Chapter 4: 2-D Kinematics

$$\Delta \vec{r} = \vec{r}_f - \vec{r}_i$$ displacement

$$\vec{v}_{AVG} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i}$$ Average velocity

$$\vec{v} = \frac{d\vec{r}}{dt}$$ Instantaneous velocity

$$\vec{a}_{AVG} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$ Average accel.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2}{dt^2}\vec{r}$$ Instantaneous acceleration

Each of these is **TWO** equations, one for x and one for y.

Vector Master Equation

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \tfrac{1}{2}\vec{a} t^2$$

this is a shorthand notation for:

$$\begin{cases} x(t) = x_0 + v_{0,x} t + \tfrac{1}{2} a_x t^2 \\ y(t) = y_0 + v_{0,y} t + \tfrac{1}{2} a_y t^2 \end{cases}$$

two independent master equations.

★ Remember: These only hold for constant vector acceleration!

$\vec{a} = \overrightarrow{\text{constant}}$ means $\begin{cases} a_x = \text{constant} \\ a_y = \text{constant} \end{cases}$

x and y Motions are INDEPENDENT

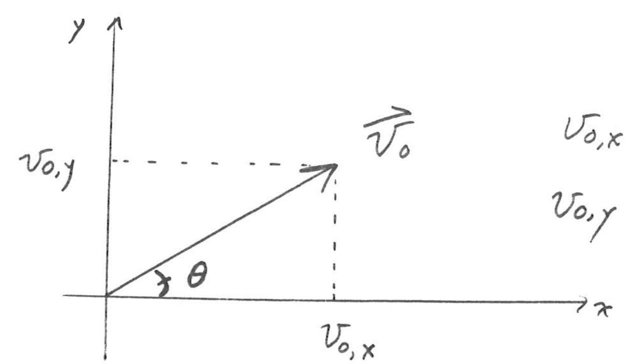
Demo: Consider two projectiles, both at height $y_0 = 2m$ and $v_{0,y} = 0$ (no initial vertical velocity) but one has $v_{0,x} = 0$ while the other has $v_{0,x} = 5$ m/s.

How long will each take to hit the floor?

Projectile Motion

The 2-D path of any object thrown near the Earth's surface is a parabola.

In general, an object's initial velocity can have an x- and a y-component.



$v_{0,x} = |\vec{v}_0| \cos\theta$

$v_{0,y} = |\vec{v}_0| \sin\theta$

Why is this a parabola?

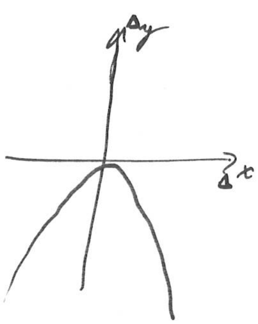
$$\Delta x = v_{ox}\, t + \tfrac{1}{2} \cancel{a_x}^{\,0} t^2$$

$$\Delta y = v_{oy}\, t + \tfrac{1}{2} a_y t^2$$

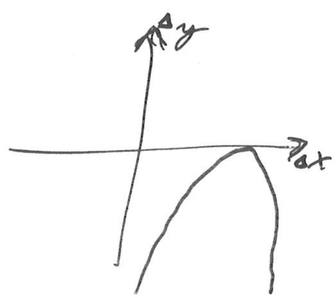
$$t = \frac{\Delta x}{v_{ox}}$$ substitute into y master equation

$$\Delta y = v_{oy}\left(\frac{\Delta x}{v_{ox}}\right) + \tfrac{1}{2}(-g)\left(\frac{\Delta x}{v_{ox}}\right)^2$$

