Scaling

Ex. Consider a sphere

$$V(r) = \underbrace{\tfrac{4}{3}\pi}\, r^3 = k_1 r^3$$

$$A(r) = \underbrace{4\pi}\, r^2 = k_2 r^2$$

If the volume of a sphere increases by a factor of 64, by what factor does the surface area change?

$$V' = 64\, V$$

$$\cancel{k_1}(r')^3 = 64\, \cancel{k_1}(r)^3$$

$$r' = 4\, r$$

$$k_2(r')^2 = 16\, k_2 (r)^2$$

$$\boxed{A' = 16\, A}$$

Chapter 3: Vectors

What is a vector?

A quantity with both a magnitude and a direction.

Ex. displacement, velocity, acceleration, momentum, force, ...

Vectors can be 1, 2, or 3 dimensional.

Not every physical quantity is a vector.

A scalar is a quantity with a magnitude only, and no direction.

Ex. mass, time, temperature, work, energy, ... density

Scalars are inherently 1 dimensional.

$\pm$ scalars can be positive or negative

$\pm$ one dimensional vectors can be positive or negative

?

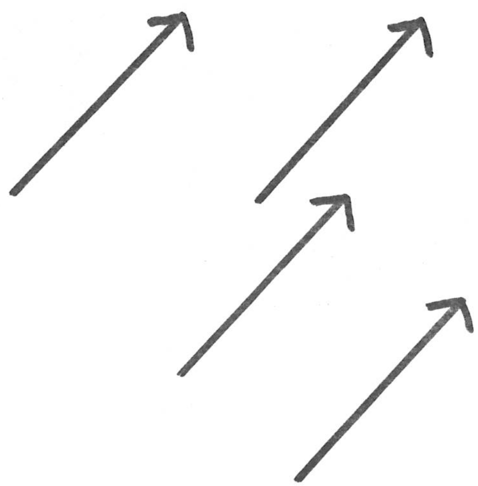
Another distinction between vectors and scalars is that

Vectors change with a different choice of coordinate axes.

Scalars do not change.

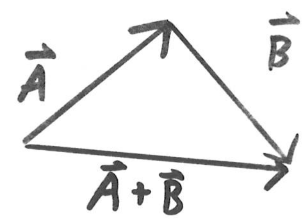
Graphical Representation of a Vector (2-D)

All of these arrows represent the <u>same</u> vector

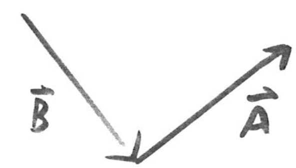


Vector Addition

What is $\vec{A}+\vec{B}$?

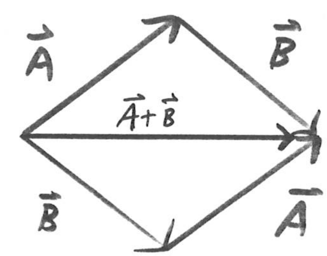


What is $\vec{B}+\vec{A}$?



Vector Addition is <u>Commutative</u>

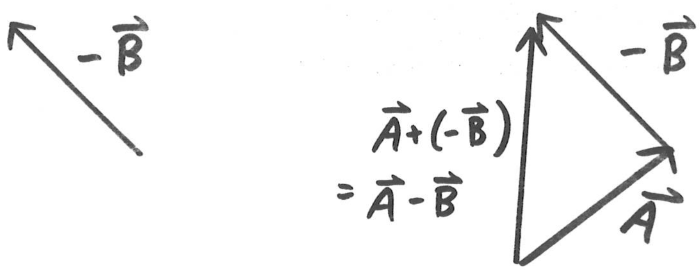
$\vec{A}+\vec{B} = \vec{B}+\vec{A}$ (Parallelogram Law)



