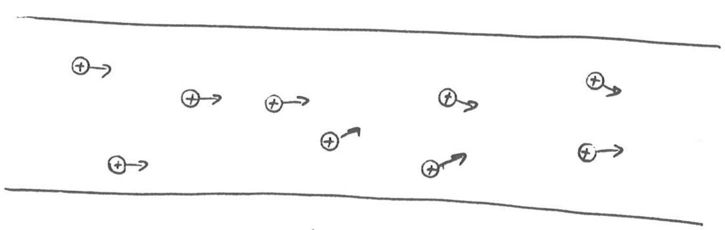
Current + Resistance

Until now, we have been studying Electrostatics (charges at rest).

<u>Current</u>: the rate at which charge moves past a hypothetical plane.

$$\boxed{i(t) = \frac{dq}{dt}}$$



$$q = \int_{t_i}^{t_f} i(t)\, dt$$

MKS Unit

The unit of current is the ampere (A). This is one of the fundamental set

$$\underset{L}{\text{meter}},\ \underset{M}{\text{kilogram}},\ \underset{T}{\text{second}},\ \underset{\text{current}}{\text{ampere}}$$

Steady State

The current is <u>not</u> a function of time — it is constant.

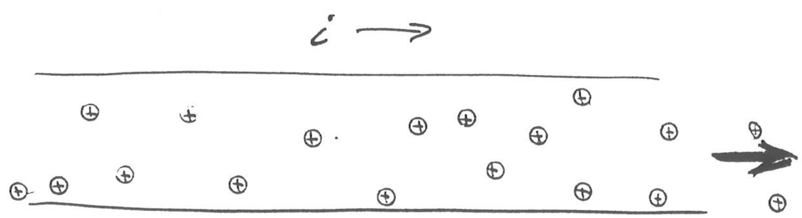
$$\boxed{i = \text{constant}}$$

Under steady state conditions, charge cannot "pile up" in the wire.

Direction of Current

Current (i) is a <u>scalar</u>, but there is an associated <u>direction</u>.

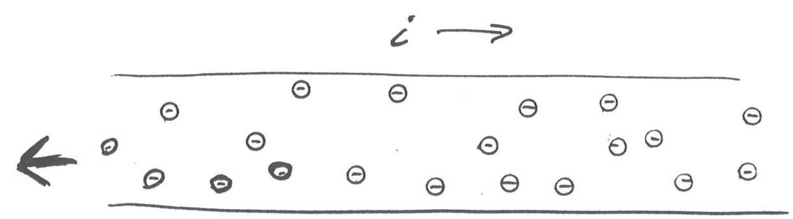
Current is defined by convention to flow in the direction that <u>positive</u> charges would move even if the moving charges are negative!



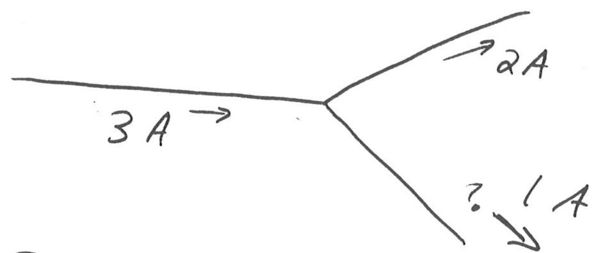
Direction of Current

Current (i) is a <u>scalar</u>, but there is an associated <u>direction</u>.

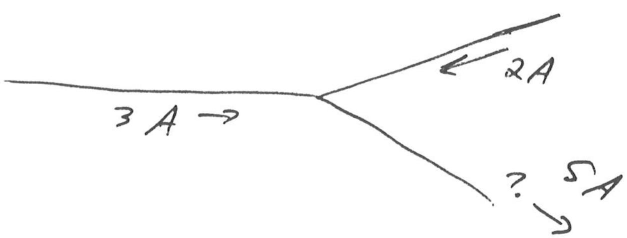
Current is defined by convention to flow in the direction that <u>positive</u> charges would move even if the moving charges are negative!



Ex

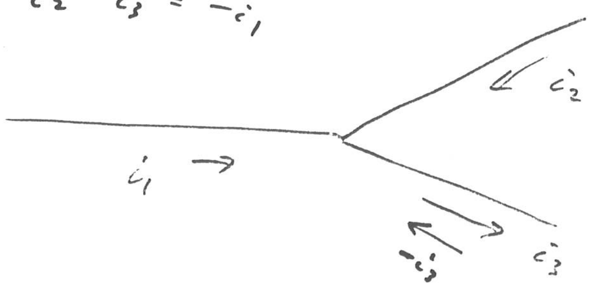


$3A + 2A + (-5A) = 0$ ✓



$i_1 + i_2 = i_3$

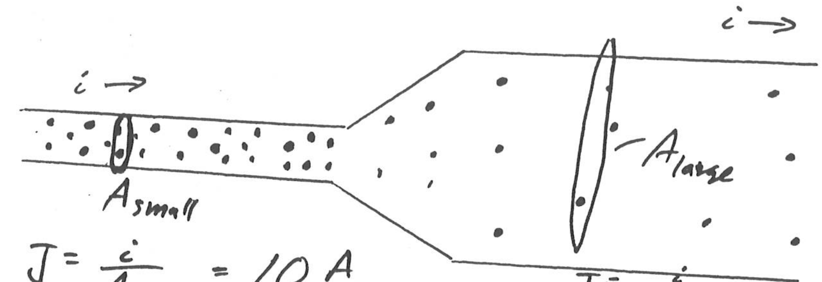
$i_2 - i_3 = -i_1$



$i_1 + i_3 + (-i_3) = 0$

Steady state current conservation is a consequence of charge conservation.

Current Density



$J = \frac{i}{A_{small}} = 10\,\frac{A}{m^2}$

$J = \frac{i}{A_{large}} = 1\,\frac{A}{m^2}$

Current Density : $J \equiv \frac{i}{Area}$ (magnitude)

$\vec{J}$ is a vector quantity.

The direction of $\vec{J}$ is the same as that of the electric field $\vec{E}$, regardless of the sign of the charge carriers.

Whoa!

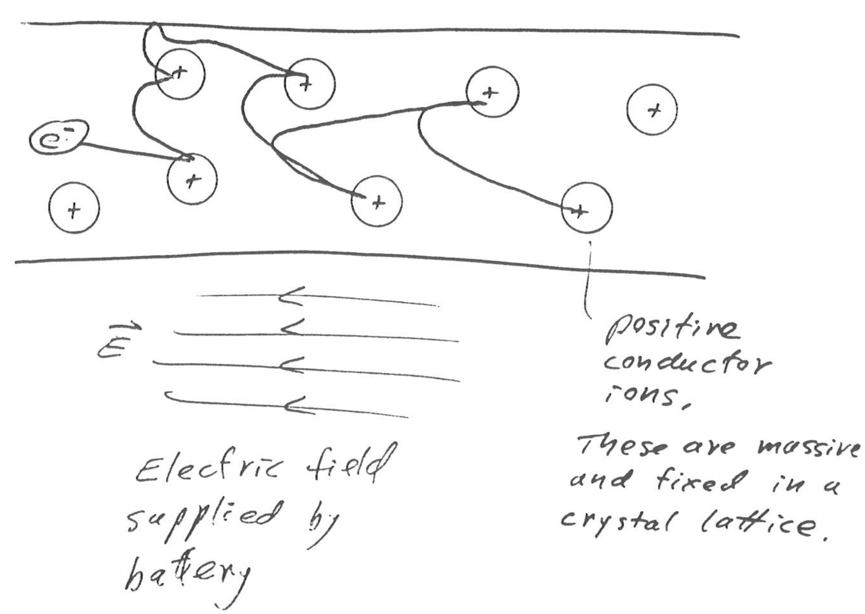
What electric field???

Something must cause the moving charges to move: An electric field in the conductor.

