Resonance

All objects have a
natural angular frequency, ω_0.

For a pendulum, $\omega_0 = \sqrt{\dfrac{g}{L}}$

For a mass on a spring, $\omega_0 = \sqrt{\dfrac{k}{m}}$

If a force is applied at a frequency
close to ω_0, the amplitude of
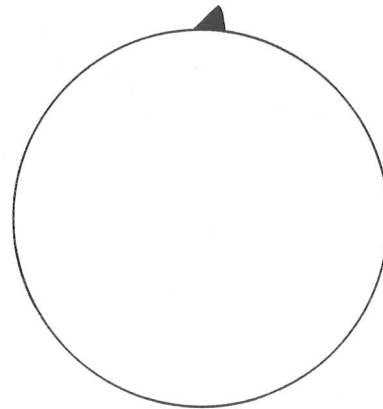oscillation will increase dramatically.

A force applied at a frequency far from ω_0
will have little effect on the amplitude.

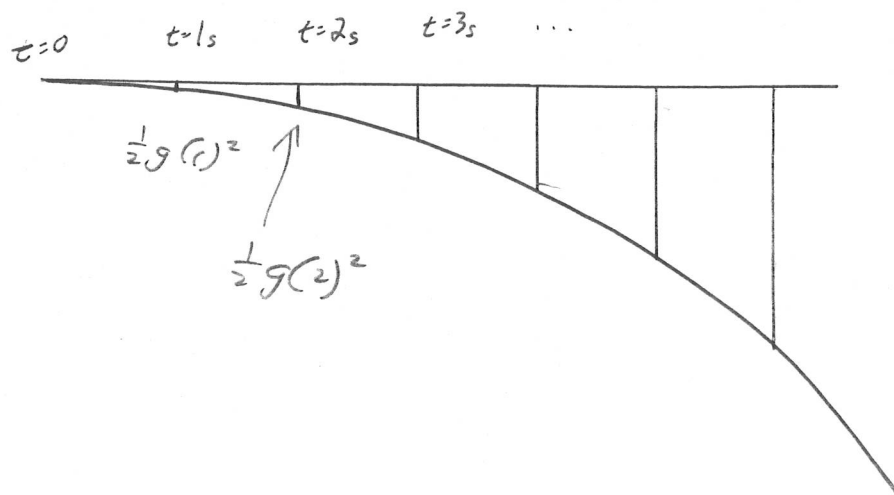# Chapter 14:

The Law of Universal Gravitation

A thought experiment:
   Suppose that you climb a mountain and
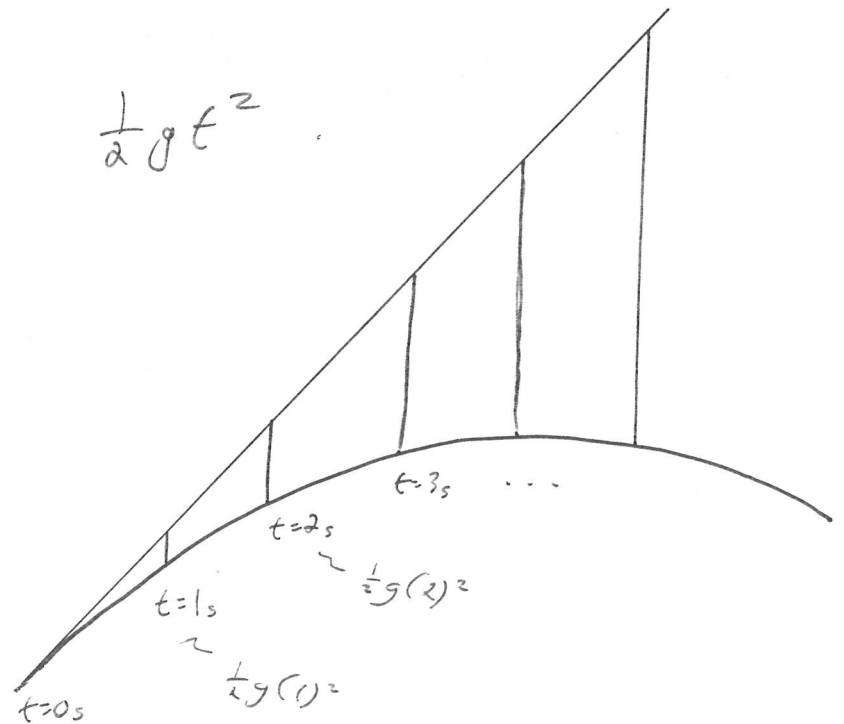fire a projectile at different speeds
horizontally.