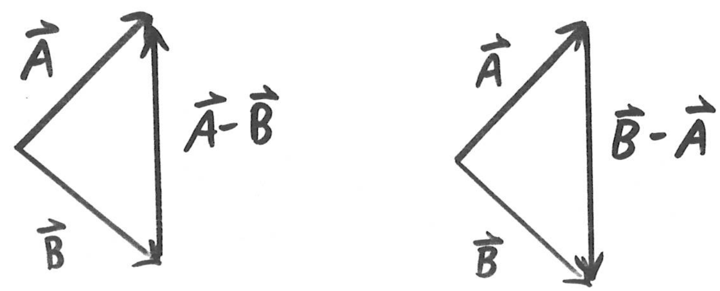
Negation of a Vector

Vector Subtraction

What is $\vec{A}-\vec{B}$? $\vec{A}+(-\vec{B})$



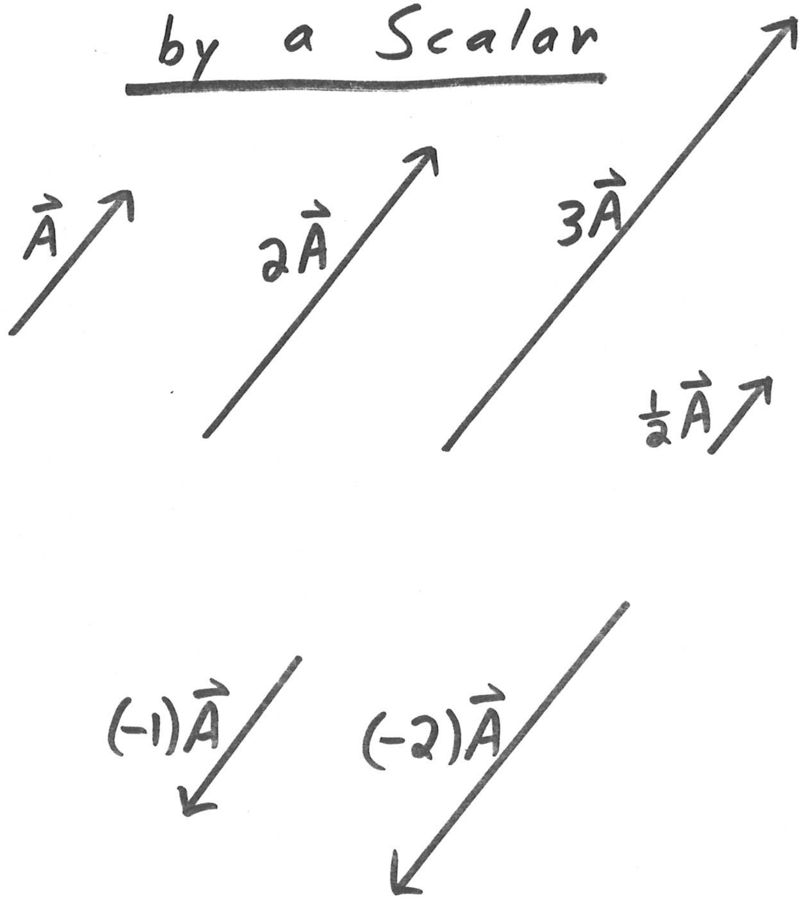
Another Method of Subtraction



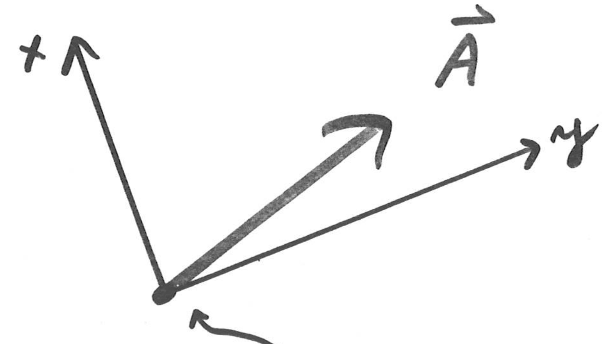
$$\vec{A}-\vec{B} = -(\vec{B}-\vec{A})$$

Vector subtraction is <u>not</u> commutative.

