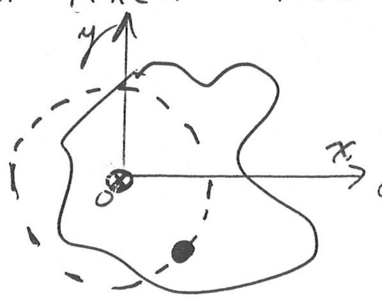
# Chapter 10: Rotation

Consider a rigid body rotating about fixed axis.



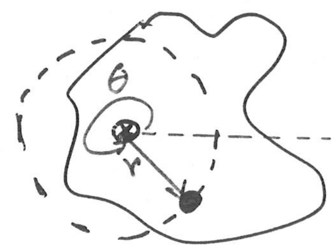

Every point moves along a circle centered on the axis.

Coordinate Systems? Cartesian

Both $x$ and $y$ change in time.

# Chapter 10: Rotation

Consider a rigid body rotating about fixed axis.



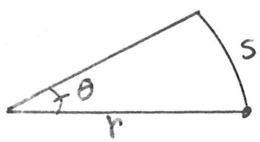

Every point moves along a circle centered on the axis.

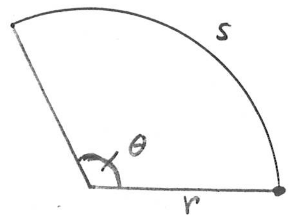
Coordinate Systems? Polar

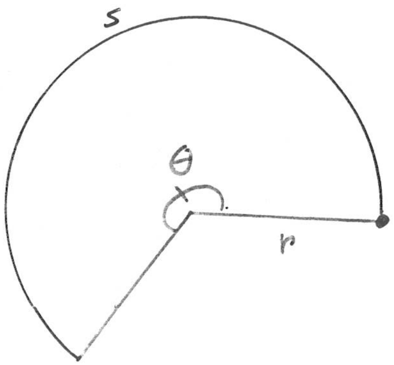
$r$ stays fixed.

$\theta$ changes with time.

The <u>distance</u> that a rotating point covers is the arc length, s.



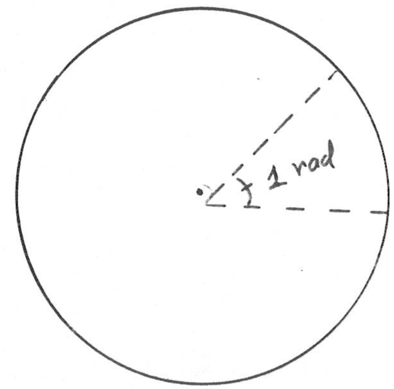




$$\boxed{s = r\theta}$$

Ex.
$C = r(2\pi)$

If s and r are measured in meters, $\theta$ is measured in radians (not degrees!).

What is 1 radian?




A bit more than 6 radius lengths fit around a circle. The exact number is $2\pi$.

$2\pi$ rad $= 360°$ (a full circle)

1 rad $= 57.3° = \frac{360°}{2\pi}$

$1° = 0.017$ rad

The speed of a point in rotation is

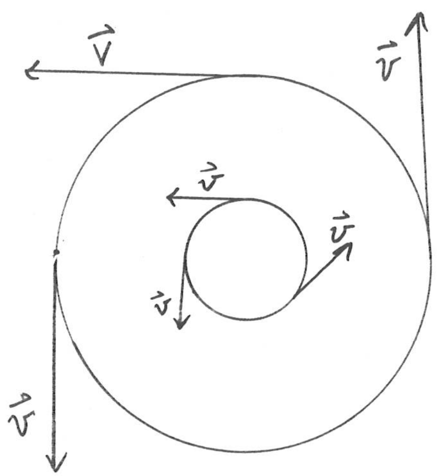
$$s = r\theta$$

$$|\vec{v}_{AVG}| = \frac{\Delta s}{\Delta t} = r\,\frac{\Delta\theta}{\Delta t}$$

$$|\vec{v}_{inst}| = \frac{ds}{dt} = r\,\underline{\frac{d\theta}{dt}}$$

Every point at radius $r$ has the same speed. But every point at radius $R$ has a different speed,

$$|\vec{v}| = R\,\underline{\frac{d\theta}{dt}}.$$




The directions of the velocity vectors are all different!