I thought $\vec{E}=0$ inside a conductor. This is true for electro statics. Now we are considering charges in motion: electro dynamics.

Doesn't an electric field cause charges to accelerate, so the current (i) will not be a constant but will increase with time?

An electric field would cause free charges to accelerate. In a conductor (like a wire), the charges accelerate for a very short time (10^{-14} seconds) then collide with atoms in the conductor, scatter, and accelerate again, ...

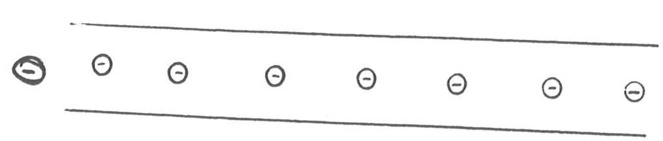


The result of the scattering and acceleration is that electrons move with a constant average velocity called the "drift velocity."

Typically, $\left|\vec{v}_{drift}\right| = 10 \frac{cm}{hour}$

A snail could race an ~~electron~~ and win!

So why doesn't it take a week to turn the lights on?



The speed of the "push" is almost the speed of light.

Resistance

It is an experimentally observed fact that for most (not all!) conductors the current is directly proportional to the potential difference across the conductor.

$$V = iR \qquad \text{Ohm's Law}$$

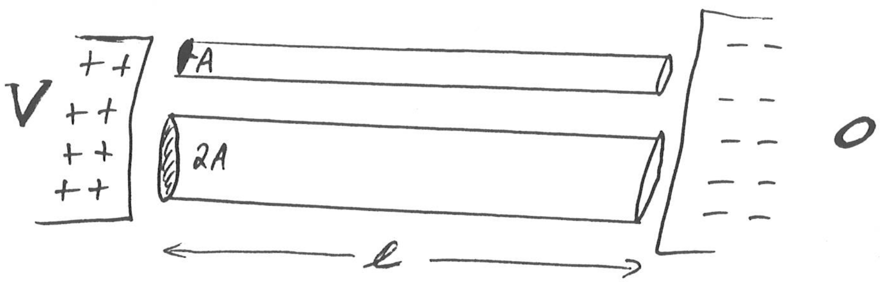
The constant of proportionality is called the resistance.

MKS Unit

$$1 \text{ ohm } (\Omega) \equiv 1 \frac{V}{A} \left(\frac{\text{volt}}{\text{ampere}}\right)$$

1Ω

I can decrease the resistance of a conductor by <u>increasing</u> its cross-sectional area.



<u>Warning</u>: increasing the radius of a cylindrical wire by a factor of 2 increases the cross-sectional area by a factor of 4. $(A = \pi r^2)$

I can also decrease the resistance of a conductor by <u>decreasing</u> its length.

The same voltage V applied over a shorter distance gives rise to a larger electric field: $E = \frac{V}{\ell}$

While individual conductors are characterized by their resistance, the material from which the conductor is made is characterized by its resistivity (ρ).

$$R = \rho \frac{\ell}{A}$$

$$\rho_{copper} = 1.69 \times 10^{-8} \ \Omega \cdot m$$

$$\rho_{iron} = 9.68 \times 10^{-8} \ \Omega \cdot m$$

Power

Dissipated in electric circuits

Resistance is a "lossy" effect, like friction. Electric potential energy (in the battery or capacitor) and kinetic energy (of the moving charges) is converted into <u>heat</u> energy.

$$dU = dq\, V = (i\, dt)\, V$$

$$\frac{dU}{dt} = \boxed{P = iV}$$

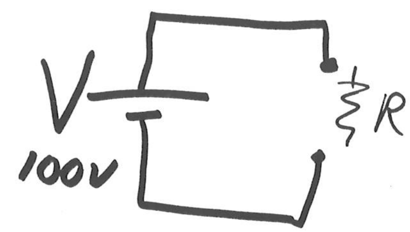
The MKS unit of power is the watt (W)
$1W = 1\,\frac{J}{s} = 1\,V \cdot A$

Using Ohm's Law: $V = iR$

$$\boxed{P = i^2 R}$$

$$\boxed{P = \frac{V^2}{R}}$$

Const. Voltage Source (wall outlet)



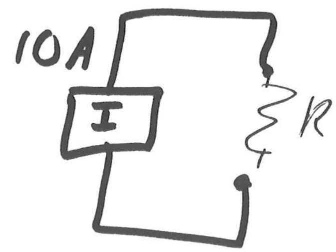
$R_1 = 1\,\Omega$

$P_1 = 10{,}000\ W = \frac{V^2}{R_1}$

$R_2 = 100\,\Omega$

$P_2 = \frac{V^2}{R_2} = 100\ W$

Const Current Supply



$R_1 = 1\,\Omega$

$P_1 = I^2 R_1 = 100\ W$

$R_2 = 100\,\Omega$

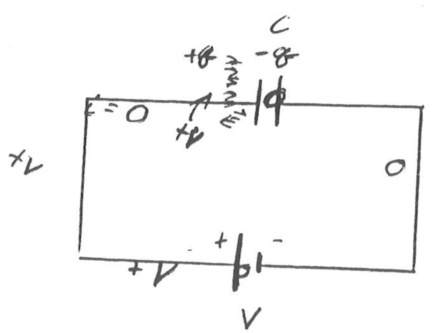
$P_2 = I^2 R_2 = 10{,}000\ W$

Consider a 40W and a 100W lightbulb in <u>parallel</u>:

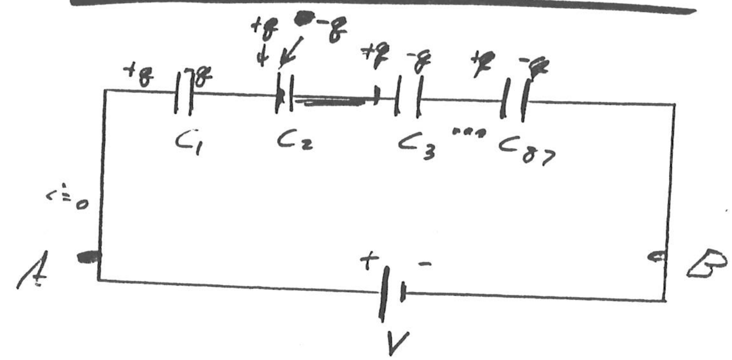
Consider a 40W and a 100W lightbulb in <u>series</u>:

Circuits

In a circuit containing batteries and capacitors, current flows until all of the capacitors are fully charged. Then no more current flows. This is the steady state. The current is not changing; it is zero (a constant).



Capacitors in Series



$q_1 = C_1 V_1$

$q_2 = C_2 V_2$

$\vdots$

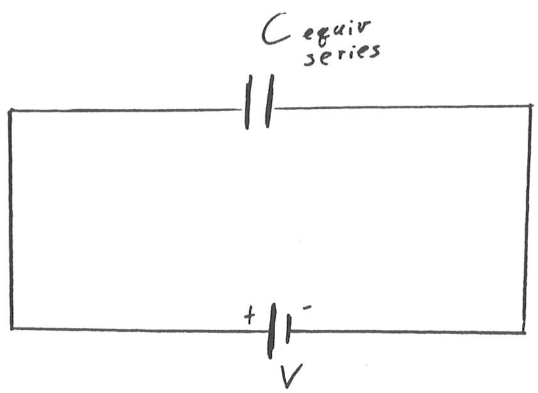
$$q_1 = q_2 = q_3 \cdots = q_{87} \equiv q$$

$$V_1 = \frac{q_1}{C_1} = \frac{q}{C_1} \qquad V_2 = \frac{q_2}{C_2} = \frac{q}{C_2}$$