Multiplication by a Scalar



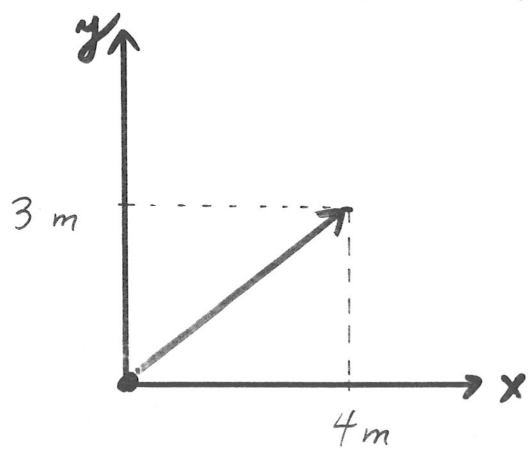
Numerical Representation of a Vector (2-D)



Choose an origin of coordinates

and directions for the coordinate axes.

Another choice

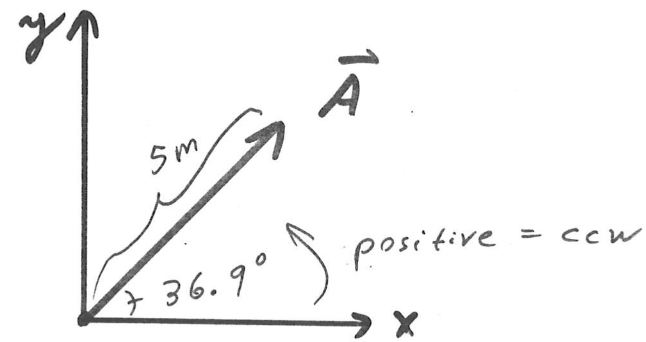


Cartesian Coordinates of the vector $\vec{A}$ form the ordered pair (x, y).

$$\vec{A} = (4\,m, 3\,m)$$

for <u>this</u> choice of axes.

For the same choice of axes, another pair of numbers also describes the vector $\vec{A}$.



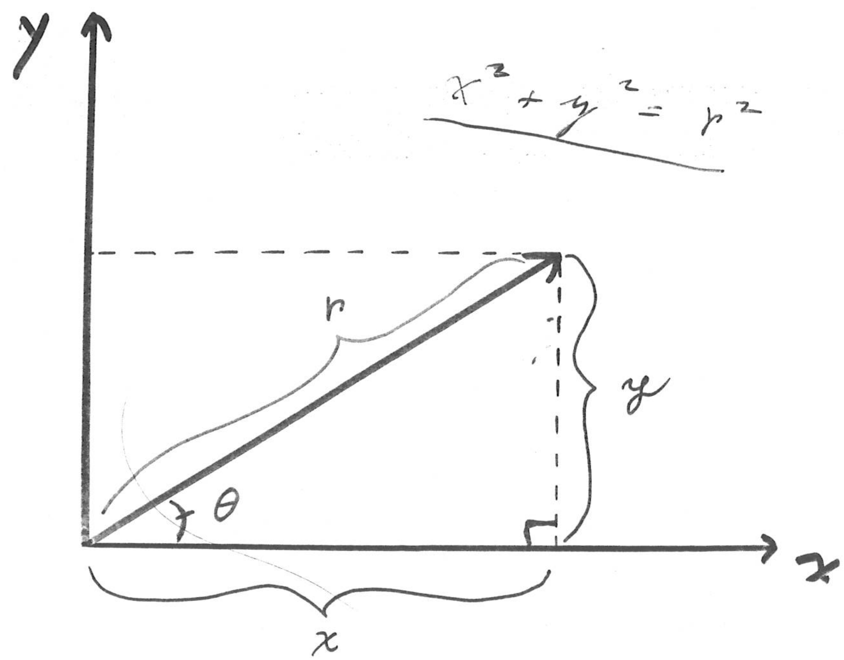
The polar coordinates of the vector $\vec{A}$ form the ordered pair (r, θ)

$$\vec{A} = (\underbrace{5\,m}_{r},\ 36.9°)$$

r is the <u>length</u> of the vector $\vec{A}$.