$$\Delta y = A\,\Delta x + B\,(\Delta x)^2$$



$A = 0$



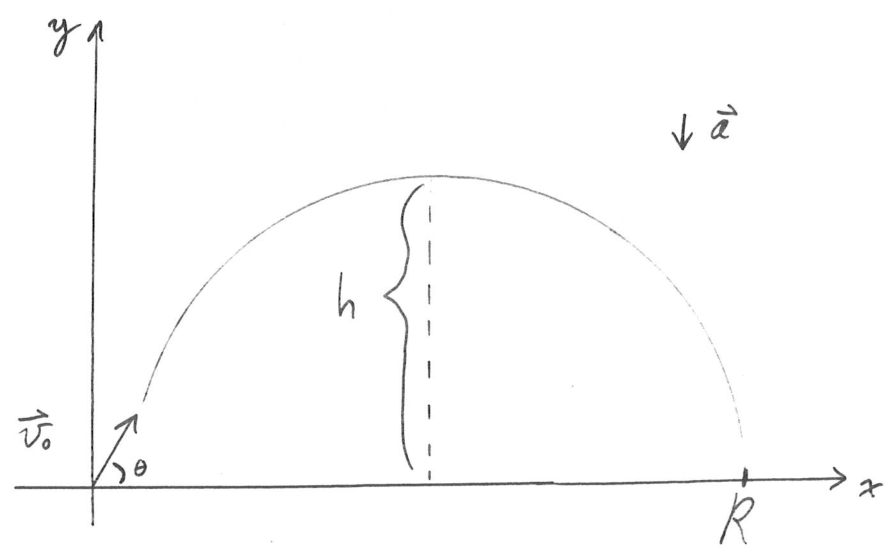
$A \neq 0$

Range and Max. Height

A projectile is fired with initial speed v_0 at an angle θ above the horizontal across a level surface.

How far does it go (range)?

How high does it go (maximum height)?

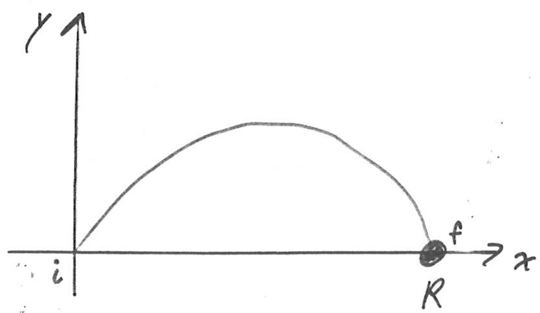


$x_0 = 0$ $\quad y_0 = 0$

$v_{0,x} = v_0 \cos\theta$ $\quad v_{0,y} = v_0 \sin\theta$

$a_x = 0$ $\quad a_y = -g$

Range



$x_f = R$ $\quad y_f = 0$

Vertical Master Equation:

$$y_f = y_0 + v_{0,y}\, t + \frac{1}{2} a_y t^2$$

$$0 = 0 + v_0 \sin\theta\, t + \frac{1}{2}(-g) t^2$$

$$t = 0 \quad \underline{\text{or}} \quad \boxed{\frac{2 v_0 \sin\theta}{g}}$$

Horizontal Master Equation:

$$x_f = x_0 + v_{0,x}\, t + \frac{1}{2} a_x t^2$$

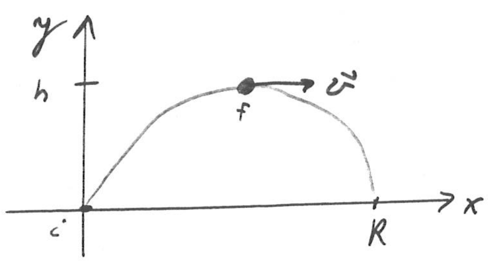
$$R = 0 + v_0 \cos\theta\, t + 0$$

$$R = v_0 \cos\theta \left(\frac{2 v_0 \sin\theta}{g}\right) = \boxed{\frac{2 v_0^2 \cos\theta \sin\theta}{g}}$$

Book: Trigonometric identity: $2\cos\theta \sin\theta = \sin(2\theta$

$$R = \frac{v_0^2 \sin(2\theta)}{g}$$

Maximum Height



$y_f = h$

$v_{y,f} = 0$

$v_{y,f} = v_{0,y} + a_y t$

$0 = v_0 \sin\theta + (-g)t$

$t = \dfrac{v_0 \sin\theta}{g}$ (half range time)

$y_f = y_0 + v_{0y} t + \frac{1}{2} a_y t^2$

$h = 0 + v_0 \sin\theta\, t + \frac{1}{2}(-g)t^2$

$h = v_0 \sin\theta \left(\dfrac{v_0 \sin\theta}{g}\right) - \frac{1}{2} g \left(\dfrac{v_0 \sin\theta}{g}\right)^2$

$$\boxed{h = \frac{v_0^2 \sin^2\theta}{2g}}$$

For what initial angle θ is the range a maximum?