Potential difference (voltage) between A and B

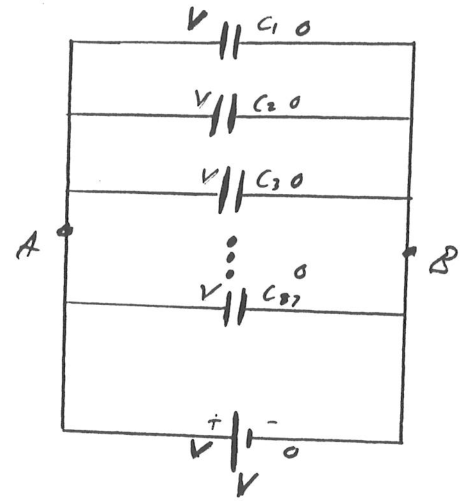
$$V_{AB} = V = V_1 + V_2 + V_3 + \dots + V_{87}$$
$$= \frac{q}{C_1} + \frac{q}{C_2} + \dots \frac{q}{C_{87}}$$
$$= q\left(\frac{1}{C_1} + \frac{1}{C_2} + \dots \frac{1}{C_{87}}\right)$$

$$q = \left[\frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \dots \frac{1}{C_{87}}}\right] V = C_{\substack{equiv \\ series}} V$$



$$\boxed{\frac{1}{C_{\substack{equiv\\series}}} = \sum_{i=1}^{N} \frac{1}{C_i}}$$

Capacitors in Parallel



$q_1 = C_1 V_1$

$q_2 = C_2 V_2$

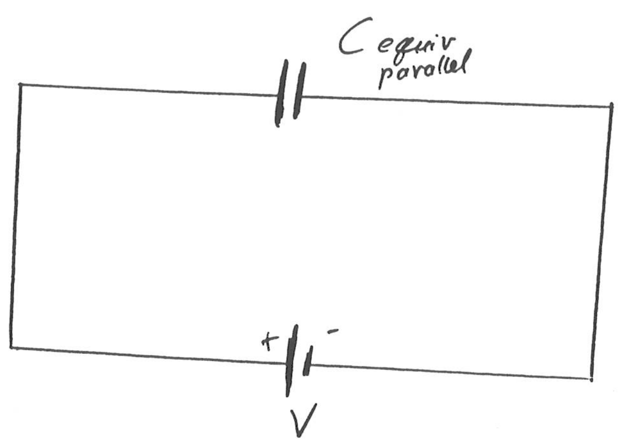
$\vdots$

V_1 = potential difference across C_1
$= V$

$V_2 = V_3 = \cdots V_{87} = V$

Total charge stored by all cap's

$$q_T = q_1 + q_2 + q_3 + \cdots q_{87}$$
$$= C_1 V_1 + C_2 V_2 + \cdots C_{87} V_{87}$$
$$= (C_1 + C_2 + \cdots + C_{87}) V$$
$$= C_{\substack{equiv\\parallel}} V$$



$$C_{\substack{equiv\\parallel}} = \sum_{i=1}^{N} C_i$$

What idea did we use?

The potential difference between two points, A and B, in the circuit can be obtained by following <u>any</u> path through the circuit.

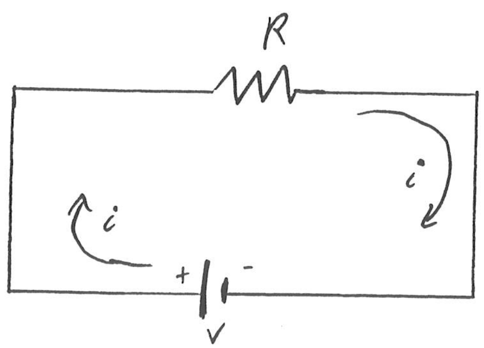
Kirchhoff's Loop Rule:

The algebraic sum of the changes in potential encountered in a complete traversal of any circuit must be zero.

Now consider circuits containing batteries and resistors, but no capacitors. In the steady state, a non-zero constant current flows through the circuit forever.

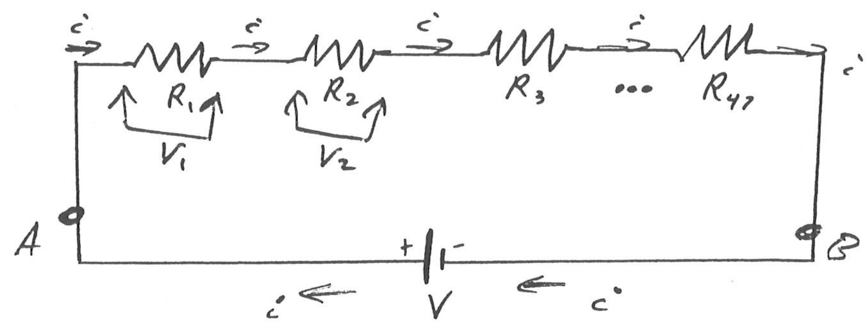
$V = iR$

"Ohm's Law"



The current must be the same in both sides of the resister because charge can not "pile up" anywhere in the circuit.

Resistors in Series



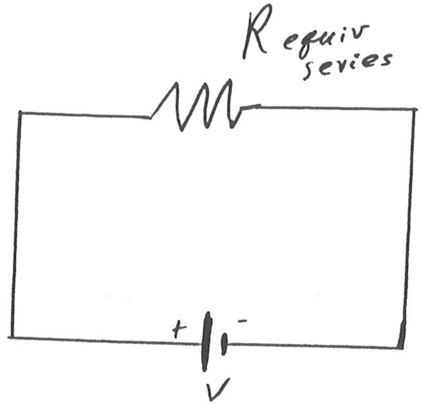
$V_{AB} = V = V_1 + V_2 + \cdots V_{47}$

potential difference between two sides of the resistor R_1

$= iR_1 + iR_2 + \cdots iR_{47}$

$= i(R_1 + R_2 + \cdots R_{47})$

$= i\, R_{equiv\ series}$



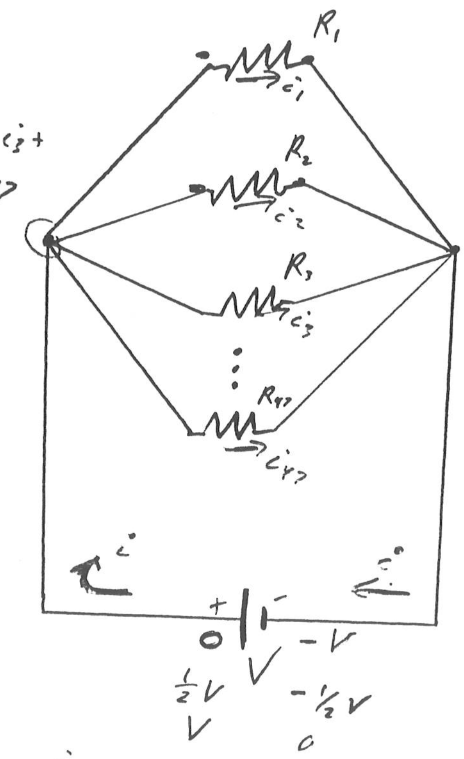
$$R_{\substack{equiv\\series}} = \sum_{i=1}^{N} R_i$$

looks like

$$C_{\substack{equiv\\parallel}} = \sum_{i=1}^{N} C_i$$

Resistors in Parallel

$i = i_1 + i_2 + i_3 + \cdots + i_{47}$



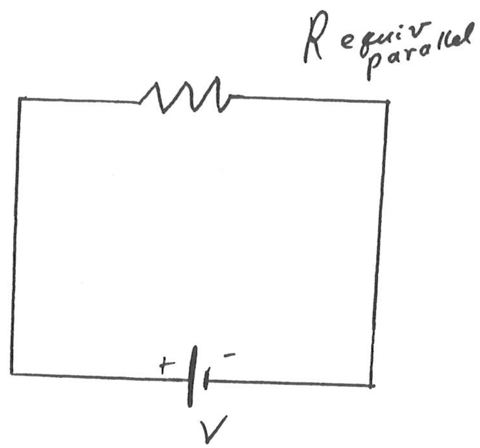
$V_1 = V = i_1 R_1$

$V_2 = V = i_2 R_2$

$\vdots$

$V_{47} = V = i_{47} R_{47}$

$$i = i_1 + i_2 + \cdots i_{47}$$
$$= \frac{V_1}{R_1} + \frac{V_2}{R_2} + \cdots \frac{V_{47}}{R_{47}}$$
$$= V\left(\frac{1}{R_1} + \frac{1}{R_2} + \cdots \frac{1}{R_{47}}\right)$$
$$V = i\left[\frac{1}{1/R_1 + 1/R_2 + \cdots 1/R_{47}}\right] = i \cdot R_{\substack{equiv\\parallel}}$$



$$\frac{1}{R_{\text{equiv parallel}}} = \sum_{i=1}^{N} \frac{1}{R_i}$$

looks like

$$\frac{1}{C_{\text{equiv series}}} = \sum_{i=1}^{N} \frac{1}{C_i}$$

What principle did we use this time?