θ is the <u>angle</u> between the vector and the positive x-axis.

How are these two different representations of vectors related?



Trigonometry:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}$$

Translation Dictionary

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$

$$y = r\sin\theta$$

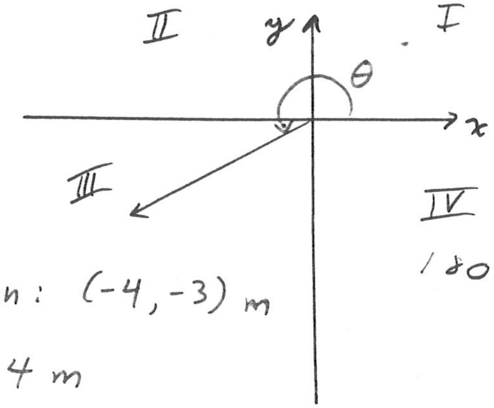
$$\boxed{\begin{aligned}\theta &= \arctan\left(\frac{y}{x}\right)\\ &= \tan^{-1}\left(\frac{y}{x}\right)\end{aligned}}$$

Warning

(Another stupid calculator trick!)

Your calculator may return a value for θ which is always between $+90°$ and $-90°$. You may need to add or subtract $180°$ to get to the correct quadrant.

Ex.



$180° < \theta < 270°$

Cartesian: $(-4, -3)$ m

$x = -4$ m

$y = -3$ m

$$r = \sqrt{x^2 + y^2} = \sqrt{(-4m)^2 + (-3m)^2} = \boxed{5\,m}$$

$$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{-3}{-4}\right)$$

Calculator $\Rightarrow$ $\theta = 36.9°$
in the first quadrant

but we know from the diagram that θ is in the third quadrant.

$$\theta = 36.9° + 180° = \boxed{216.9°}$$

The Cartesian coordinates

$(-4, -3)$ m

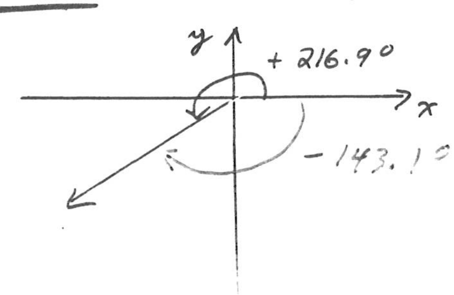
represent the same vector as the polar coordinates

$(5m, 216.9°)$

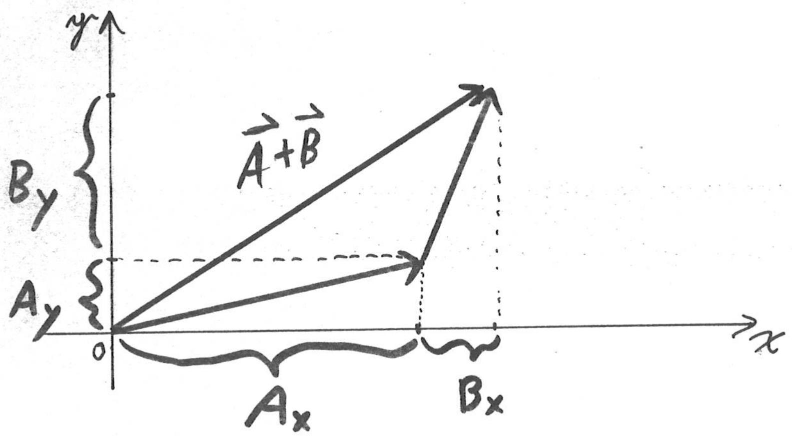
where positive θ means measured counter-clockwise from the positive x-axis.

What about $\theta = 36.9° - 180° = -143.1°$

Negative θ means measured clockwise from the positive x-axis.