$R = \dfrac{v_0^2 \sin 2\theta}{g}$ $\theta = 45°$

$R_{max} = \dfrac{v_0^2}{g}$ at $45°$ $\sin(90°) = 1$

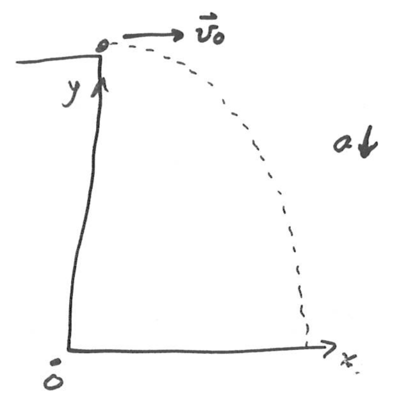
For what initial angle θ is the maximam height greatest?

$h = \dfrac{v_0^2 \sin^2\theta}{2g}$ 90

$h_{max} = \dfrac{v_0^2}{2g}$

Ex Standing on the roof of Fondren Science (height = 15 m), I toss a rock with an initial velocity 4 m/s <u>horizontally</u>.

How far from the base of the building will the rock land?



- Picture
- choose an origin
- choose positive $\vec{x}$
  positive $\vec{y}$

Need to solve <u>two</u> problems simultaneously:

- Find the <u>time</u> it takes the rock to hit.
- Find the <u>horizontal distance</u> the rock travels in this time.

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

$$\vec{r}_0 = (\qquad \hat{\imath} + \qquad \hat{\jmath})\,\text{m}$$

$$\vec{v}_0 = (\qquad \hat{\imath} + \qquad \hat{\jmath})\,\text{m/s}$$

$$\vec{a} = (\qquad \hat{\imath} + \qquad \hat{\jmath})\,\text{m/s}^2$$

We <u>could</u> continue to solve the problem in vector notation, but it is easier to consider x and y components separately.

$\hat{\imath}$ terms

$$x(t) = v_{0x}\, t = (4\ \text{m/s})\, t \qquad (a_x = 0)$$

$\hat{\jmath}$ terms

$$y(t) = y_0 + \cancel{v_{0y} t}^{\,0} + \tfrac{1}{2} a_y t^2$$

$$0 = 15\text{m} + \tfrac{1}{2}(-9.8\ \text{m/s}^2)\, t^2$$

How long until it hits the ground?

$$y(t) = 0 \quad \Rightarrow \quad 0 = 15\text{m} - \tfrac{1}{2}(9.8\ \text{m/s}^2)\, t^2$$

$$t = \sqrt{\frac{2(15\text{m})}{9.8\ \text{m/s}^2}} = \pm 1.7\text{s} \qquad \text{choose } +1.7\text{s}$$

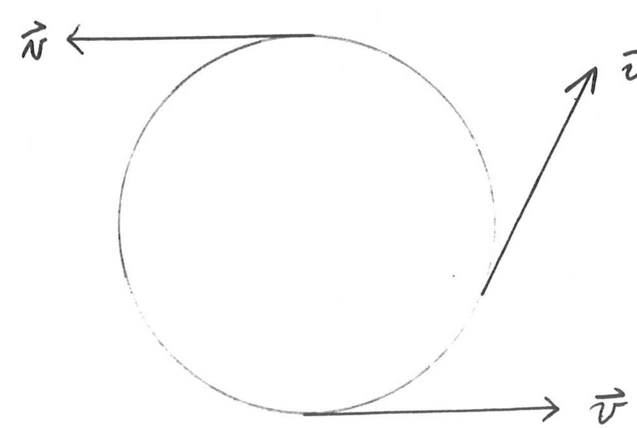
How far does it go horizontally in this time?

$$x(t) = (4\ \text{m/s})\, t = 4(\text{m/s})(1.7\text{s})$$

$$= \boxed{6.8\ \text{m}}$$

Uniform Circular Motion

A particle moves along a circle with constant speed. That is, the magnitude of the velocity vector is constant, but the direction changes.