Electric charge is conserved, so it can't "pile up" anywhere in the circuit. Whatever charge flows <u>into</u> a junction must flow <u>out</u>.

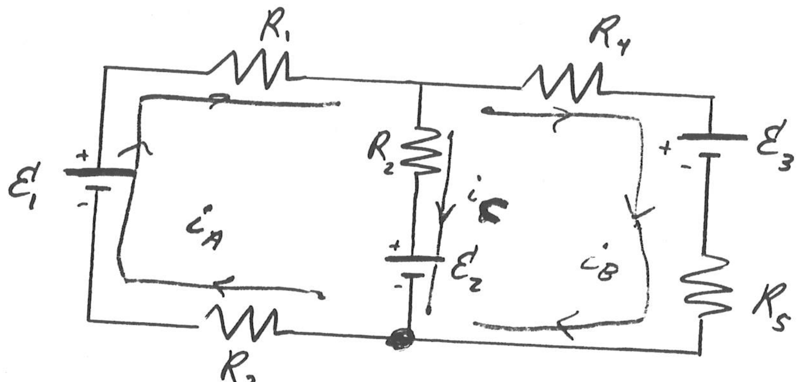
Kirchhoff's Junction Rule:

The sum of the currents approaching any junction must be equal to the sum of the currents leaving that junction.

Steady state: Charge Conservation implies Current Conservation

Circuits containing more than one Battery

P. 800

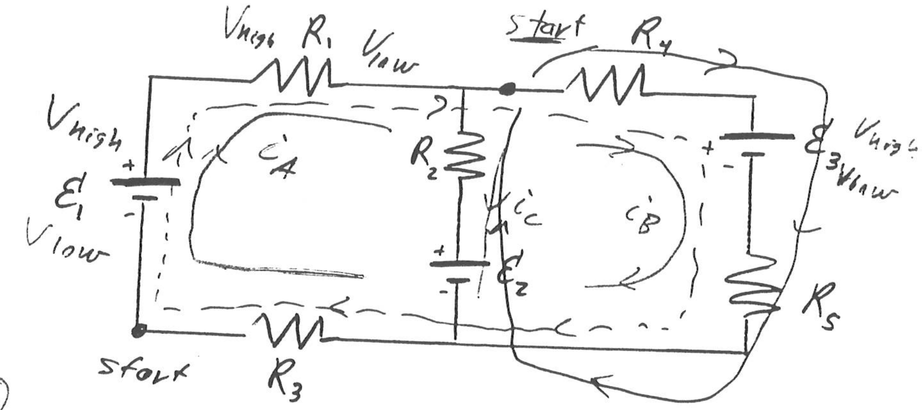


$i_B + i_C = i_A$ KJR

1) Try to replace resistors by equivalent resistances (series or parallel)
2) Draw currents with directions
3) Kirchhoff's Junction rule
4) Kirchhoff's Loop rule

None of the resistors are in series or parallel. Now what?

Circuits containing more than one Battery

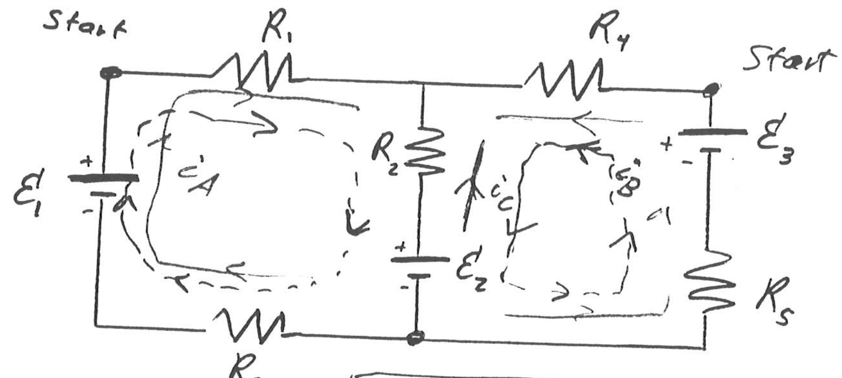


(1) $0 = +\mathcal{E}_1 - i_A R_1 - i_B R_4 - \mathcal{E}_3 - i_B R_5 - i_A R_3$

(2) $i_A = i_B + i_C$

(3) $0 = -i_B R_4 - \mathcal{E}_3 - i_B R_5 + \mathcal{E}_2 + i_C R_2$

Circuits containing more than one Battery

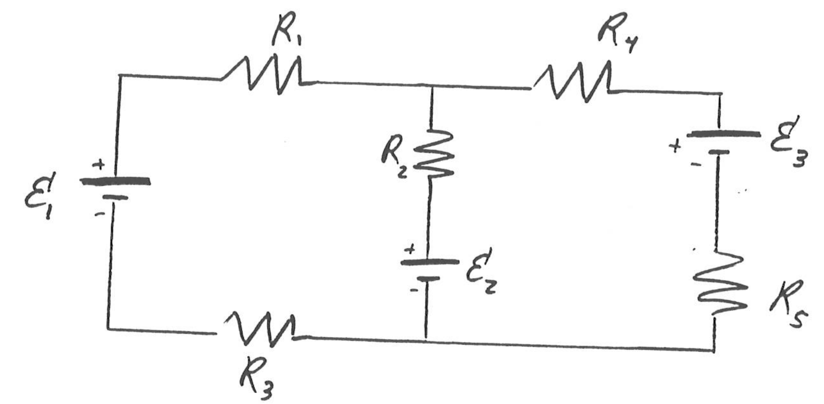


$\boxed{0 = i_A + i_B + i_C}$ ①

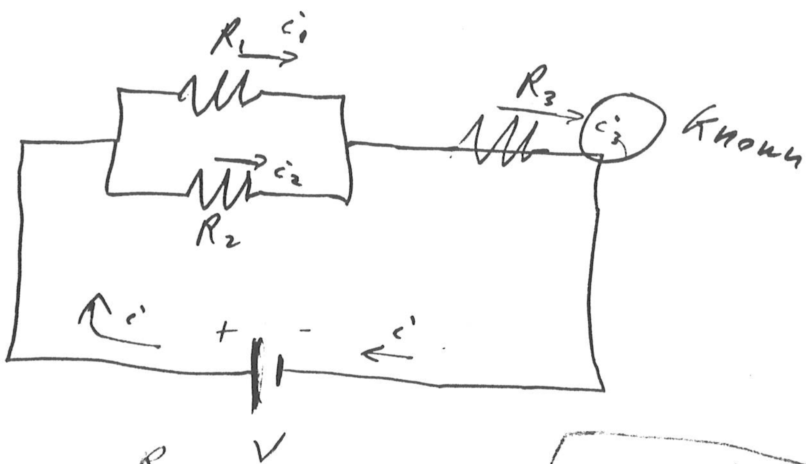
② $0 = -i_A R_1 + i_C R_2 - \mathcal{E}_2 - i_A R_3 + \mathcal{E}_1$