Adding vectors in Cartesian form

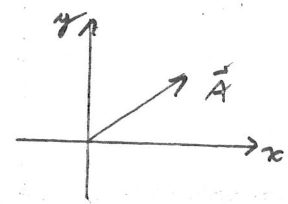


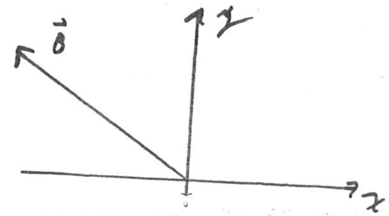
$$(A_x, A_y) + (B_x, B_y) = (A_x + B_x\, ,\ A_y + B_y)$$

Adding vectors in Polar form

Convert to Cartesian form where it's **EASY!** Then convert back to polar form.

Ex $\vec{A} = (1\text{cm}, 45°)$ $\quad \vec{B} = (2\text{cm}, 135°)$

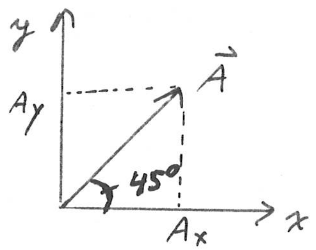


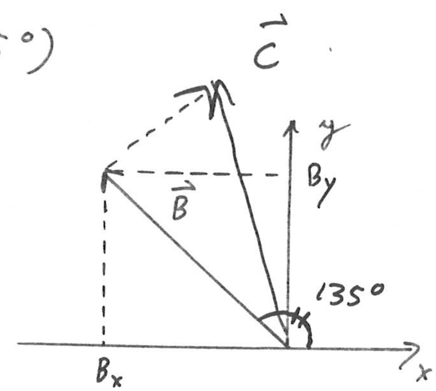


What is $\vec{A} + \vec{B}$?

Ex. $\vec{A} = (1\text{ cm}, 45°)$

$\vec{B} = (2\text{ cm}, 135°)$





What is $\vec{A} + \vec{B} = \vec{C}$?

$A_x = 1\text{ cm} \cos 45° = 0.71\text{ cm}$

$A_y = 1\text{ cm} \sin 45° = 0.71\text{ cm}$

$B_x = 2\text{ cm} \cos 135° = -1.4\text{ cm}$

$B_y = 2\text{ cm} \sin 135° = +1.4\text{ cm}$

$C_x = A_x + B_x = -0.71\text{ cm}$

$C_y = A_y + B_y = +2.1\text{ cm}$

$r = \sqrt{C_x^2 + C_y^2} = 2.2\text{ cm}$

$\theta = \tan^{-1}\left(\frac{C_y}{C_x}\right) = 108° = \arctan\left(\frac{2.1}{-.71}\right)$

Multiplication by a (positive) Scalar

Cartesian: $\vec{A} = (A_x, A_y)$

$2\vec{A} = (2A_x, 2A_y)$

Ex. $4\frac{m}{s}(1\text{ m}, 2\text{ m}) = (4\frac{m^2}{s}, 8\frac{m^2}{s})$

Polar: $\vec{A} = (r, \theta)$

$2\vec{A} = (2r, \theta)$

same angle!

twice as long

Ex. $3(2\text{ miles}, 130°) = (6\text{ miles}, 130°)$

Negation

Cartesian: $\vec{A} = (A_x, A_y)$

$$-\vec{A} = (-A_x, -A_y)$$

Ex. $-(3\text{ ft}, 4\text{ ft}) = (-3\text{ ft}, -4\text{ ft})$

Polar: $\vec{A} = (r, \theta)$

$$-\vec{A} = (r, \theta \pm 180°)$$

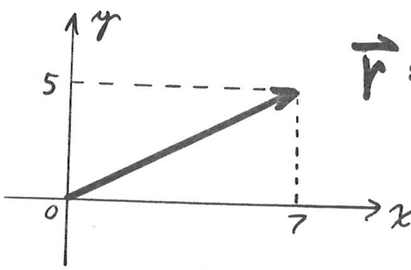
same length, opposite direction

Ex. $-(5\text{ m}, 20°) = (5\text{m}, 200°)$

$= (5\text{m}, -160°)$

Unit Vectors

Another way of representing vectors in cartesian form.