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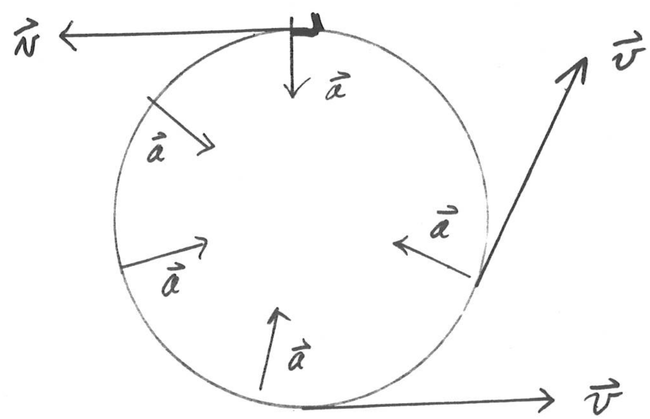


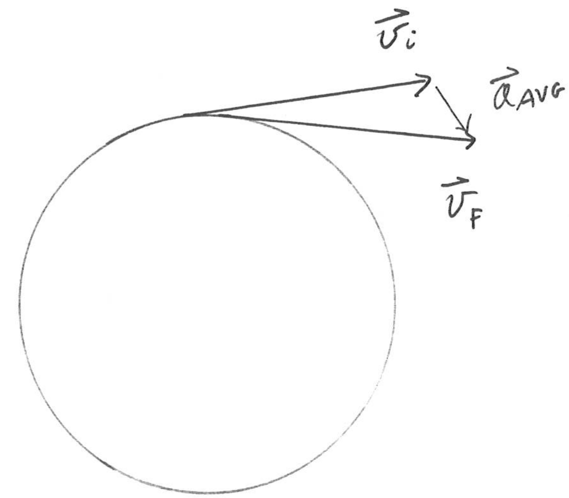
For uniform circular motion, the acceleration vector points in toward the center of the circle.

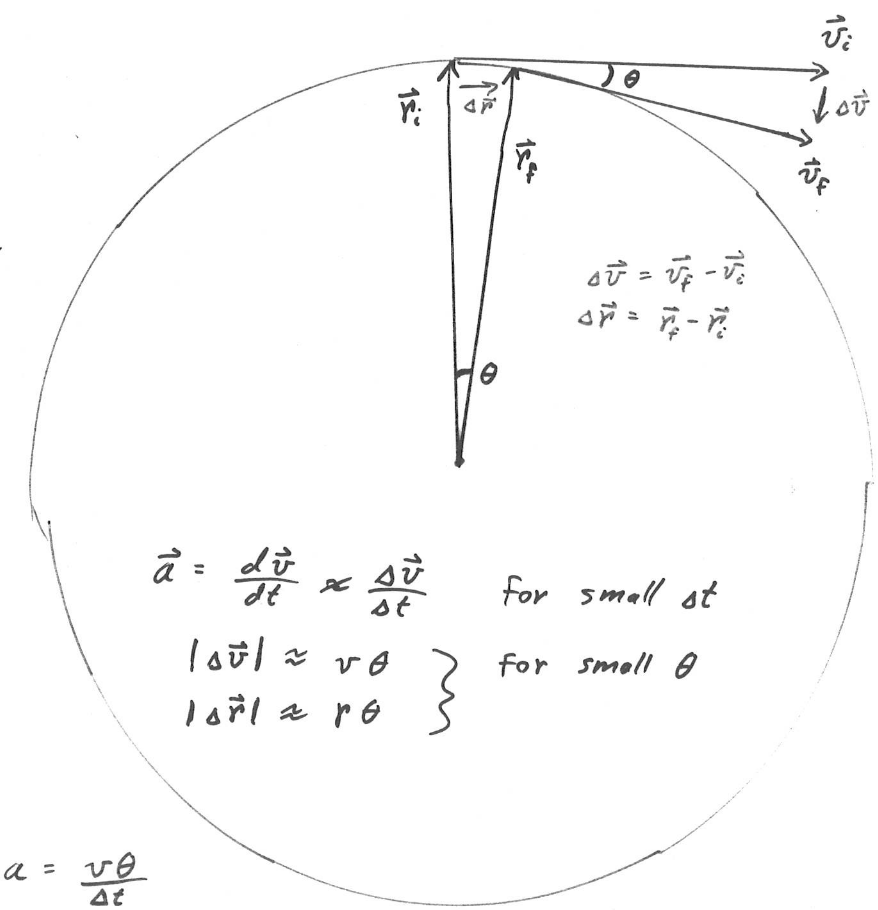
$$\vec{a} = \frac{d\vec{v}}{dt}$$ vector equation

Is the velocity vector a constant? No!

Then there must be an acceleration.

