Chapter 7: Work + Energy

The work done <u>by</u> a force <u>on</u> an object is a special combination of that force and the displacement of the object.

If net work is done on an object, that object's Kinetic energy will change.

The kinetic energy of an object is the energy due solely to its motion.

Kinds of Energy

- Electro-magnetic
- Chemical
- Nuclear
- Thermal
- Mechanical $\left\{ \begin{array}{l} \text{Kinetic} \\ \text{Potential} \end{array} \right.$

One of the iron-clad rules of physics:

Energy is <u>conserved</u> — it cannot be created nor destroyed. Energy can, however, be <u>converted</u> from one form to another.

Kinetic Energy

The energy of an object due to its motion.

$$K = \frac{1}{2} m v^2$$

K is always positive or zero.

K is zero only when $v = 0$.

K is a scalar quantity:

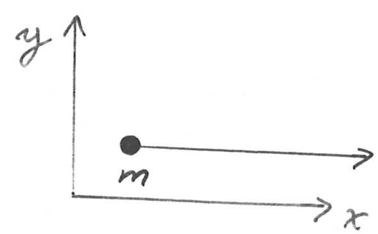
- There is no direction associated with K.
- K will <u>not</u> change when I change my coordinate system.

Ex.

$\vec{v}$ $\quad |\vec{v}| = 3\ m/s$

$m = 2\ kg$

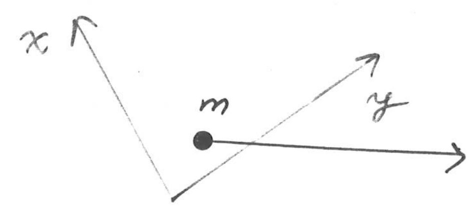
One coordinate choice:



$\vec{v} = 3\ m/s\ \hat{i}$

$\underline{K} = \frac{1}{2} m v^2 = \frac{1}{2}(2)(3)^2 = 9\ J$

A different coordinate choice:



$\vec{v} = (-2.12\ \hat{i} + 2.12\ \hat{j})\ m/s$

$\underline{K} = \frac{1}{2} m v^2 = \frac{1}{2} m |\vec{v}|^2 = \frac{1}{2}(2)(3)^2 = 9\ J$

$|\vec{v}| = \sqrt{2.12^2 + 2.12^2} = 3$

Dimensions + Units of Energy

$$K = \frac{1}{2} m v^2$$

$$[K] = [\tfrac{1}{2}][m][v]^2$$

$$= 1 \cdot M \cdot \frac{L^2}{T^2} = \frac{ML^2}{T^2}$$

The S.I. unit of energy is the joule (J).

$$1\ \text{J} = 1\ \text{N}\cdot\text{m} = 1\ \frac{\text{kg m}^2}{\text{s}^2}$$

Work

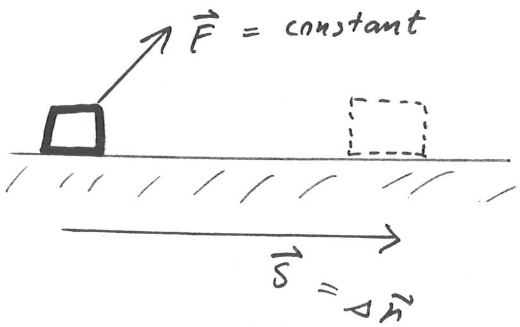
A special case

The work done by a constant force $\vec{F}$ on an object which undergoes a displacement $\vec{s}$ is

$$W = |\vec{F}| \cdot |\vec{s}| \cos\theta = \vec{F} \cdot \vec{s}$$

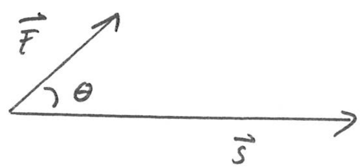
where θ is the angle between the vectors $\vec{F}$ and $\vec{s}$ drawn tail to tail.