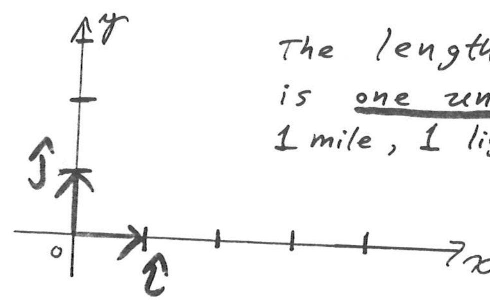


$\vec{r} = (7, 5)\text{ m}$

$$\vec{r} = 7\hat{\imath}\,\text{m} + 5\hat{\jmath}\,\text{m}$$

$\hat{\imath}$ is a unit vector in the x direction

$\hat{\jmath}$ is a unit vector in the y direction



The length of $\hat{\imath}$ or $\hat{\jmath}$ is <u>one unit</u> (1 meter, 1 mile, 1 lightyear...)

Unit vectors keep track of the directions for us.

$\vec{r}_A = (200, -150)$ miles

$= [200\,\hat{\imath} - 150\,\hat{\jmath}]$ miles

$= 200\,\hat{\imath}$ miles $- 150\,\hat{\jmath}$ miles

$\vec{r}_B = (-700, 200)$ miles

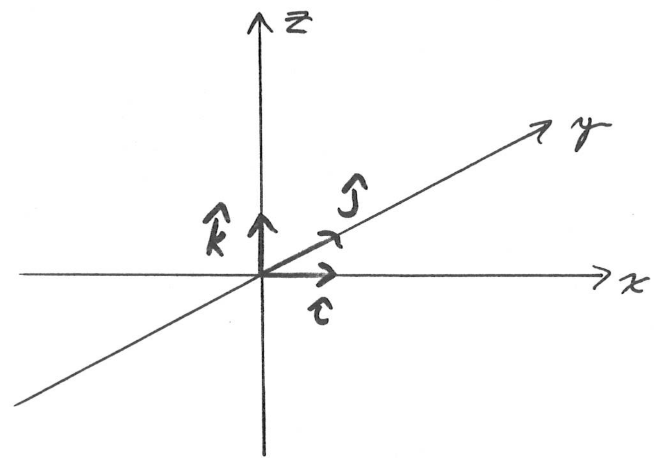
$= [-700\,\hat{\imath} + 200\,\hat{\jmath}]$ miles

$\vec{r}_B - \vec{r}_A = -700\,\hat{\imath}$ miles $+ 200\,\hat{\jmath}$ miles $- [200\,\hat{\imath}$ miles $- 150\,\hat{\jmath}$ miles$]$

$= -900\,\hat{\imath}$ miles $+ 350\,\hat{\jmath}$ miles

Usual rules of algebra apply.

Three Dimensions



$\vec{r} = (-10, 6, 4)$

$= -10\,\hat{\imath} + 6\,\hat{\jmath} + 4\,\hat{k}$

10 units in the negative x direction
6 units along the y direction
4 units along the z axis

4 Ways of Representing Vectors

Graphically 

Cartesian Coordinates (a, b)