How far does an object fall
under the influence of gravity?

$$\frac{1}{2}g t^2$$

$t=0$ $t=1s$ $t=2s$ $t=3s$ . . .

$\frac{1}{2}g(1)^2$

$\frac{1}{2}g(2)^2$

How far does an object "fall" under
the influence of gravity?

$$\frac{1}{2}g t^2$$

$t=3s$ . . .

$t=2s$ $\frac{1}{2}g(2)^2$

$t=1s$

$\frac{1}{2}g(1)^2$

$t=0s$

Isaac Newton calculated how far the Moon "fell" every second.

$T = 27.3$ days ≈ 1 month

$R = 240,000$ miles

$v = \dfrac{2\pi R}{T} \approx 3000$ ft/s

$a_n = \dfrac{v^2}{R} \approx 0.1$ in/s^2

How far does the Moon "fall" in 1 second?

$d = \tfrac{1}{2} a_n t^2 = \dfrac{1}{20}$ of an inch

How far does an apple fall in 1 sec

$d = \tfrac{1}{2} g t^2 = 16$ ft

$\dfrac{d_{apple}}{d_{Moon}} \approx 3600$

Based on the relationship between the radius of a planet's orbit and the length of its year (its period) Kepler's Third Law, Newton assumed that the acceleration due to gravity, g, is inversely proportional to the square of the distance from the center of the Earth.

$R_{Moon} = 240,000$ miles

$R_{apple} = $ Radius of Earth $= 4000$ miles

$$\frac{R_{Moon}}{R_{Apple}} = 60 \qquad \left(\frac{R_{Moon}}{R_{Apple}}\right)^2 = 3600 \, !$$

The acceleration of any object toward the Earth (g) is not a constant $9.8 \, m/s^2$ but is

$$g(r) = \frac{k}{r^2}$$

where r is measured from the center of the Earth.

Universal Gravitation

Every particle in the universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

$$F = \frac{G \, m_1 \, m_2}{r^2}$$

Direction: along the line joining the centers of gravity of the two masses.

dimension & units of G?

$$F = \frac{G m_1 m_2}{r^2}$$

$$[F] = \frac{[G][m_1][m_2]}{[r]^2}$$

$$[G] = \frac{[F][r]^2}{[m_1][m_2]} = \frac{\left(\frac{ML}{T^2}\right)(L)^2}{(M)^2} = \frac{L^3}{MT^2}$$

Units

$$\frac{m^3}{kg \, s^2} = \frac{N \cdot m^2}{kg^2}$$

Experimentally measured value:

$$G = 6.672 \times 10^{-11} \frac{N \cdot m^2}{kg^2}$$

A very small number ⟹ gravity is weak!

Ex. A block of mass m sits near the surface of the Earth. What is the force that the Earth exerts on the block?

$$F = mg$$

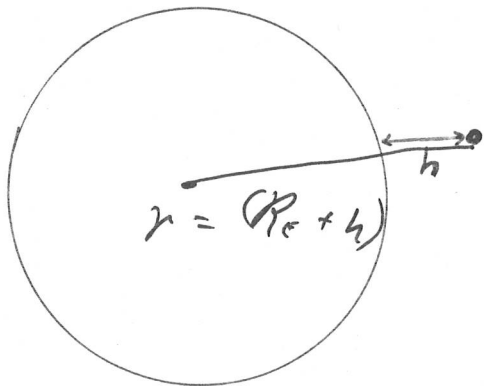$$F = \frac{G M_E \, (m)}{r^2} = \left(\frac{G M_E}{R_E^2}\right) m$$

$$g = \frac{G M_E}{R_E^2} = 9.8 \, m/s^2$$

$$= \frac{\left(6.672 \times 10^{-11} \frac{N \cdot m^2}{kg^2}\right)\left(6 \times 10^{24} kg\right)}{\left(6 \times 10^6 \, m\right)^2}$$

Ex. What is the acceleration due to gravity at a height h above the surface of the Earth?