Demo

It would be very efficient if one quantity could describe the rotational motion of the whole body.

Angular velocity (and angular speed)

$$\omega_{AVG} \equiv \frac{\Delta\theta}{\Delta t}$$

$$\omega_{inst} \equiv \frac{d\theta}{dt}$$

(omega, not double-you)

When a body spins about an axis, __every__ point in the body has the __same__ angular speed.

If the angular speed $\omega$ is constant, then the polar angle $\theta$ changes linearly with time.

$$\omega = \text{constant} = \frac{d\theta}{dt}$$

Integrate:

$$\boxed{\theta_f = \theta_0 + \omega t}$$

Remember:

$$x_f = x_0 + v t$$

---

If the angular speed is not constant?

Angular acceleration:

$$\alpha_{AVG} = \frac{\Delta\omega}{\Delta t}$$

$$\alpha_{inst} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Is this begining to look familiar?

| Linear Kinematics | Angular Kinematics |
|---|---|
| $x$ | $\theta$ |
| $v$ | $\omega$ |
| $a$ | $\alpha$ |
| $t$ | $t$ |
| $v = \text{const.}$ | $\omega = \text{const.}$ |
| $x_f = x_0 + v t$ | $\theta_f = \theta_0 + \omega t$ |
| $a = \text{const.}$ | $\alpha = \text{const.}$ |
| $v_f = v_0 + at$ | $\omega_f = \omega_0 + \alpha t$ |
| $x_f - x_0 = \frac{1}{2}(v_0 + v_f)t$ | $\theta_f - \theta_0 = \frac{1}{2}(\omega_0 + \omega_f)t$ |
| $x_f - x_0 = v_0 t + \frac{1}{2}at^2$ | $\theta_f - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$ |
| $v_f^2 = v_0^2 + 2a(x_f - x_0)$ | $\omega_f^2 = \omega_0^2 + 2\alpha(\theta_f - \theta_0)$ |

# Relation Between Linear + Angular

$$\underline{s = r\,\theta}$$

Speed $|\vec{v}| = \underline{v} = \frac{ds}{dt} = r\frac{d\theta}{dt} = \underline{r\omega}$

There are <u>two</u> kinds of acceleration

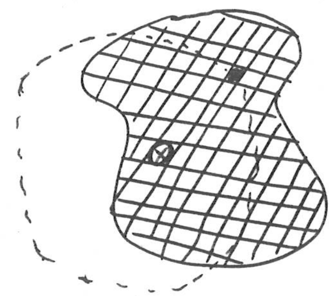
$$\underline{a_t} = \frac{d|\vec{v}|}{dt} = r\frac{d^2\theta}{dt^2} = r\frac{d\omega}{dt} = \underline{r\alpha}$$

$$a_r = \frac{v^2}{r} = r\omega^2$$

$$s = r\theta$$
$$v = r\omega$$
$$a_t = r\alpha$$

# Rotational Kinetic Energy

Break a rotating body into many pieces and consider the sum of all the pieces.



$$K_i = \frac{1}{2} m_i v_i^2$$

$$v_i = r_i \omega_i$$

$$K = \sum_i K_i = \sum_i \left(\frac{1}{2} m_i v_i^2\right)$$

$$= \sum_i \left(\frac{1}{2} m_i r_i^2 \omega_i^2\right)$$

$$= \left[\sum_i \left(\frac{1}{2} m_i r_i^2\right)\right]\omega^2$$

$\omega_i = \omega$ same for <u>all</u> points $i$.