Ex:



The work done on the object by the constant force $\vec{F}$ is

$$W = |\vec{F}| \cdot |\vec{s}| \cos\theta$$



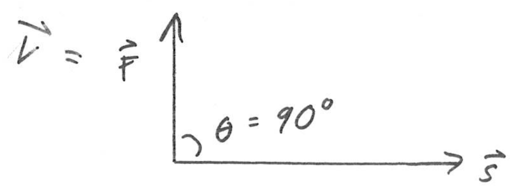
Notice: W is a scalar.

The work done on an object by a force can be zero.

$\vec{F}$ can be zero.

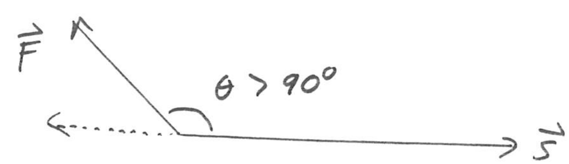
$\vec{S}$ can be zero. Ex holding a mass still.

$\vec{F}$ and $\vec{s}$ can be perpendicular.



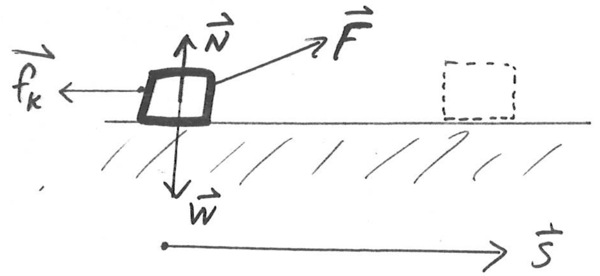
Ex. Normal force

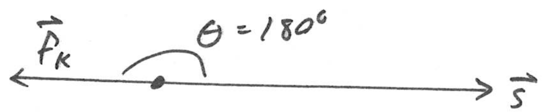
The work done on an object by a force can be negative.



Ex frict.

Ex. The work done by kinetic friction on an object is negative.





$$W_{f_K} = |\vec{f}_K| \cdot |\vec{s}| \cos 180°$$
$$= -f_K \, s$$

What about the work done on an object by static friction?

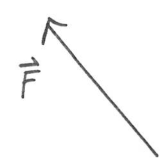
$W_{f_s} = 0$ because $\vec{s} = 0$

Dot Product

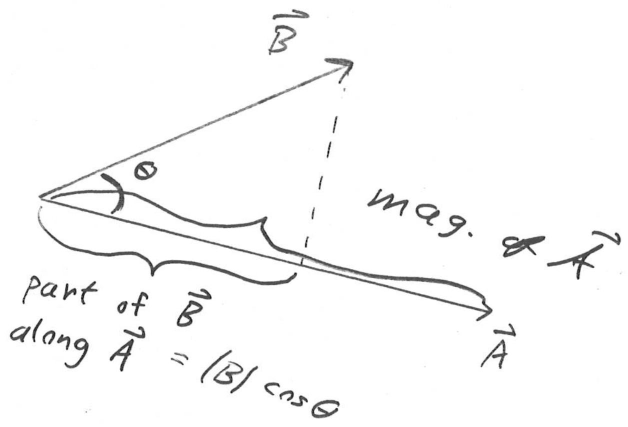
(Scalar Product)

The dot product of two vectors is a scalar. It is defined to be the length of one vector times the length of the second vector times the cosine of the angle between them, drawn tail to tail.

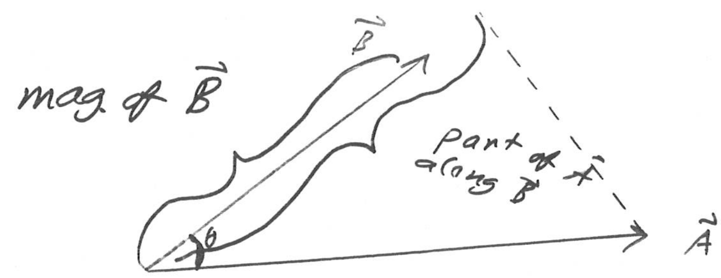
$$W = \underline{\vec{F} \cdot \vec{s}} = |\vec{F}| \cdot |\vec{s}| \cos \theta$$



Other Definitions



The dot product of $\vec{A}$ and $\vec{B}$ is equal to the magnitude of $\vec{A}$ times the part of $\vec{B}$ that lies along $\vec{A}$.