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} \qquad \vec{v}_f - \vec{v}_i = \Delta \vec{v}$$





$\vec{a} = \frac{d\vec{v}}{dt} \approx \frac{\Delta\vec{v}}{\Delta t}$ for small Δt

$$\left.\begin{array}{l} |\Delta\vec{v}| \approx v\theta \\ |\Delta\vec{r}| \approx r\theta \end{array}\right\} \text{ for small } \theta$$

$a = \frac{v\theta}{\Delta t}$

$\theta = \frac{\Delta r}{r}$ $\qquad a = \frac{v}{r}\left(\frac{\Delta r}{\Delta t}\right) = \frac{v^2}{r}$

Ex: What is the centripetal acceleration of a person at the equator due to the Earth's rotation?

$r = \underline{6.37 \times 10^6\ m} \approx 4000$ miles

$v = \frac{2\pi r}{1\ day} = 463\ m/s \approx 1000$ mph

$a_r = \frac{v^2}{r} = 0.034\ m/s^2 = 0.003\ g_{earth}$

Ex: Merry-go-round

$r = 10\ m \approx 30\ ft$

$v = 10\ mph = 4.5\ m/s$

$a_r = \frac{v^2}{r} = 2\ m/s^2 = \frac{1}{5}\ g_{earth}$

$a_{MGR} = 70\ a_{rotation}$

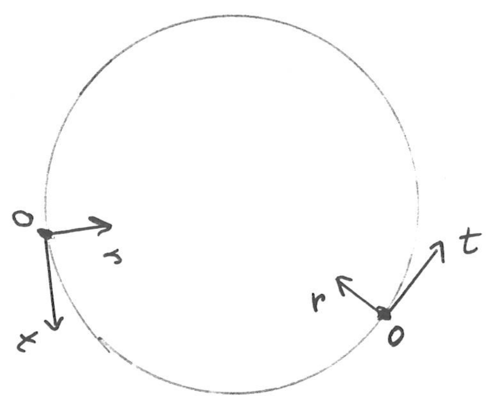
Magnitude

For <u>uniform</u> circular motion, the length of the acceleration vector is

$$|\vec{a}| = \frac{v^2}{r}$$

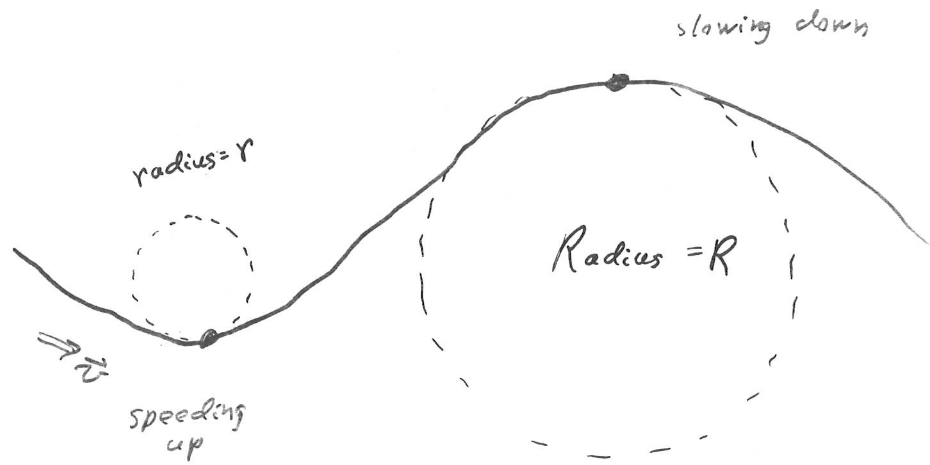
Uniform only!

New Coordinate Axes



r - radial (in)
t - tangential

General Curvilinear Motion



$$a_r = \frac{|\vec{v}_1|^2}{r} \qquad a_r = \frac{|\vec{v}_2|^2}{R}$$

$$a_t = \frac{d|\vec{v}_1|}{dt} \qquad a_t = \frac{d|\vec{v}_2|}{dt}$$

$$|\vec{a}| = \sqrt{a_r^2 + a_t^2}$$

a_r is the instantaneous speed squared, divided by the radius of the circle that approximates the path of the particle.

a_t is the time derivative of the speed (not the velocity).