$$K = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

$$= \frac{1}{2} I \omega^2$$

$\omega \rightsquigarrow v$

$I \rightsquigarrow m$

$I = \sum_i m_i r_i^2$ is called the <u>moment of inertia</u> of the body.
(or rotational inertia)

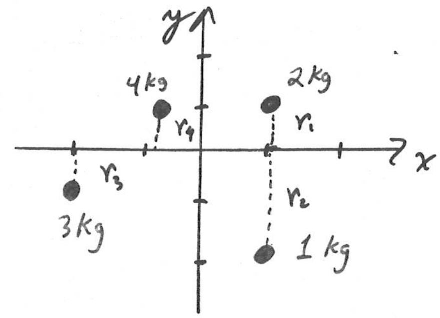
In our table of analogues,

$\underline{\underline{I}}$ is playing the role of $\underline{\underline{m}}$.

$x \to \theta$
$t \to t$
$v \to \omega$
$m \to I$ ("angular mass")

# Calculating Moments of Inertia

Ex.




What is the moment of inertia?
About the x-axis?

$$I_x = \sum_i m_i r_i^2$$

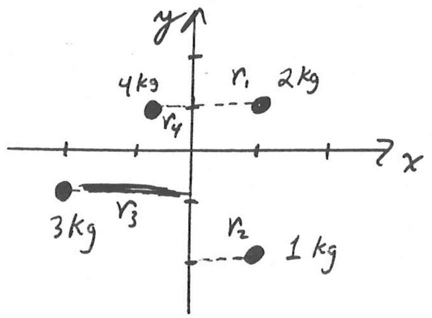
$$= (2\,kg)(1\,m)^2 + (1\,kg)(2\,m)^2$$

$$+ (3\,kg)(1\,m)^2 + (4\,kg)(1\,m)^2$$

$$= 13\ kg\cdot m^2$$

# Calculating Moments of Inertia

Ex.




What is the moment of inertia? About the y-axis?

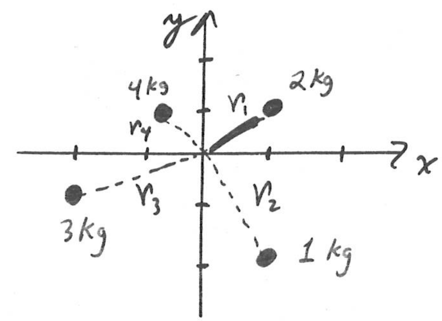
$$I_y = \sum_i m_i r_i^2$$

$$= (2\,kg)(1\,m)^2 + (1\,kg)(1\,m)^2 + (3\,kg)(2\,m)^2 + 4\,kg\,(1\,m)^2$$

$$= 19\ kg\cdot m^2$$

# Calculating Moments of Inertia

Ex.




$$r_1 = \sqrt{1\,m^2 + 1\,m^2}$$

$$r_1^2 = 1\,m^2 + 1\,m^2$$
$$= 2\,m^2$$

What is the moment of inertia? About the z-axis?

$(1\,m)^2$ $(2\,m)$

$$I_z = \sum_i m_i r_i^2$$

$$= 2\,kg\,(1\,m^2 + 1\,m^2) + (1\,kg)(1^2\,m^2 + 2^2\,m^2) + 3\,kg\,(2^2\,m^2 + 1\,m^2) + 4\,kg\,(1^2\,m^2 + 1^2\,m^2)$$

$$= 32\ kg\cdot m^2 = I_x + I_y$$

If the collection of mass is <u>continuous</u> instead of discrete:

$$I = \sum_i m_i r_i^2 \xrightarrow[i \to \infty]{} \int r^2 \, dm$$

The text has done the integrals for the common shapes for us.