The dot product of $\vec{A}$ and $\vec{B}$ is also equal to the magnitude of $\vec{B}$ times the part of $\vec{A}$ that lies along $\vec{B}$. $(= |A| \cos\theta)$

Unit Vectors

$$\hat{\imath}\cdot\hat{\imath} = |\hat{\imath}|\,|\hat{\imath}|\cos\theta$$
$$= 1\cdot 1\cdot\cos(0) = \boxed{1}$$
$$\hat{\jmath}\cdot\hat{\jmath} = \boxed{1}$$
$$\hat{k}\cdot\hat{k} = \boxed{1}$$

$$\hat{\imath}\cdot\hat{\jmath} = |\hat{\imath}|\,|\hat{\jmath}|\cos\theta$$
$$= 1\cdot 1\cdot\underbrace{\cancel{\cos(90^\circ)}}_{0} = \boxed{0}$$
$$\hat{\jmath}\cdot\hat{k} = \boxed{0}$$
$$\hat{k}\cdot\hat{\imath} = \boxed{0}$$

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

$$\vec{A}\cdot\vec{A} = |A|\,|A|\cos(0) = |A|^2$$

Cartesian Form

$$\vec{A} = (A_x, A_y, A_z)$$
$$\vec{B} = (B_x, B_y, B_z)$$

then

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z \leftarrow$$

The dot product is commutative.

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

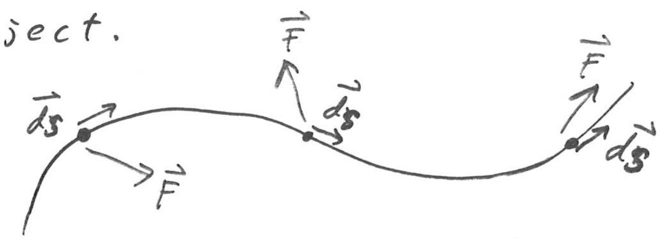
Real Definition of Work

The work done by a <u>non-constant</u> force on an object as it moves along any <u>curved path</u> is

$$W = \int \vec{F} \cdot \vec{ds}$$

Whoa! What is $\vec{ds}$?

$\vec{ds}$ is an infinitesimal vector that points along the path of the object.



How do I integrate $\int \vec{F} \cdot \vec{ds}$?

Use the definition of dot product.

$$\vec{F} = F_x \hat{\imath} + F_y \hat{\jmath} + F_z \hat{k}$$

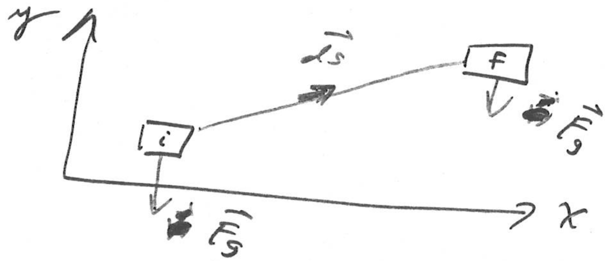
$$\vec{ds} = dx\, \hat{\imath} + dy\, \hat{\jmath} + dz\, \hat{k}$$

$$\vec{F} \cdot \vec{ds} = F_x\, dx + F_y\, dy + F_z\, dz$$

$$W = \int \vec{F} \cdot \vec{ds}$$

$$= \left(\int_{x_i}^{x_f} F_x\, dx \right) + \left(\int_{y_i}^{y_f} F_y\, dy \right) + \left(\int_{z_i}^{z_f} F_z\, dz \right)$$

