}_o$ is exactly the same!

Why?



$\vec{L}_o = \vec{R} \times \vec{p}$

Only perpendicular components matter.

$\vec{R} =$ 1500 m south + 300 m up

$\vec{p} = mv$ south $= 750 \times 10^3$ kg·m/s south

south × south = ? 0

up × south = ? East

$\vec{L}_o = 225 \times 10^6$ kg·m²/s East

This makes sense...

$\vec{p} = m\vec{v}$ is a constant,

$\Downarrow$

so there is no net external force acting on the plane, $\vec{a} = 0$

$\Downarrow$

so there is no net external torque acting on the plane,

$\Downarrow$

so the plane's angular momentum about the airport must be a constant.

Ex If I sit on a frictionless chair at rest, holding a spinning bicycle wheel and flip the wheel 180°, what will happen? The wheel is initially spinning ccw as seen from above.

a) Nothing

b) Spin clockwise

c) Spin counterclockwise

d) Get dizzy and fall off chair

why? $\vec{L}_i$ is up.

No external torques. $\Rightarrow \vec{L}_i = \vec{L}_f$

Correct Ans = Spin counterclockwise

Why?

Angular momentum is conserved,

$$\vec{L}_i = \vec{L}_f \quad \text{as a vector.}$$

$$\vec{L}_i = \underset{\text{wheel}}{\vec{L}_i} + \underset{\text{me}}{\vec{L}_i}$$

↑ up   0

After the flip

$$\vec{L}_f = \underset{\text{wheel}}{\vec{L}_f} + \underset{\text{me}}{\vec{L}_f}$$

↓ down   ↑ up $\Rightarrow$ ccw

Compare $|\vec{L}_{f\,me}|$ to $|\vec{L}_{i\,wheel}|$.

Remember that we can add vectors graphically.

<u>Before flip</u> $\quad \vec{L}_i = \uparrow \vec{L}_{i\,wheel}$

<u>After flip</u>

$$\vec{L}_f = \downarrow \vec{L}_{f\,wheel} + \vec{L}_{f\,me}$$

$$\underset{\vec{L}_{i\,wheel}}{\uparrow} = \underset{\vec{L}_{f\,wheel}}{\downarrow} + \underset{\vec{L}_{f\,me}}{\Big\uparrow}$$

$$|\vec{L}_{f\,me}| = 2|\vec{L}_{i\,wheel}|$$

We know

$$\tau = I\alpha \qquad (F = ma)$$

A torque provides an angular acceleration which changes the angular velocity. This can change the angular momentum.

<u>But</u>

$\vec{\tau}$, $\vec{L}$, $\vec{\alpha}$, $\vec{\omega}$, $d\vec{\theta}$

are all vector quantities. Direction is important!

$$\vec{L} = I\vec{\omega} \qquad (\vec{p} = m\vec{v})$$

$$\vec{L} = \vec{r} \times \vec{p}$$

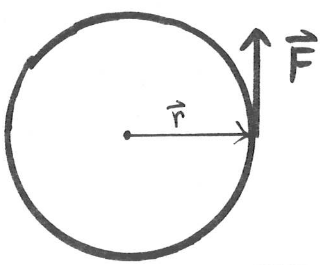
Torque is a vector:

$$\vec{\tau} = I\vec{\alpha} \qquad (\vec{F} = m\vec{a})$$

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

new

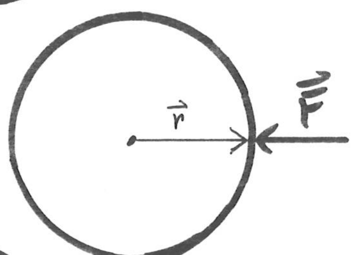
The Cross Product again!



$\theta = 90°$

$|\vec{\tau}| = |\vec{r}|\,|\vec{F}| \sin\theta$

Direction? out of screen

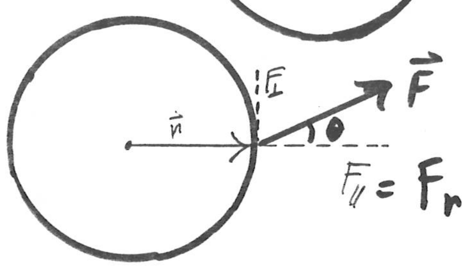


$\theta = 180°$

$\sin 180° = 0$

$|\vec{\tau}| = 0$

why?



$$\boxed{F_t = F\sin\theta}$$

$$|\vec{\tau}| = |\vec{r}|\,F_\perp = |\vec{r}|\,F_t$$

Let's find a relation between angular momentum $\vec{L}$ and torque $\vec{\tau}$.