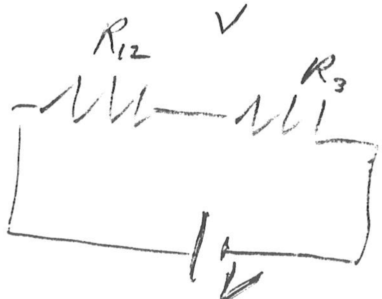
③ $0 = -i_B R_4 + i_C R_2 - \mathcal{E}_2 - i_B R_5 + \mathcal{E}_3$

Circuits containing more than one Battery



A simple Circuit





$$\boxed{R_{12} = \frac{1}{1/R_1 + 1/R_2}}$$

$$\frac{1}{R_{12}} = \frac{1}{R_1} + 1/R_2$$

$$R_{123} = R_{12} + R_3$$

$$= \frac{1}{1/R_1 + 1/R_2} + R_3$$

$$V = i(R_{123})$$

$$i = \frac{V}{R_{123}}$$

KJR

$$\boxed{i_1 + i_2 = i_3}$$

KLR

$$0 = -i_1 R_1 - i_3 R_3 + V$$

$$i_1 = \frac{V - i_3 R_3}{R_1}$$

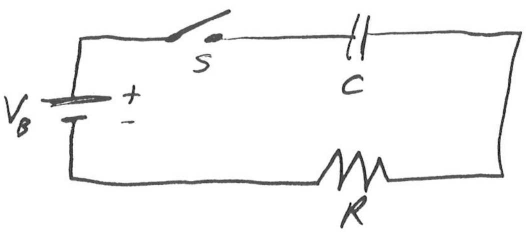
i_1 known

$$i_2 = i_3 - i_1$$

i_2 known

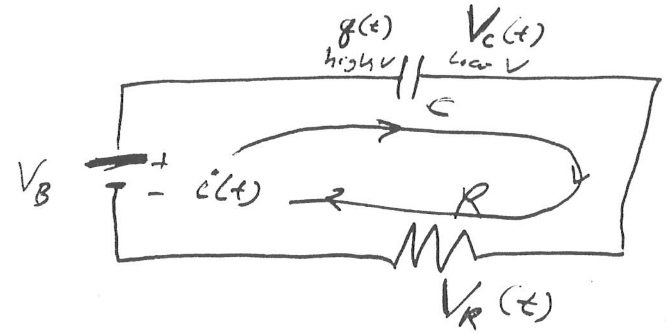
RC Circuits

Circuits containing a Resistor, a Capacitor, and possibly a battery, in series.



If you wait long enough, the steady state is boring: $i = 0$.

But this is our first opportunity to study a time-dependent current $i(t)$.



At any time t, Kirchhoff's Loop Rule gives:

$$0 = V_B - V_c(t) - V_R(t)$$

$$0 = V_B - \frac{q(t)}{C} - i(t)R$$

$q(t) = CV_c$

$V_R(t) = i(t)R$

but the current is the time rate of change of the charge:

$$i(t) = \frac{d}{dt}\left[q(t)\right]$$

This is a differential equation.

A solution will tell you the charge at any time. That is, the solution is not a number, but rather a <u>function</u> of time $q(t)$.

$$0 = V_B - \frac{q(t)}{C} - R\frac{d}{dt}\left[q(t)\right]$$

Guess:

Try a solution of the form:

$$q(t) = Ae^{-t/RC} + B$$

constants

$$i(t) = \frac{d}{dt}\left[q(t)\right] = -\frac{A}{RC}e^{-t/RC}$$

$$\frac{d}{dt}e^{-\frac{t}{z}} = \left(\frac{-1}{z}\right)e^{-t/z}$$

Substitute:

$$0 = V_B - \frac{Ae^{-t/RC} + B}{C} + R\left[\frac{A}{RC}e^{-t/RC}\right]$$

$$0 = V_B - \underbrace{\frac{A}{C}e^{-t/RC}} - \frac{B}{C} + \underbrace{\frac{A}{C}e^{-t/RC}}$$

$$V_B = \frac{B}{C} \Rightarrow B = CV_B$$

"A" is still not fixed

$$q(t) = Ae^{-t/RC} + CV_B$$

Fix A with a so-called <u>boundary condition</u>: the charge on the capacitor at $t=0$ is zero.

The capacitor is neutral before the switch is closed.

$$q(0) = 0$$

$$0 = q(0) = Ae^{-\frac{0}{RC}} + CV_B = A + CV_B$$

$$\Rightarrow \quad A = -CV_B = -B$$

<u>Solution:</u>

$$q(t) = CV_B\left(1 - e^{-t/RC}\right)$$

$$i(t) = \frac{d}{dt}[q(t)] = \frac{CV_B}{RC}e^{-t/RC} = \frac{V_B}{R}e^{-t/RC}$$

$$V_C(t) = \frac{q(t)}{C} = V_B\left(1 - e^{-t/RC}\right)$$

$$V_R(t) = i(t)R = V_B e^{-t/RC}$$

Check the solution!

$$0 \stackrel{?}{=} V_B - V_C(t) - V_R(t)$$

$$= V_B - V_B\left(1 - e^{-t/RC}\right) - V_B e^{-t/RC} = 0 \checkmark$$

Consider some special times:

$\boxed{t=0}$ $\quad q(0) = 0 \qquad V_C(0) = 0$

$\quad i(0) = \frac{V_B}{R} \qquad V_R(0) = V_B$

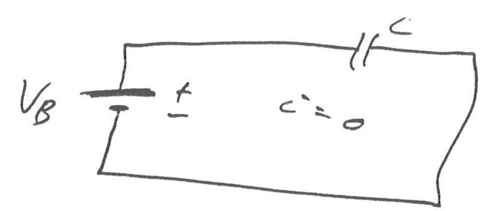
What does this mean?

no capacitor!

$\boxed{t=\infty}$ $\quad q(\infty) = CV_B \qquad V_C(\infty) = V_B$

$\quad i(\infty) = 0 \qquad V_R(\infty) = 0$

What does this mean?



no resistor!