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a_r is non-zero whenever the path is curved. (direction changes)

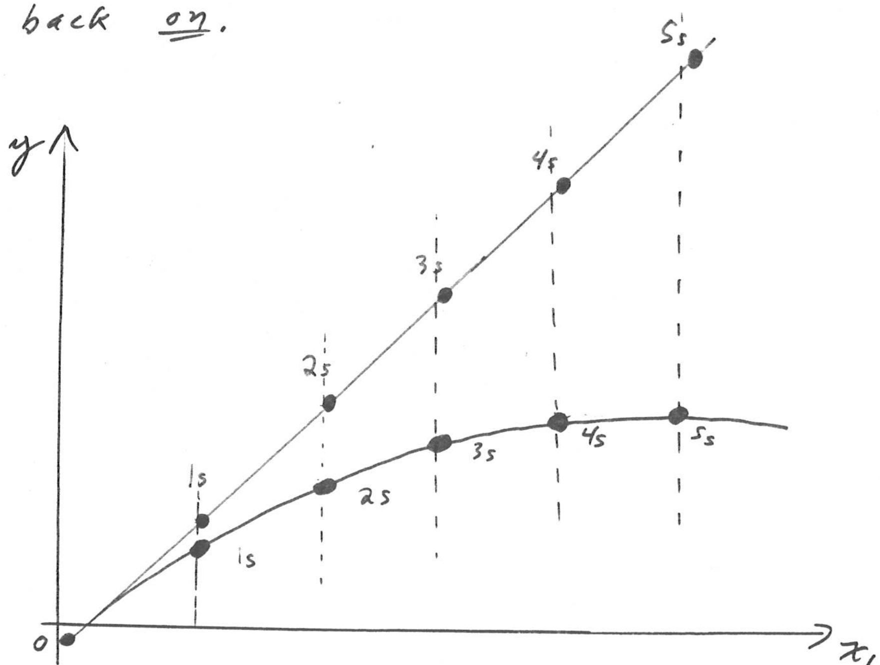
a_t is non-zero whenever the particle is speeding up or slowing down.

$a_t = 0$ for uniform circular motion!

Ex. A train slows down as it rounds a sharp horizontal turn, slowing from 90 km/hr to 50 km/hr in 15 s. The radius of the curve is 150 m. What is the total acceleration when the speed of the train is 60 km/hr?

X and y Independence Again

A particle is fired with initial x-velocity v_{ox} and initial y-velocity v_{oy}. First turn gravity off. Then turn gravity back on.



x-motion is the same in both cases!

Shoot the Monkey

We are on safari and we want to shoot a monkey with a tranquilizer dart. But when the monkey hears the dart gun fire, it is frightened and lets go of the tree branch. Where should I aim?

- Below the monkey?
  - How far below?

- At the monkey?

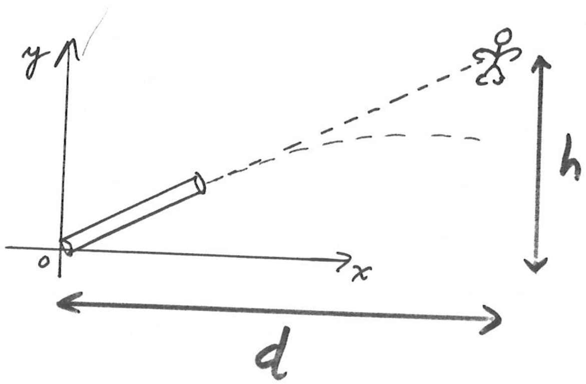
Demo

Why does this work?

Look at the two equations of motion:

Dart $\quad \vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$

Monkey $\quad \vec{m}(t) = \vec{m}_0 \qquad\quad + \frac{1}{2}\vec{a}t^2$



How long does the dart take to reach the monkey?

Horizontal part: $\qquad (a_x = 0)$

$$x(t) = x_0 + v_{0x} t$$

$$d = 0 + v_{0x} t$$

$$\boxed{t = \frac{d}{v_{0x}}}$$

Now the vertical part:

First with gravity off. $\quad (a_y = 0$

What is the y-position of the dart at the time found above?

$$y(t) = y_0 + v_{0y} t + \cancel{\tfrac{1}{2} g_y t^2}$$

$$h = 0 + v_{0y}\left(\frac{d}{v_{0x}}\right)$$

Now turn gravity back on.