Start with: $\vec{\tau} = \vec{r} \times \vec{F}$

and substitute: $\vec{F} = \frac{d\vec{p}}{dt}$ Newton $= m\vec{a}$

$$\boxed{\vec{\tau} = \vec{r} \times \frac{d\vec{p}}{dt}}$$

Remember this while we explore angular momentum.

$$\vec{L} = \vec{r} \times \vec{p}$$

Take the time derivative of both sides.

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

consider just this term

$$\frac{d\vec{r}}{dt} = \vec{v} \quad \text{and} \quad \vec{p} = m\vec{v}$$

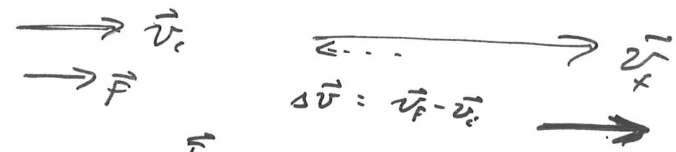
this gives $\vec{v} \times m\vec{v} = 0$

so $\boxed{\dfrac{d\vec{L}}{dt} = \vec{r} \times \dfrac{d\vec{p}}{dt}}$

$$\boxed{\vec{\tau}_{net} = \frac{d\vec{L}}{dt}}$$

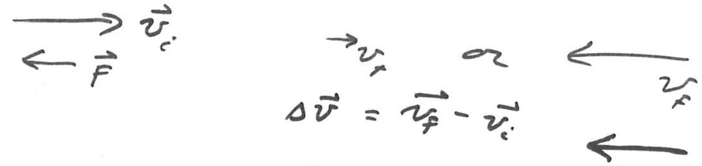
Analog of $\vec{F}_{net} = \dfrac{d\vec{p}}{dt}$

Linear Analogs

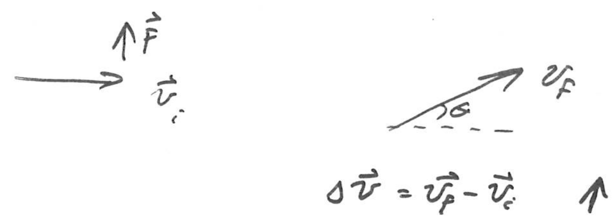


$\Delta\vec{v} = \vec{v}_f - \vec{v}_i$

$\vec{F}$ and $\Delta\vec{v}$ same direction



$\Delta\vec{v} = \vec{v}_f - \vec{v}_i$



diff. direc
same speed

$\Delta\vec{v} = \vec{v}_f - \vec{v}_i$

$\vec{F}$ and $\Delta\vec{v}$ same direction

Simple notation:

$\vec{\tau}$ and $\vec{L}$ are in the same direction. E.g. bicycle wheel speeding up

But the equation

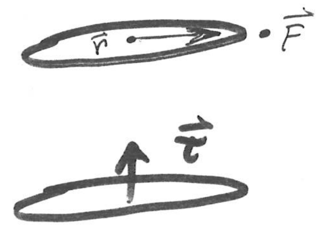
$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

does **not** say that $\vec{\tau}$ and $\vec{L}$ are in the same direction. It says that torque and the **change in** angular momentum are in the same direction.

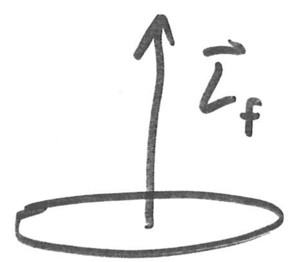
$\vec{\tau}$ in same direct $\Delta\vec{L}$

Bicycle wheel speeding up:

Now apply a torque



The force $\vec{F}$ points <u>into</u> the screen



$\Delta\vec{L} = \vec{L}_f - \vec{L}_i$ points up

$\uparrow \Delta\vec{L}$

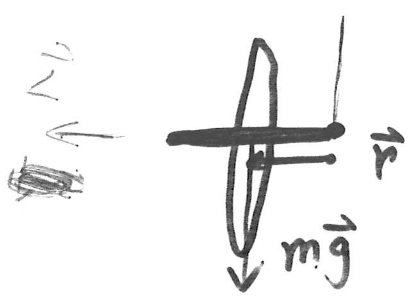
in the same direction as the torque.