Table 11-2 $\qquad I_{cm}$

$$dm = \rho \, dV, = \rho \, dx \, dy \, dz$$

(a)
(b)
(c)
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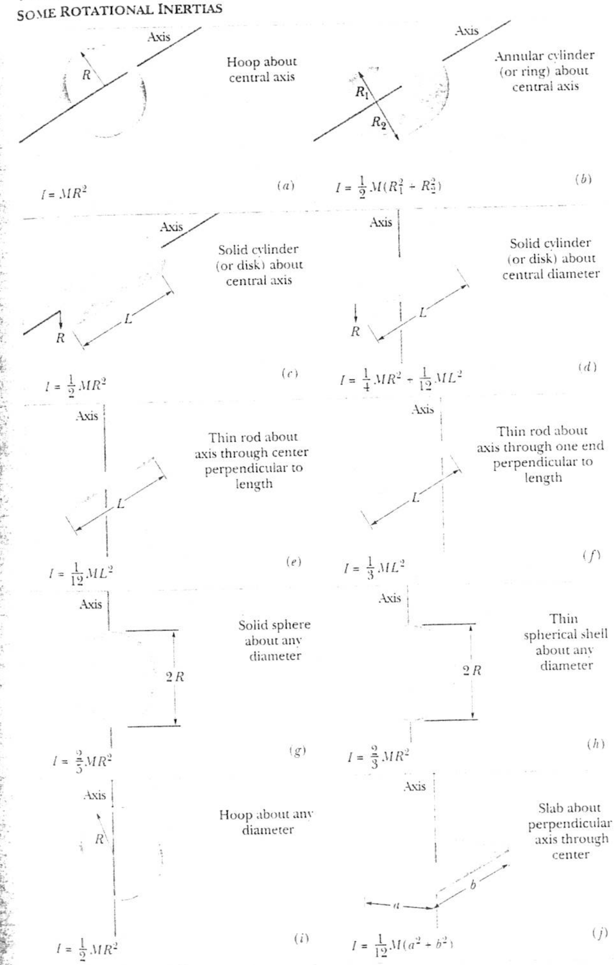
TABLE 11-2
SOME ROTATIONAL INERTIAS

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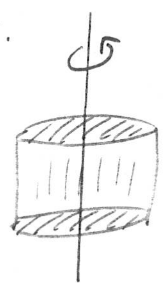
Steiner

# Parallel-Axis Theorem

If you know the moment of inertia of a body about an axis through the center of mass, then you can easily obtain the moment of inertia about another axis parallel to the first.

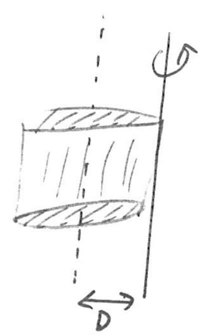
Ex.