Ex How much work is done by gravity as I move a mass of 1kg up 1m and across 2m? (in a straight line)



$$W = \int_i^f \vec{F_g} \cdot \vec{ds}$$

work

$\vec{F_g} = -9.8N\,\hat{j}$

$\vec{ds} = dx\,\hat{i} + dy\,\hat{j}$

$$= \int_{x_i}^{x_f} \cancel{(F_g)_x}^{\,0}\, dx + \int_{y_i}^{y_f} (F_g)_y\, dy$$

$$= \int_0^{1m} (-9.8\,N)\, dy = -9.8\, y \Big|_0^{1m}$$

$\downarrow \vec{F_g}$ $\uparrow \vec{dy}$

$$= -9.8(1) - (-9.8)0 = \boxed{-9.8\ J}$$

Ideal Springs

A spring that obeys Hooke's Law:

$$\boxed{F_x = -kx}$$

x is the displacement of the end of the spring from equilibrium.

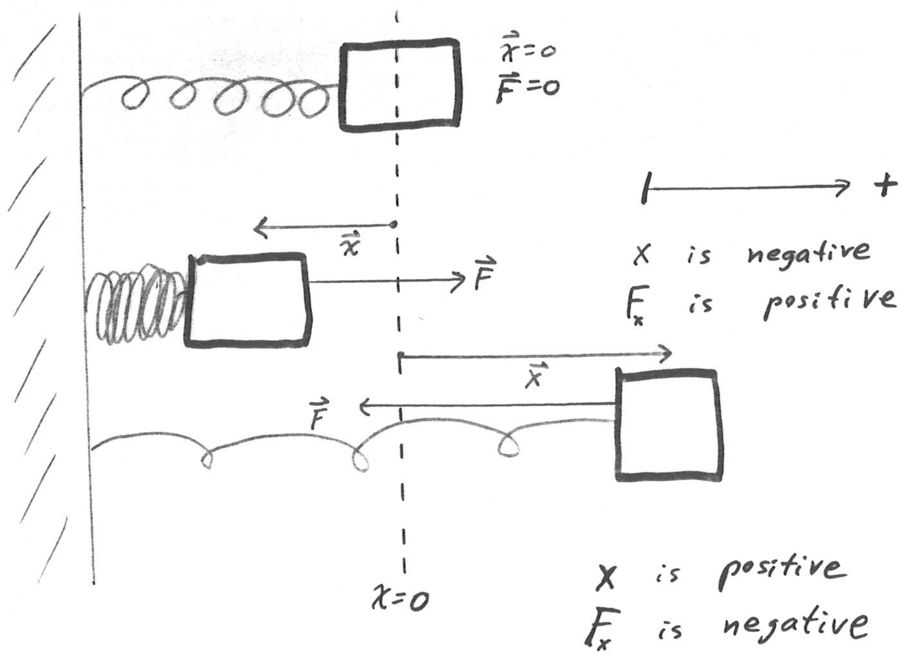
k is a constant of proportionality called the spring constant.

k small $\Rightarrow$ easily stretched

k large $\Rightarrow$ very stiff

The minus sign means that no matter how you choose coordinates the spring force will always point in a direction opposite to the displacement from equilibrium.

(Restoring force)



Work done by a spring

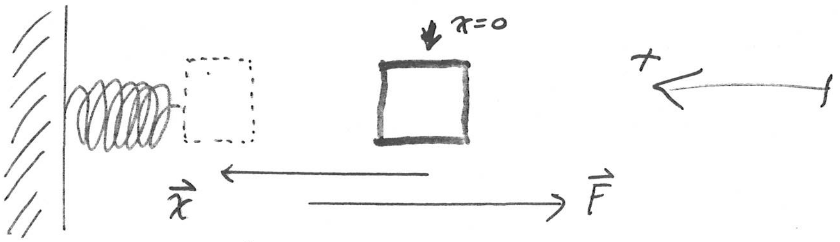
$$W = \int \vec{F}\cdot\vec{ds} \longrightarrow \int_{x_i}^{x_f} F_x\,dx$$