Capacitive Time Constant

RC has the dimension of time

$[RC] = T$

In MKS units: $1\,\Omega \cdot F = 1$ second

In the time RC, the charge on the capacitor has increased from

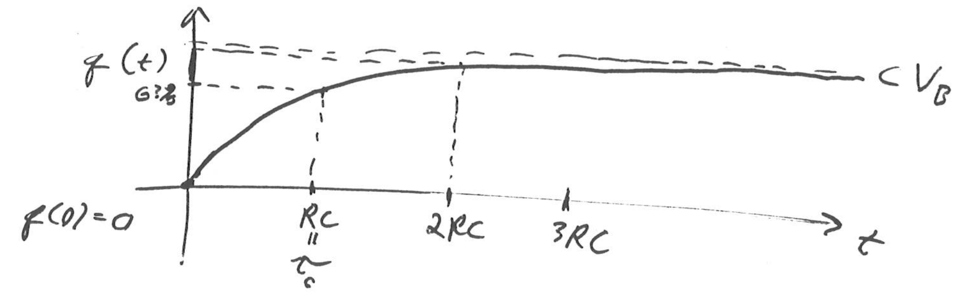
$q(0) = 0$ to

$$q(RC) = CV_B\left(1 - e^{-\frac{RC}{RC}}\right)$$

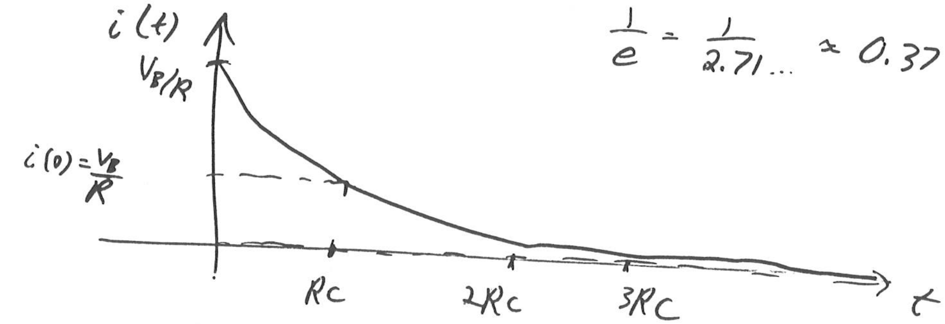
$$= CV_B\left(1 - e^{-1}\right)$$

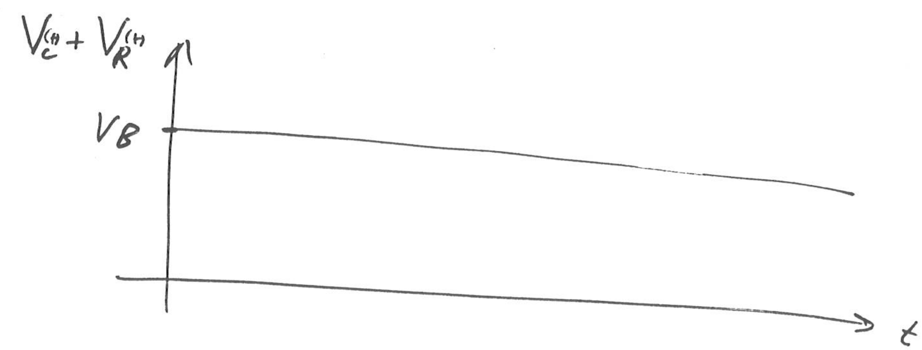
$$\approx 63\%\ CV_B$$

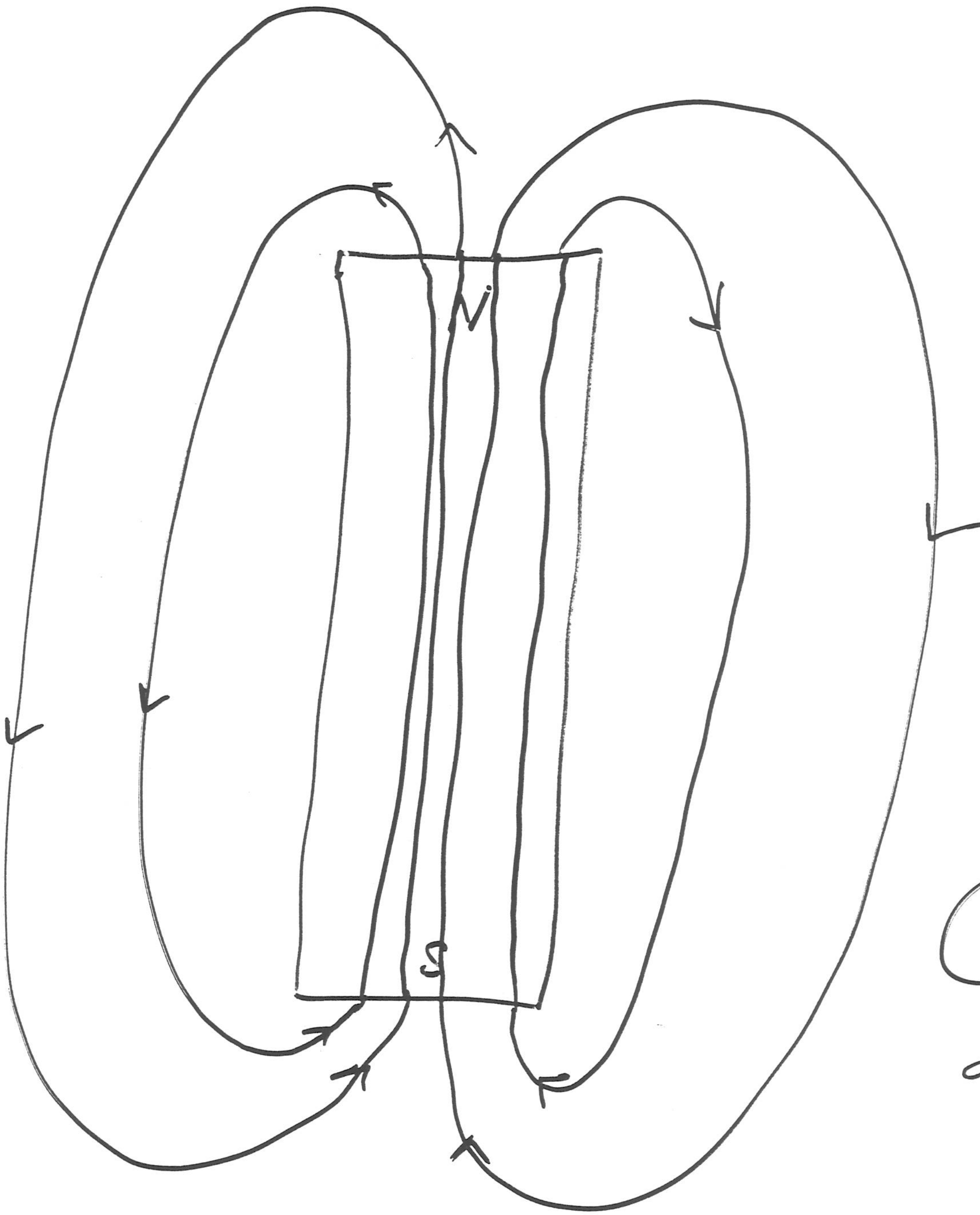
that is 63% of its fully charged value.

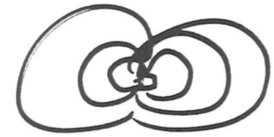


$$\frac{1}{e} = \frac{1}{2.71\ldots} \approx 0.37 = 37\%$$









dipole

All $\vec{B}$ lines are <u>closed</u> loops.

The Magnetic Field

The magnetic field is **not** produced by "magnetic charges" called magnetic monopoles.

Instead, <u>electric</u> charges in motion, that is, <u>currents</u> produce the magnetic field $\vec{B}$.

Recall: the electric force is

$$\vec{F}_e = q\vec{E}$$

force on charge q

field produced by all charges **except** q.

We derived this from Coulomb's Law

$$\vec{F}_{\substack{\text{of } 1 \\ \text{on } 2}} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{(r_{12})^2} \hat{r}_{1\to 2}$$

from experiment

We determine the magnetic force from experiment also:

$$\vec{F}_m = q\vec{v} \times \vec{B}$$

force on charge q

velocity of charge q

magnetic field produced by all moving charges **except** q

"cross product" or "vector product"

The cross-product is <u>perpendicular</u> to both vectors is the product.

$\vec{F}_m$ is $\perp$ to $\vec{v}$

$\vec{F}_m$ is $\perp$ to $\vec{B}$

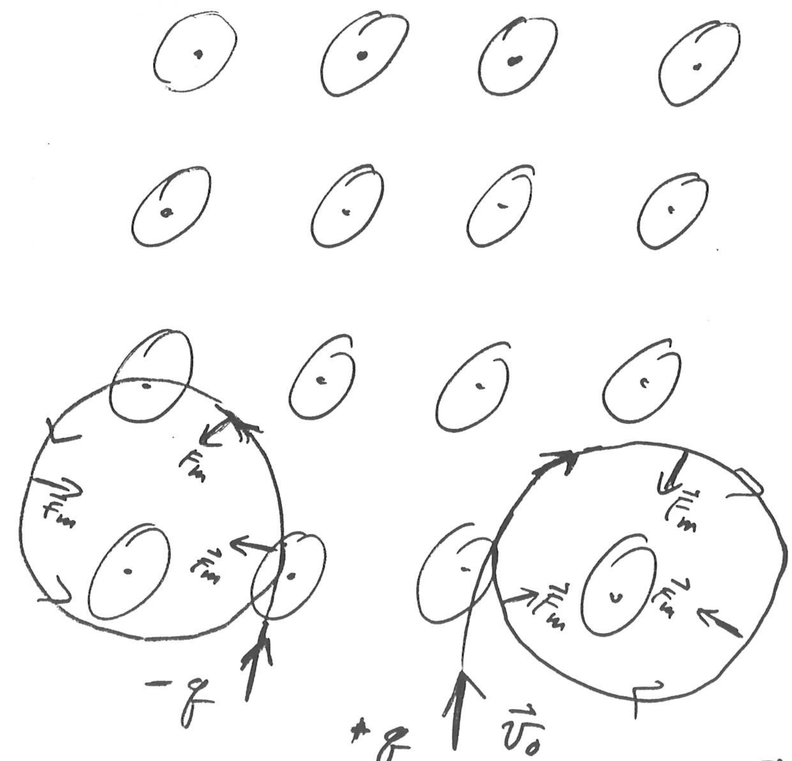
$\vec{v}$ and $\vec{B}$ can be $\perp$ or $\parallel$ or anything in between.