Smallest

$\rightarrow I_{cm} = \frac{1}{2} M R^2$

Solid Cylinder

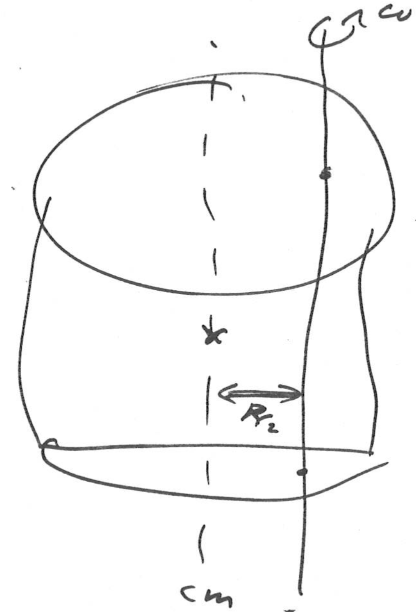



$I_{new} = I_{cm} + M D^2$

bigger than $I_{cm}$

$= \frac{1}{2} M R^2 + M R^2$

$= \frac{3}{2} M R^2$




$D = \frac{R}{2}$

$I_{cm} = \frac{1}{2} M R^2$

$I_{new} = I_{cm} + M D^2$

$= \frac{1}{2} M R^2 + M \left(\frac{R}{2}\right)^2$

$= \frac{1}{2} M R^2 + \frac{1}{4} M R^2$

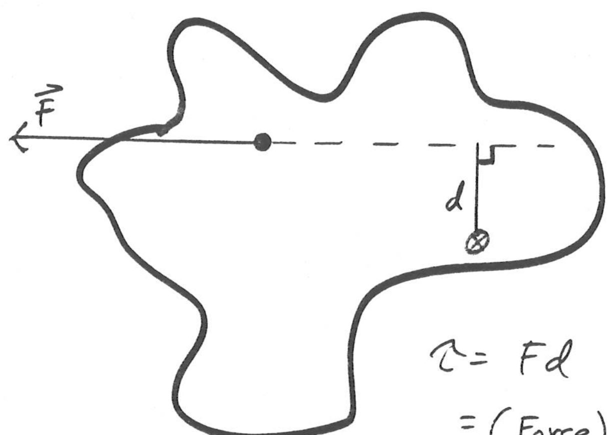
$= \frac{3}{4} M R^2$

# Torque

Demo: door, weights

("angular force")

Just as in the definition of moment of inertia, you must specify the axis about which torques are to be calculated.




$\tau = Fd$

$= (\text{Force})(\text{Lever Arm})$

counterclockwise ≡ positive

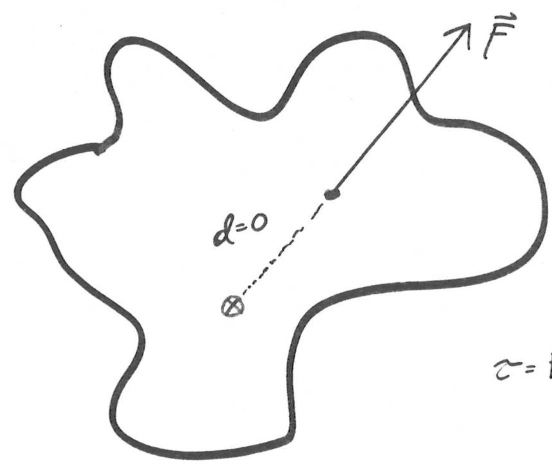
# Torque

Demo: door, weights

("angular force")

$\tau$ - tau

Just as in the definition of moment of inertia, you must specify the axis about which torques are to be calculated.



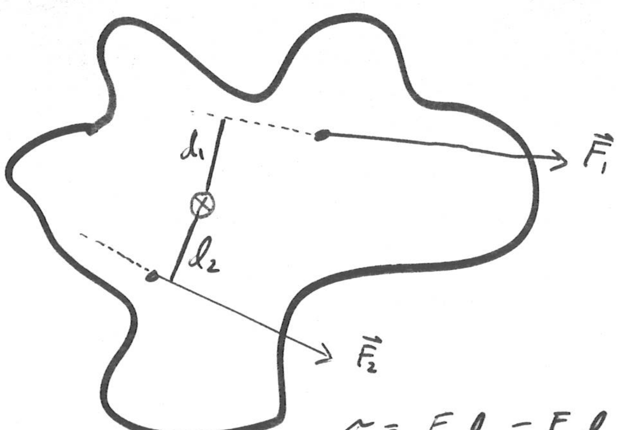

$\tau = Fd = 0$

# Torque

Demo: door, weights

("angular force")

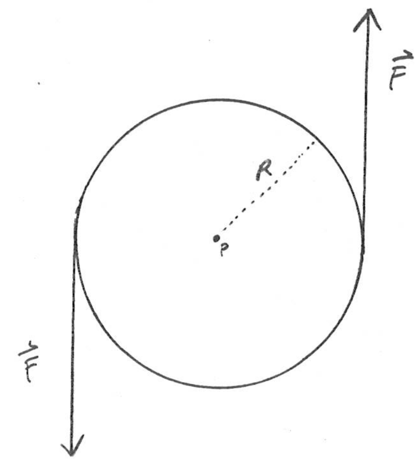
Just as in the definition of moment of inertia, you must specify the axis about which torques are to be calculated.