$$W_s = \int_{x_i}^{x_f} (-kx)\,dx = -k\frac{x^2}{2}\Big|_{x_i}^{x_f}$$

$$= -\frac{1}{2}k x_f^2 - \left(-\frac{1}{2}k x_i^2\right)$$

$$= -\frac{1}{2}k x_f^2 + \frac{1}{2}k x_i^2$$

Ex. What work is done <u>by the spring</u> when the spring is compressed 2m from equilibrium? ($k = 10\ \frac{N}{m}$)



$$W_s = \int_{x_i=0}^{x_f=+2} (-kx)\,dx = -\frac{1}{2}kx^2\Big|_0^2$$

$$= -\frac{1}{2}(10\ N/m)(+2m)^2 = \boxed{-20J}$$

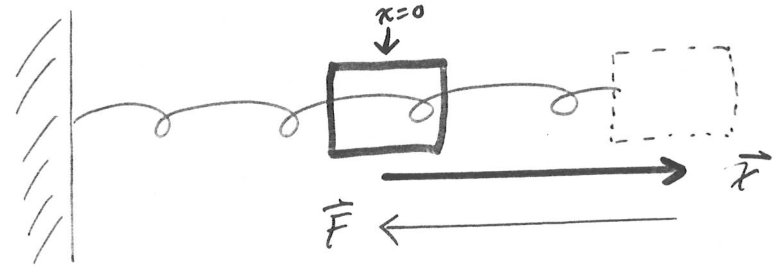
$\longrightarrow +$

$$W_s = \int_{x_i=0}^{x_f=-2} (-kx)\,dx = -\frac{1}{2}kx^2\Big|_0^{-2}$$

$$= -\frac{1}{2}(10\ N/m)(-2m)^2 = \boxed{+20J}$$

W_s is a scalar

Ex What work is done <u>by the spring</u> when the spring is stretched 2m from equilibrium? ($k = 10\ N/m$)



$\longrightarrow +$

$$W_s = \int_0^2 (-kx)\,dx = -\frac{1}{2}kx^2\Big|_0^2$$

$$= \boxed{-20J}$$

$\left.\begin{array}{ll} \vec{ds} & \longrightarrow \\ \vec{F} & \longleftarrow \end{array}\right\} W < 0$

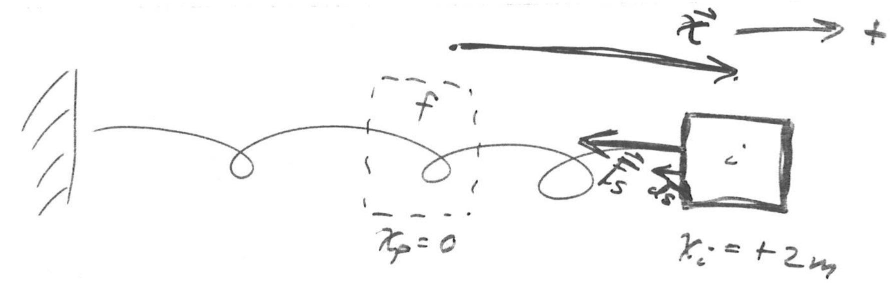
$\left.\begin{array}{ll} \vec{ds} & \longleftarrow \\ \vec{F} & \longleftarrow \end{array}\right\} W > 0$

Summary:

The work done by a spring on an object as the object is moved from equilibrium (stretched or compressed) is negative.

Can a spring do positive work on an object?

The work done by the spring is positive when the object is moved from a stretched or compressed state toward equilibrium.