Suppose $\vec{\tau}$ and $\vec{L}$ are <u>not</u> in the same direction.

Demo

$\vec{L}$ changes, not in <u>magnitude</u> but in <u>direction</u>. There must be a torque acting. What is the source of the torque?

Gravity

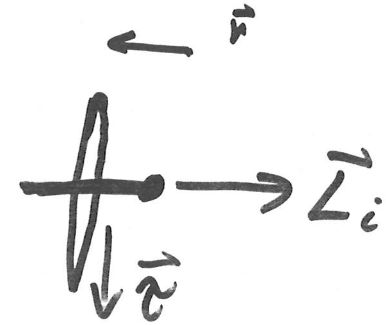


Direction?

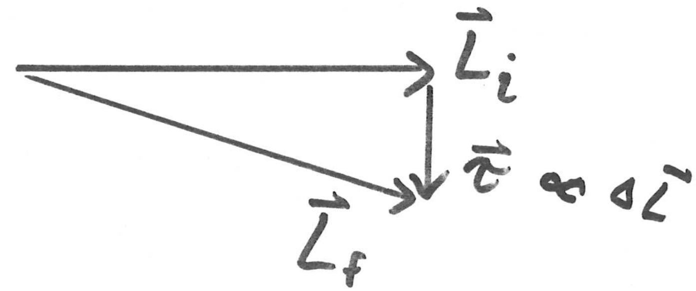
$\vec{\tau}$ Out

What the heck is going on here

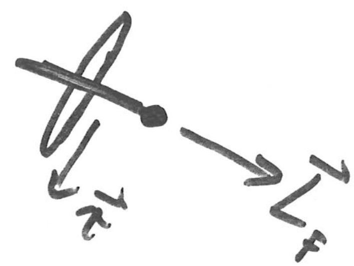
top view



the force $m\vec{g}$ is <u>into</u> the screen



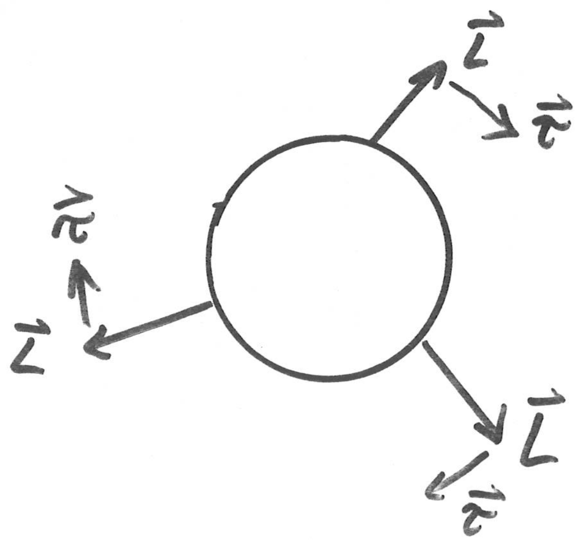
torque is <u>always</u> perpendicular to the angular momentum in this case



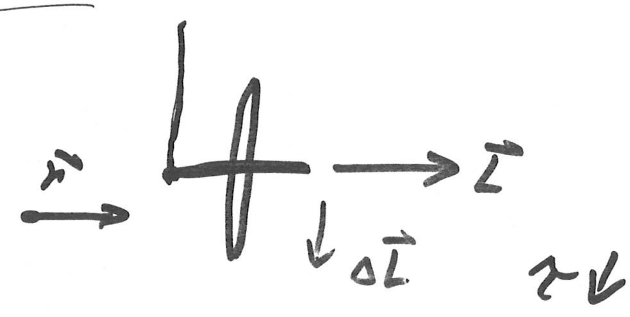
Result:

Precession

The angular momentum vector $\vec{L}$ rotates in a circle.



Why can't the spinning wheel fall down?



To fall down, $\Delta\vec{L}$ must be down. Since $\vec{\tau} = \frac{d\vec{L}}{dt}$, the torque must be down. What is the direction of the force if the torque is down?

$\vec{F}$ must be out of the screen. There is no force in this direction