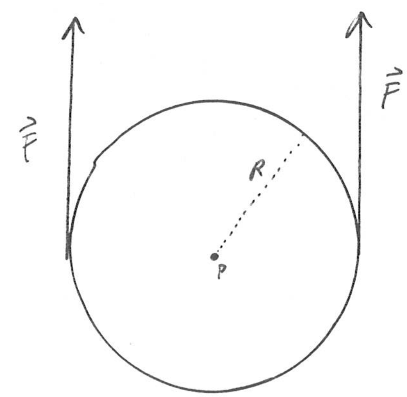

$$\tau = F_2 d_2 - F_1 d_1$$

$F_2 d_2$: ccw, +

$F_1 d_1$: cw, −




$\tau_p =$



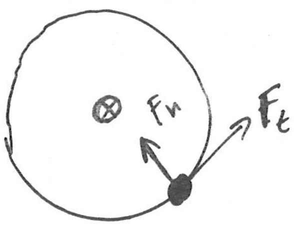

$\tau_p =$

# Torque and Angular Acceleration

Consider a single particle first.

$a_r = \frac{v^2}{r}$

$F_r = m a_r$ ✗ $\quad a_t = \alpha r$




$\underline{F_t} = \underline{m a_t}$ ✓

$I = \Sigma m_i r_i^2$
$= m r^2$

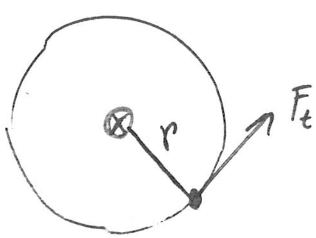
$F_t = m a_t = m(\alpha r)$

multiply both sides by $r$

$r F_t = (m r^2)\alpha$

$\boxed{\tau = I \alpha}$

Analog of:
$F = ma$




$I$ is defined for the whole body.

$\alpha$ is the same for the whole body.

$F \to \tau$

$$\boxed{\sum \tau_i = \tau_{net} = I\alpha}$$

## Work + Energy

$W_f = \int_i^f \vec{F} \cdot d\vec{s} = \int F \cdot (r\, d\theta) = \int (F r)\, d\theta$

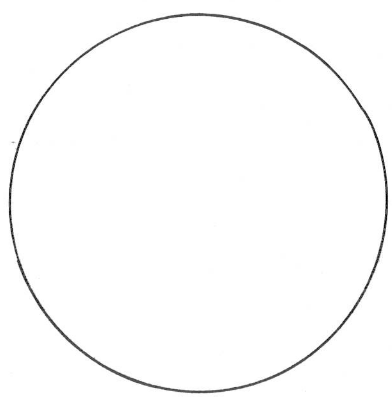
$$\boxed{W = \int \tau\, d\theta}$$

$W_{net} = \Delta K = K_f - K_i$

$$\boxed{W_{net} = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2}$$

# Kinetic Energy of a Rolling Body

First, consider a body spinning about an axis and not rolling

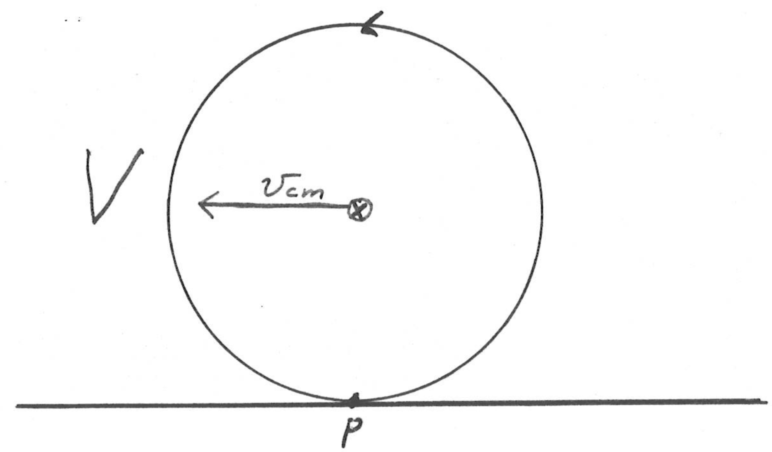


what do the velocities of all the points in the body look like?

$$K_{\substack{\text{not}\\ \text{rolling}}} = \sum \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2} I_{cm}\, \omega^2$$

$\boxed{\omega = \frac{v}{r}}$

Now let's describe the velocities of points in a rolling body.