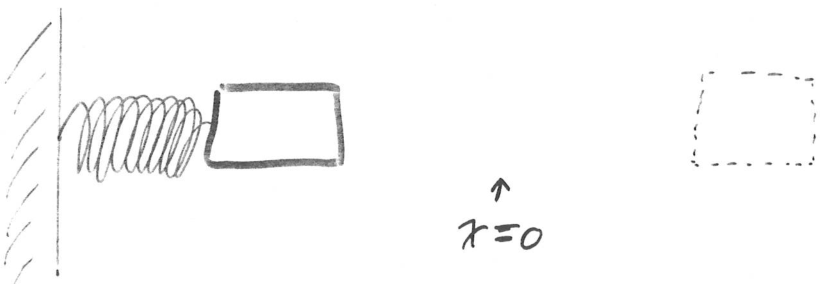
$$W_s = \int_{x_i}^{x_f} F_x\,dx = \int_{+2}^{0} (-kx)\,dx = -\frac{1}{2}kx^2\Big|_{+2}^{0}$$

$$-\frac{1}{2}\left(10\,\frac{N}{m}\right)0^2 - \left[-\frac{1}{2}\,10\,\frac{N}{m}\,(2m)^2\right]$$

$$0 + 20\,J$$

Start at $x=0$ (equilibrium) then $W_s < 0$

Ex How much work is done by our favorite spring ($k = 10$ N/m) as the mass is moved from an initial position where the spring is compressed 2m to a final position where the spring is extended 2m?



$$W_s = 0$$

The Work-Energy Theorem

$$W_{\vec{F}} = \int \vec{F} \cdot d\vec{s}$$

$$K = \frac{1}{2} m v^2$$

$$\boxed{W_{total} = \Delta K = K_f - K_i}$$

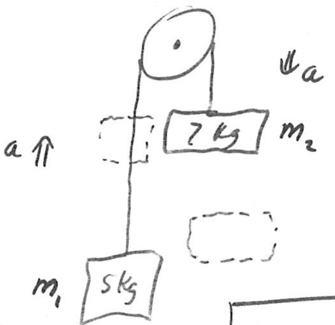
environment

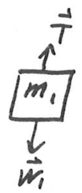
particle motion

Just like Newton's Second Law

$$\sum \vec{F} = m\vec{a}$$

Ex (Atwood's machine) What is the speed of the 7kg mass when it has fallen from rest a distance of 1m ?





$\Sigma F_y = m_1 a_1$ $\qquad$ $\Sigma F_y = m_2 a_2$

$T - W_1 = m_1 a$ $\qquad$ $T - W_2 = m_2(-a)$

$$\boxed{a = \frac{m_2 - m_1}{m_1 + m_2}\, g} \qquad T = \frac{2 m_1 m_2\, g}{m_1 + m_2}$$

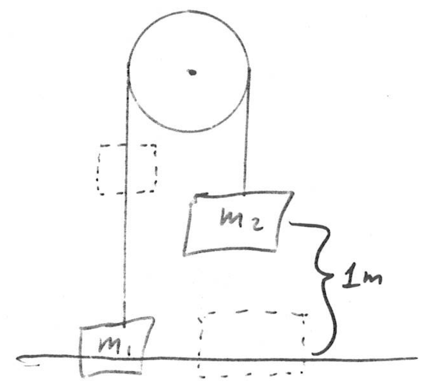
$$v_f^2 = v_0^2 + 2a(\Delta y)$$

$$= 0 + -2\left(\frac{7-5}{7+5}\right) 9.8\, \text{m/s}^2 (-1\,\text{m})$$

a is negative

$$= \boxed{1.8\ \text{m/s}}$$

Now with the work-energy theorem:



It is critical that we consider the entire system (both masses)!

$$W_{net} = \Delta K = K_f - K_i$$

What is the work done by the force of gravity on m_1 ?

$$W_{g_1} = \int_{y_i}^{y_f} F_y\, dy = \int_0^{+1m} (-m_1 g)\, dy$$

$$= -m_1 g y \Big|_0^1 = -(5\,\text{kg})(9.8\,\text{m/s}^2)(1\,\text{m})$$

$$= \boxed{-49\ \text{J}} = \vec{F}_{g_1} \cdot \vec{s}$$

$$= |F_{g_1}|\,|s| \cos 180°$$

What is the work done by the force of gravity on m_2?