What is the y-position of the dart at $t = \frac{d}{v_{0x}}$?

$$y(t) = y_0 + v_{0y} t + \frac{1}{2} a_y t^2$$

$$? = 0 + \underbrace{v_{0y}\left(\frac{d}{v_{0x}}\right)}_{h} + \frac{1}{2}(-g)\left(\frac{d}{v_{0x}}\right)^2$$

$$h - \frac{g}{2}\left(\frac{d}{v_{0x}}\right)^2$$

What is the y-position of the monkey at $t = \frac{d}{v_{0x}}$?

$$m(t) = \underline{m_0} + \frac{1}{2} a_y t^2$$

$$= h + \frac{1}{2}(-g)\left(\frac{d}{v_{0x}}\right)^2$$

$$= h - \frac{g}{2}\left(\frac{d}{v_{0x}}\right)^2$$

<u>The lesson:</u>

Aim at the monkey as if there were no acceleration due to gravity. This result does <u>not</u> depend on:

$\lvert\vec{v}_0\rvert$	initial speed of dart *
h	height
d	distance
g	gravitational acceleration

Why not?

Because it works <u>without</u> gravity, and gravity changes the y positions of both the dart and the monkey by the same amount.

Reference Frames

Problem: I am on a train traveling 60 mph. I walk toward the rear of the train at 3 mph. How fast do I move with respect to the ground?

Easy - 1 dimensional problem.

60 mph − 3 mph = 57 mph

What are we doing?

$\vec{v}_{t,g}$ velocity of train with respect to the ground. $60\hat{\imath}$ mph

$\vec{v}_{m,t}$ velocity of me with respect to the train. $-3\hat{\imath}$ mph

→ $\vec{v}_{m,g}$ velocity of me with respect to the ground.

$$\vec{v}_{m,g} = \vec{v}_{t,g} + \vec{v}_{m,t}$$

$$= \vec{v}_{m,t} + \vec{v}_{t,g}$$

$$= -3\hat{\imath} \text{ mph} + 60\hat{\imath} \text{ mph} = 57\hat{\imath} \text{ mph}$$

What is $\vec{v}_{g,t}$? Velocity of the ground with respect to the train?

$$\vec{v}_{g,t} = -\vec{v}_{t,g} = -60\hat{\imath} \text{ mph}$$

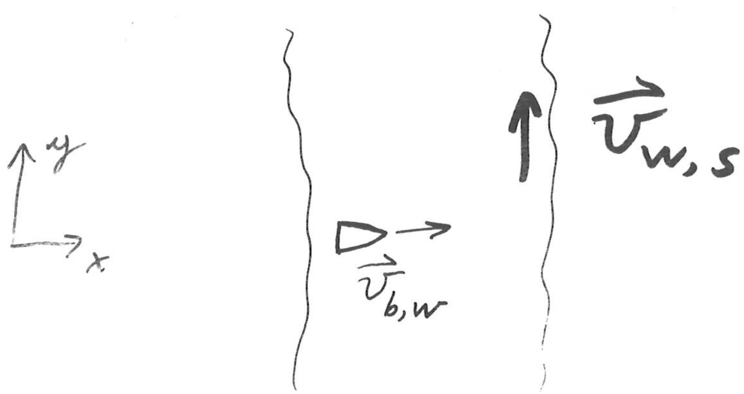
Same idea for multiple dimensions but now vectors.

DEMO

Example:

A boat is crossing a river of width 160 m which flows with a uniform speed of 1.5 m/s.

The boat points perpendicular to the river and travels 2 m/s with respect to the water.



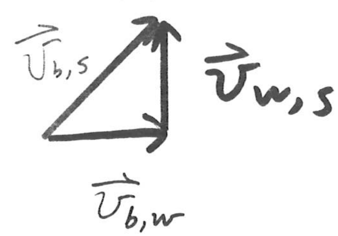
What is the velocity of the boat with respect to the shore? ($\vec{v}_{b,s}$) When it crosses the river, how far downstream has it gone?

$$\vec{v}_{w,s} = (0\hat{i} + 1.5\hat{j})\ \text{m/s}$$

$$\vec{v}_{b,w} = (2\hat{i} + 0\hat{j})\ \text{m/s}$$

$$\vec{v}_{b,s} = \vec{v}_{b,w} + \vec{v}_{w,s}$$

$$= \boxed{(2\hat{i} + 1.5\hat{j})\ \text{m/s}}$$



How far downstream has it gone?

Can I get across the river in <u>less than</u> 80s if I do not change the water-speed of the boat, only the direction?

2-4

Can I cross the river in a straight horizontal line?