It looks like the body is rotating around the fixed axis through P.

$$K_{\text{rolling}} = \sum \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2} I_P\, \omega^2$$

$\boxed{\omega = \frac{v}{r}}$

not $I_{cm}$ this time!

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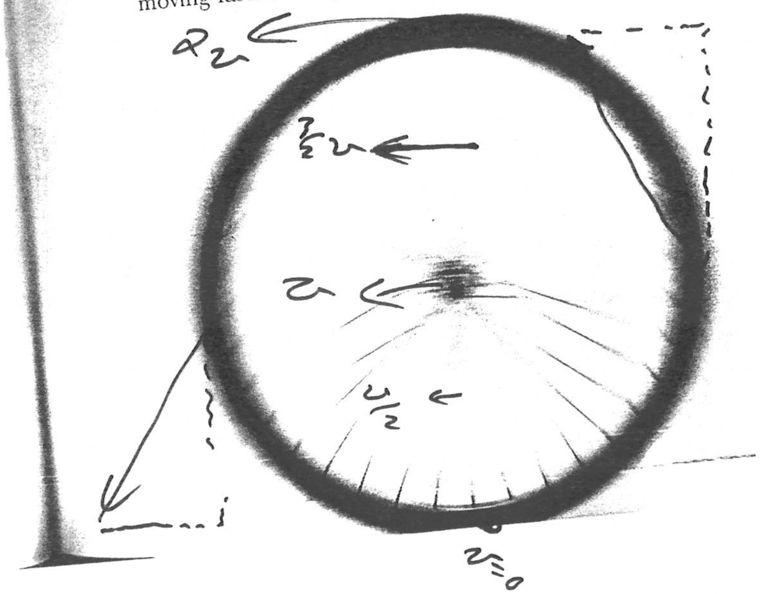
FIGURE 12-4 A photograph of a rolling bicycle wheel. The spokes near the top of the wheel are more blurred than those near the bottom of the wheel because they are moving faster, as Fig. 12-3c shows.




<u>But</u>, we can use the parallel-axis theorem to write

$$I_p = I_{cm} + Mr^2$$

$$r^2\omega^2 = v_{cm}^2$$

then

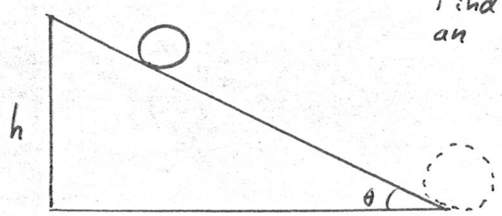
$$K_{rolling} = \frac{1}{2} I_p \omega^2 = \frac{1}{2}(I_{cm} + Mr^2)\omega^2$$

$$= \underbrace{\frac{1}{2} I_{cm}\omega^2}_{K_{rot}} + \underbrace{\frac{1}{2} M (v_{cm})^2}_{K_{trans}}$$

$$= K_{rot} + K_{trans}$$

The kinetic energy of a <u>rolling</u> body can be written as the sum of rotational KE about the center of mass and translational KE of the c-o-m.

$$K_{rolling} > K_{spinning}$$




Find the final speed of an object rolling down a slope.

which object in table 10.2 wins the race?

Do these results depend on the masses or radii of the objects?

No!

- Two spheres of the same radius and different masses tie
- Two spheres of different radii and the same mass tie
- Two spheres of different radii and different masses tie

Only the number in front of $MR^2$ matters. That describes the distribution of mass in the body.

Any sphere beats any cylinder beats any hollow sphere beats any hoop.