$$+\uparrow \quad W_{g_2} = \int_{y_i}^{y_f} F_y \, dy = \int_{+1}^{0} (-m_2 g) \, dy$$

$$= -m_2 g y \Big|_1^0 = -(7\text{kg})(9.8)(-1\text{m})$$

$$= \boxed{+68.6 \text{ J}}$$

$$\downarrow \vec{F} \quad \downarrow \vec{ds}$$

What is the work done by the tension in the rope on m_1 ~~and~~ m_2?

internal force

$$W_T = \underbrace{\int_0^1 T \, dy}_{m_1} + \underbrace{\int_1^0 T \, dy}_{m_2} = 0$$

$$W_{net} = \Delta K = K_f - K_i = \begin{matrix} K_{f_1} + K_{f_2} \\ -(K_{i_1} + K_{i_2}) \end{matrix}$$

$$-49 \text{ J} + 68.6 \text{ J} = \tfrac{1}{2} m_1 v_f^2 + \tfrac{1}{2} m_2 v_f^2 - \left(\tfrac{1}{2} m_1 \cancel{v_0^2}_{\,0} + \tfrac{1}{2} m_2 \cancel{v_0^2}_{\,0}\right)$$

$$19.6 \text{ J} = \tfrac{1}{2}(5\text{kg}) v_f^2 + \tfrac{1}{2}(7\text{kg}) v_f^2$$

$$= \tfrac{1}{2}(12 \text{ kg}) v_f^2$$

$$v_f = \sqrt{\frac{(19.6\text{J})\,2}{12 \text{ kg}}} = \boxed{1.81 \text{ m/s}}$$

Rule of Thumb

When solving for acceleration or tension, it is generally easier to use $\Sigma\vec{F} = m\vec{a}$ (Newton II).

When solving for displacement or velocity, it is generally easier to use $W_{net} = \Delta K$ (work-energy theorem).

But the physics is the same. Any problem can be solved by either method.

Power

Power is the rate at which work is done.

$$P_{AVG} = \bar{P} = \frac{\Delta W}{\Delta t} = \frac{W_f - W_i}{t_f - t_i}$$

$$P_{inst} = P = \frac{dW}{dt} = \vec{F}_{inst} \cdot \vec{v}_{inst}$$

$[F][v]$

Dimension: $[P] = \frac{ML}{T^2} \cdot \frac{L}{T} = \frac{ML^2}{T^3}$

Unit: $1 \text{ watt} = 1 \text{ W} = 1 \frac{\text{kg m}^2}{\text{s}^3}$

Vector or Scalar?

Ex: I want to cook a steak by dragging it behind my car.

$$P_{inst} = \vec{F}\cdot\vec{v} = \vec{f}_{K_{inst}}\cdot\vec{v}_{inst} = -|f_K||v|$$

$$f_K = \mu_K N = \mu_K mg$$

$$P = -\mu_K mgv \qquad \boxed{P = \frac{dW}{dt}}$$

$$W_{Friction} = \int_0^T P\,dt = -\mu_K mgvT$$

Heat energy $= -W_{friction} = \mu_K mgvT$

$\simeq (.7)(1\ kg)(9.8\ m/s^2)(27\ m/s)(3600\ s)$

$= 667{,}000\ J = 159$ Food Cal.

Ex: A baseball pitcher throws a ball, accelerating it from rest to 90 mph. Is the work done on the ball by his arm's force positive or negative?

$$W_{net} = \Delta K = K_f - K_i$$

$$K_f = \tfrac{1}{2}mv^2$$

$$K_i = 0$$

$W_{arm} = \frac{1}{2}mv^2 - 0$ = positive

The catcher catches the ball. Does the force of the catcher's arm do positive or negative work?

$W_{net} = \Delta K = K_f - K_i \qquad K_f = 0$

$K_i = \frac{1}{2}mv^2$

$W_{catch} = 0 - \frac{1}{2}mv^2$ negative