- b └ y component
- a └ x component

Polar Coordinates $\left(\sqrt{a^2+b^2},\ \tan^{-1}(b/a)\right)$

- $\tan^{-1}(b/a)$ └ angle θ
- $\sqrt{a^2+b^2}$ └ length r

Unit Vectors $a\hat{\imath} + b\hat{\jmath}$

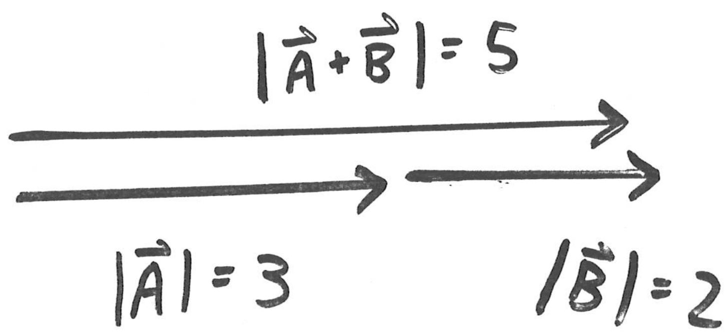
Magnitude of a Vector

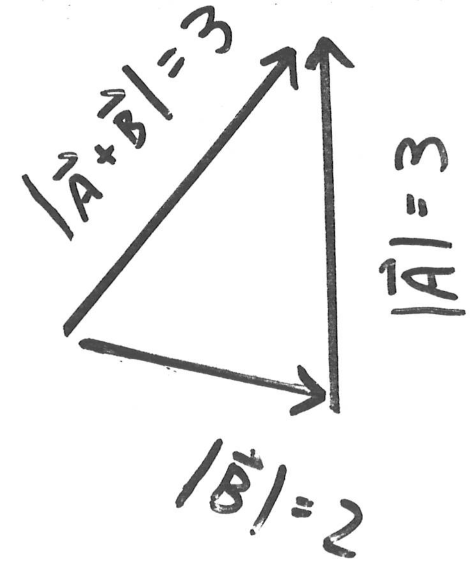
The magnitude of a vector is the positive number representing its length.

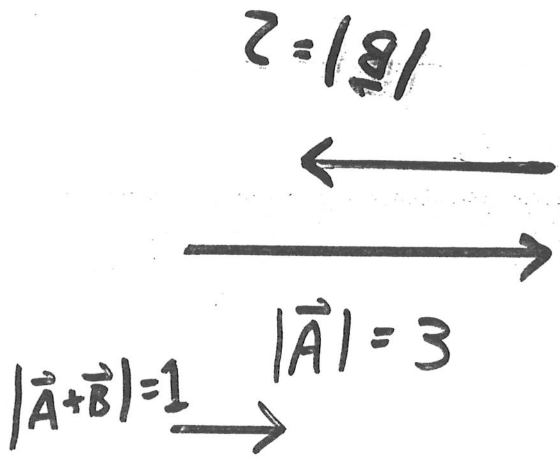
1-D: $|\vec{A}| = |A_x|$ Absolute Value

2-D: $|\vec{A}| = \sqrt{(A_x)^2+(A_y)^2} = r$ polar coord.