$r = (R_E + h)$

$$g(r) = \frac{GM_E}{r^2} = \frac{GM_E}{(R_E + h)^2}$$
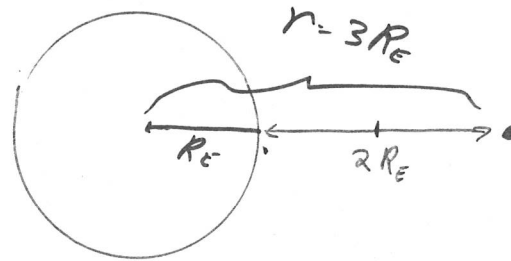
$$\leq 9.8 \, \text{m/s}^2$$

What is the weight of an astronaut orbitting the Earth at an altitude h of 2 Earth-radii?

$m = 100 \, kg$

$W_{surface} = mg$
$= 980 \, N$

$r = 3R_E$

R_E $2R_E$

$$F_{gravity} = W = \frac{GM_E \, m}{r^2} = \frac{GM_E \, m}{(3R_E)^2}$$

$$= \frac{1}{9} \left(\frac{GM_E}{R_E^2} \right) m = \frac{1}{9} \, mg_{surf}$$

$$= \frac{1}{9} (100 kg)(9.8 \, \text{m/s}^2) = \underline{109 \, N}$$

Kepler's Laws

1) All planets move in elliptical orbits with the Sun at one of the focal points

2) The radius vector drawn from the Sun to any planet sweeps out equal areas in equal time intervals.

3) The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit. $T^2 \propto r^3$

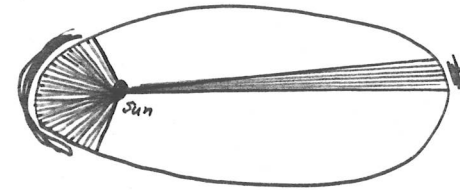These are all _empirical_ laws ; they describe the observations but do not explain them.

All three of Kepler's Laws can be _derived_ from Newton's Law of Universal Gravitation:

$$F = G \frac{M_1 M_2}{r^2}$$

1) All planets move in elliptical orbits with the Sun at one focus.
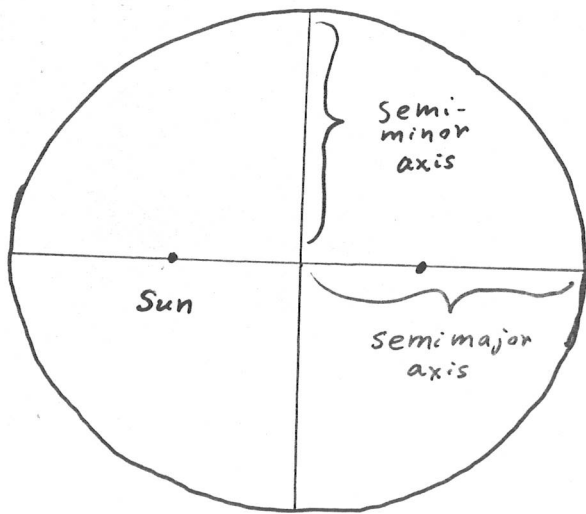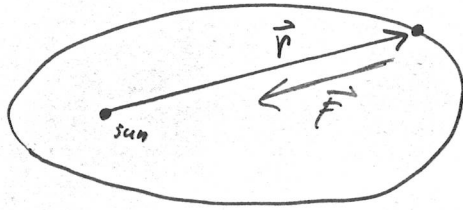

2) The radius vector drawn from the Sun to any planet sweeps out equal areas in equal time intervals.

Both green-shaded areas represent one month.

Planets (and comets) move faster when they are closer to the Sun.