3-D: $|\vec{A}| = \sqrt{(A_x)^2+(A_y)^2+(A_z)^2}$


$|\vec{A}+\vec{B}| = 5$
$|\vec{A}| = 3$
$|\vec{B}| = 2$

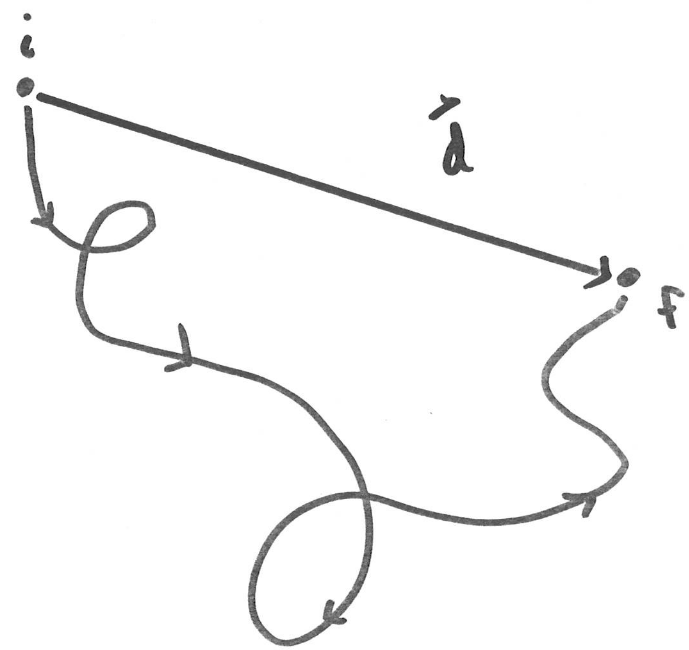

$|\vec{A}+\vec{B}| = 3$
$|\vec{A}| = 3$
$|\vec{B}| = 2$

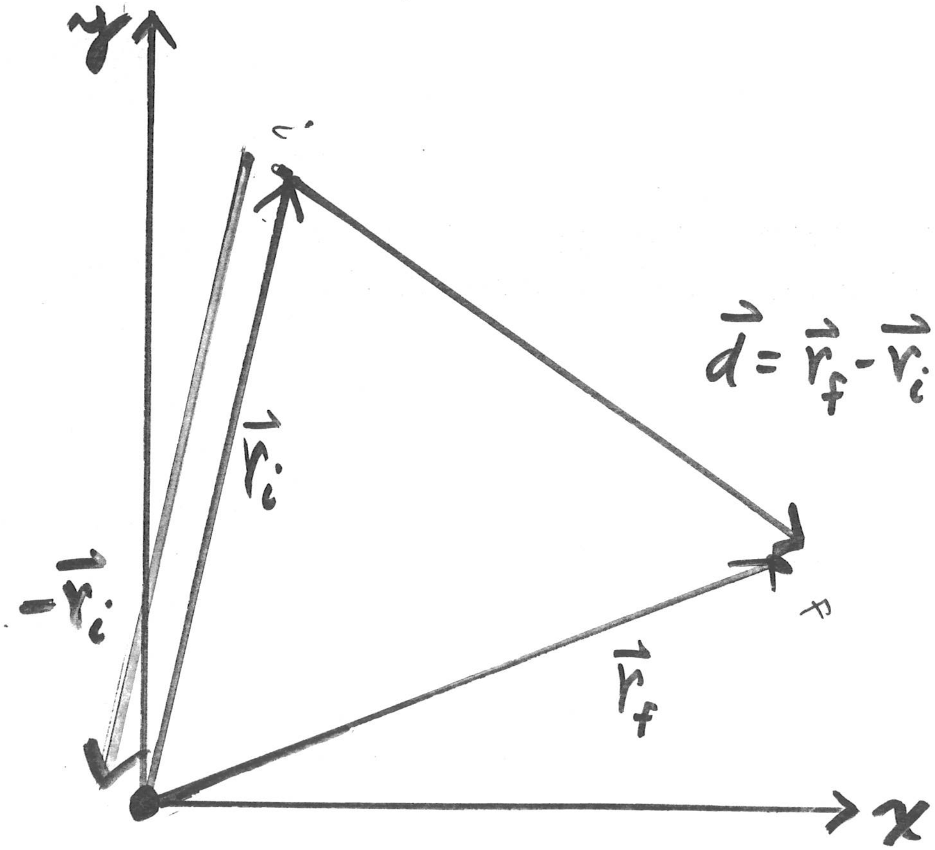


$$\left| |\vec{A}| - |\vec{B}| \right| \leq |\vec{A} + \vec{B}| \leq |\vec{A}| + |\vec{B}|$$

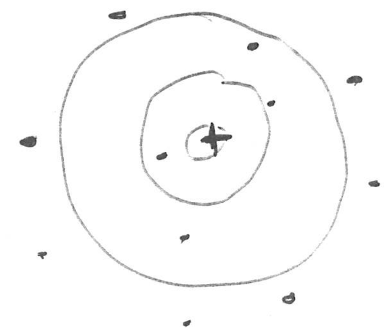
Displacement Vector

Describes the change in position from an initial position to a final position, independent of the path.



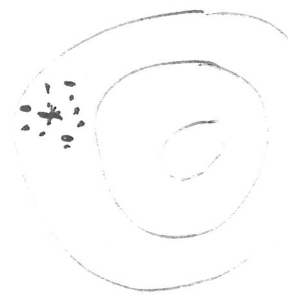


Accuracy



but not very precise

Precision



but not very accurate

Precise and accurate