In which direction does the force of gravity on the planet point?

$$\vec{\tau}_o = \vec{r} \times \vec{F} = ? = 0$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} = 0 \qquad \Rightarrow \vec{L} \text{ is } \underline{constant.}$$

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

3) The square of the orbital period (T) (the planet's "year") is proportional to the cube of the semi major axis of $\underline{\text{the elliptical orbit.}}$ ^radius

We will consider the special case of a $\underline{circular}$ orbit.

$$\Sigma F_n = m_p \, a_r$$

$$F = \frac{G M_s \, m_p}{r^2} = \frac{m_p v^2}{r}$$

$$v_{orb} = \frac{2\pi r}{T}$$

$$\frac{G M_s}{r^2} = \left(\frac{2\pi r}{T}\right)^2 \frac{1}{r}$$

$$T^2 = \left(\frac{4\pi^2}{G M_s}\right) r^3$$

This relation suggested to Newton that the force law was inverse-square.

_Ex._ At what altitude must a satellite orbit to attain geosynchronous orbit?

Gravitational Potential Energy

If g is _not_ constant, then

$$U_g \neq mgy + \text{constant}$$

$$\vec{F} = \frac{G m_1 m_2}{r^2} \quad , \quad \text{directed } \underline{\text{toward}} \text{ the origin}$$

(one of the masses)

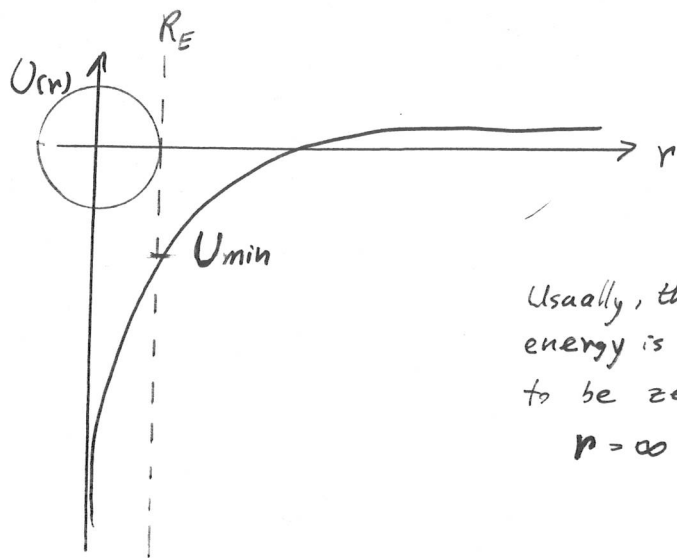$\vec{dr}$ is an infinitesimal displacement vector directed _away from_ the origin

$$\vec{F} \cdot \vec{dr} = - \frac{G m_1 m_2}{r^2} dr$$

$$U(r) = -\int \vec{F} \cdot \vec{dr} = -\int \left(-\frac{Gm_1 m_2}{r^2} dr \right)$$

$$= \int \frac{Gm_1 m_2}{r^2} dr$$

$$= Gm_1 m_2 \int r^{-2} dr$$

$$= Gm_1 m_2 \left(-\frac{1}{r} \right) + constant$$

Usually, the potential energy is chosen to be zero at

$$r = \infty$$

Ex Suppose that a small mass m "falls" to Earth from infinitely far away. What is the change in the mass' potential energy?

$$\Delta U = U_F - U_i = U(R_E) - U(\infty)$$

$$-\frac{GM_E m}{R_E} - \left(-\frac{GM_E m}{\infty} \right)^{0}$$

$$= -\frac{GM_E m}{R_E}$$

What happened to this energy?

becomes $-\Delta K$

Ex Suppose the mass has no initial
velocity at r=∞. What is
its speed when it strikes the Earth?

$$\Delta K + \Delta U = 0$$

$$K_F - \cancel{K_i}^0 + \left(- \frac{GM_E m}{R_E}\right) = 0$$

$$\frac{1}{2} m v_f^2 = \frac{GM_E m}{R_E}$$

$$v_f = \sqrt{\frac{2GM_E}{R_E}} = 1.12 \times 10^4 \, m/s$$

$$\text{escape velocity} = 25,000 \, mph$$

How fast must I throw a rock
to get it out to r=∞ with zero speed?