Consequence:

The magnetic force <u>does no work</u>.

<u>Power</u> $P_{inst} = \vec{F}\cdot\vec{v} = \dfrac{dW}{dt}$

$$W_m = \int_0^T P(t)\,dt = \int_0^T \cancel{\vec{F}_m\cdot\vec{v}}^{\,0} = 0$$

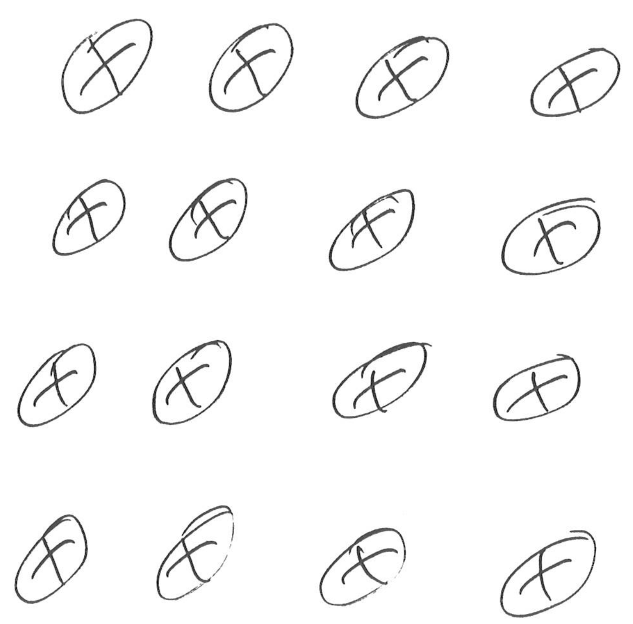


$\vec{B}$ out of page

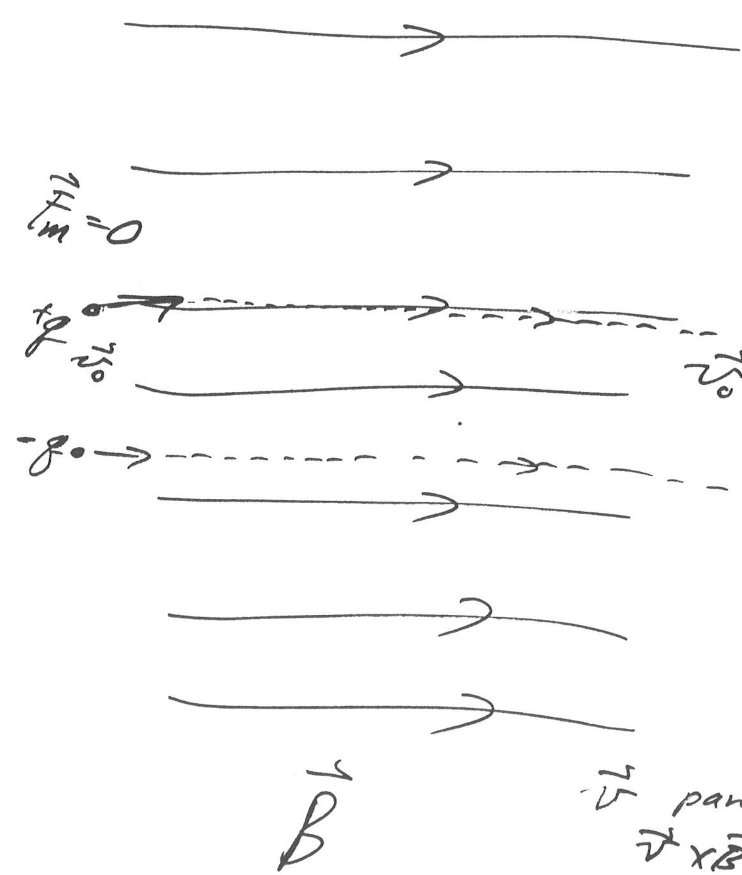
$\vec{F}_m = q\vec{v}\times\vec{B}$

$\vec{F}_m$ centripetal force

speed = constant = $|\vec{v}_0|$



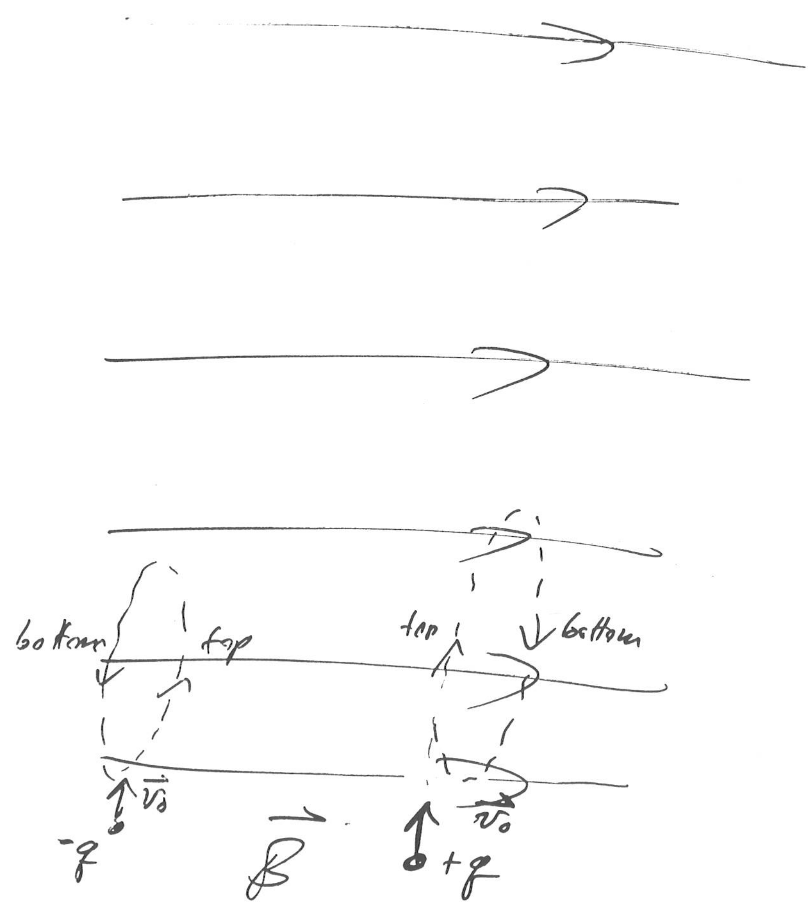
$\vec{B}$ into page

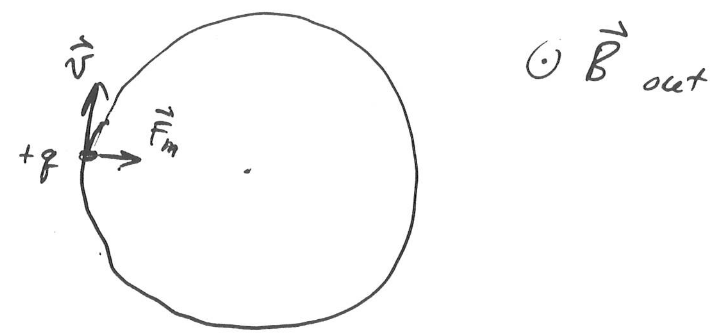


$\vec{v}$ parallel to $\vec{B}$

$\vec{v} \times \vec{B} = 0$

Radius of orbit in a $\vec{B}$ field





$$\sum F_r = m a_r$$

$$F_m = qvB\sin\theta = m\frac{v^2}{R}$$

$$= qvB(1) = \frac{mv^2}{R}$$

$$R = \frac{mv}{qB}$$

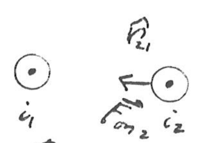
From Experiment:

The force due to current i_1 on i_2 in a wire of length ℓ is

$$\vec{F}_{1 \text{ on } 2} = \left(\frac{\mu_0}{4\pi}\right) \frac{2\ell\, i_1 i_2 \cos\theta}{r_2}\, \hat{r}_{21}$$

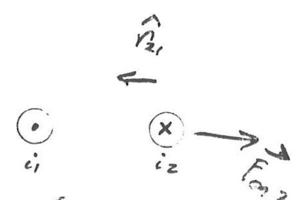
θ is the angle between i_1 and i_2.

$\hat{r}_{21}$ is a unit vector from i_2 to i_1.



$\theta = 0$

$\cos\theta = 1$



$\theta = 180°$

$\cos\theta = -1$

<u>attractive</u> if i_1 and i_2 are <u>parallel</u>.

<u>repulsive</u> if i_1 and i_2 are <u>antiparallel</u>.

<u>zero</u> if i_1 and i_2 are <u>perpendicular</u>.

The constant

$$\mu_0 \equiv 4\pi \times 10^{-7}\ T\cdot m/A$$

is called the <u>permeability of free space</u>.

and "T" stands for "Tesla", the MKS unit of magnetic field $\vec{B}$.

$$1\,T = 1\,\frac{N}{A\cdot m}$$

The electric field unit does not have a special name. The MKS unit of $\vec{E}$ is

$$\frac{V}{m} = \frac{N}{C}$$

$$\boxed{1V = \frac{1J}{C}}$$

From the last chapter, the force on a charge $\underline{q}$ moving with velocity $\underline{\vec{v}}$ in a magnetic field $\underline{\vec{B}}$ is:

$$\vec{F}_m = q\,\vec{v}\times\vec{B}$$

Can we reconcile this with:

$$\vec{F}_{\text{of 1 on 2}} = \frac{\mu_0}{4\pi}\,\frac{2\ell\, i_1 i_2 \cos\theta}{r_2}\,\hat{r}_{21} \quad ?$$

- if current i_2 flows for time T, then charge $q_2 = i_2 T$ has passed by.
- if the charge q_2 moves with speed v, then it flows a distance $\ell = vT$.

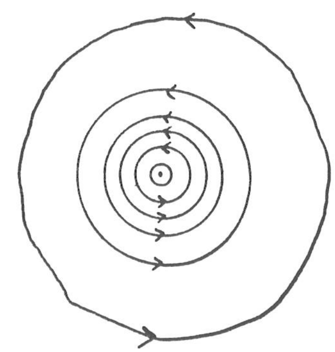
$$\vec{F}_{\text{of 1 on 2}} = \frac{\mu_0}{4\pi}\,\frac{2(v\cancel{T})\, i_1 \left(\frac{q_2}{\cancel{T}}\right)\cos\theta}{r_2}\,\hat{r}_{21}$$

So the magnetic field due to current i_1 in a <u>straight</u> wire is

$$B_1 = \frac{\mu_0}{4\pi}\,\frac{2 i_1}{r} \qquad \text{(magnitude)}$$

How about <u>direction</u>?

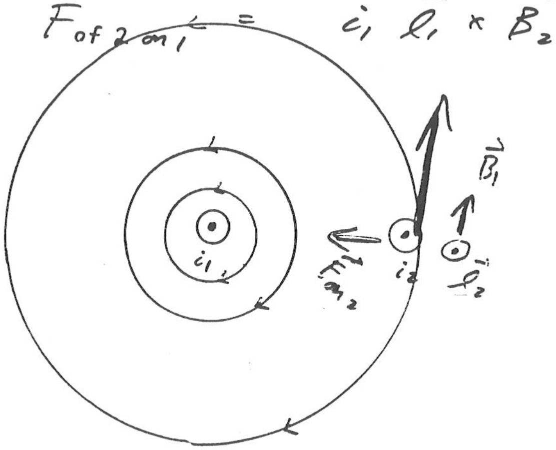
To recover the experimental laws of attraction and repulsion for parallel and antiparallel currents, the $\vec{B}$ field must look like:



- more dense (stronger $\vec{B}$ field) close to the wire
- right hand rule
- lines of $\vec{B}$ never end (no magnetic charges - monopoles)

$\vec{F}_{of\ 1\ on\ 2} = i_2\ \vec{\ell}_2 \times \vec{B}_1$ $\qquad$ $\vec{F}_{of\ 1\ on\ 2} = q_2 \vec{E}_1$

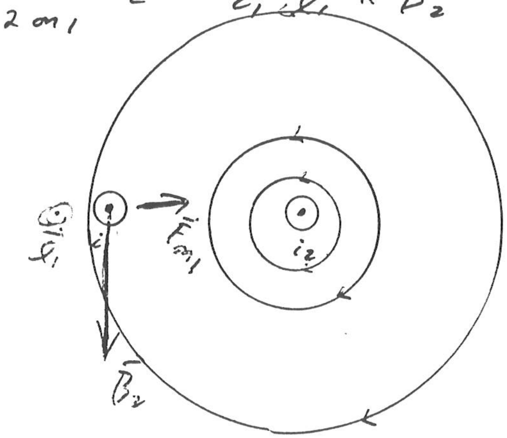
$\vec{F}_{of\ 2\ on\ 1} = i_1\ \vec{\ell}_1 \times \vec{B}_2$ $\qquad$ $\vec{F}_{of\ 2\ on\ 1} = q_1 \vec{E}_2$



$\vec{\ell}_2$ points along i_2

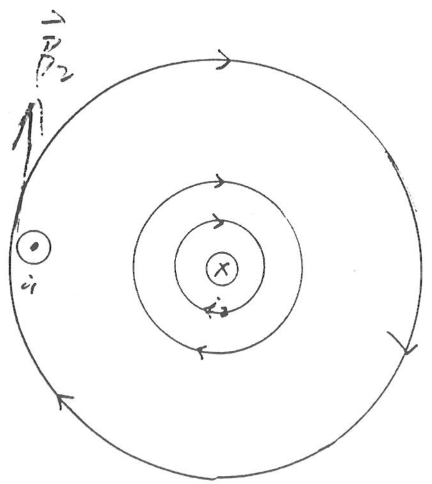
$\vec{F}_{of\ 1\ on\ 2} = i_2\ \vec{\ell}_2 \times \vec{B}_1$ $\qquad$ $\vec{F}_{of\ 1\ on\ 2} = q_2 \vec{E}_1$

$\vec{F}_{of\ 2\ on\ 1} = i_1\ \vec{\ell}_1 \times \vec{B}_2$ $\qquad$ $\vec{F}_{of\ 2\ on\ 1} = q_1 \vec{E}_2$



$$\vec{F}_{\text{of 1 on 2}} = i_2 \vec{\ell}_2 \times \vec{B}_1 \qquad \vec{F}_{\text{of 1 on 2}} = q_2 \vec{E}_1$$

$$\vec{F}_{\text{of 2 on 1}} = i_1 \vec{\ell}_1 \times \vec{B}_2 \qquad \vec{F}_{\text{of 2 on 1}} = q_1 \vec{E}_2$$



What if the wire is not straight?

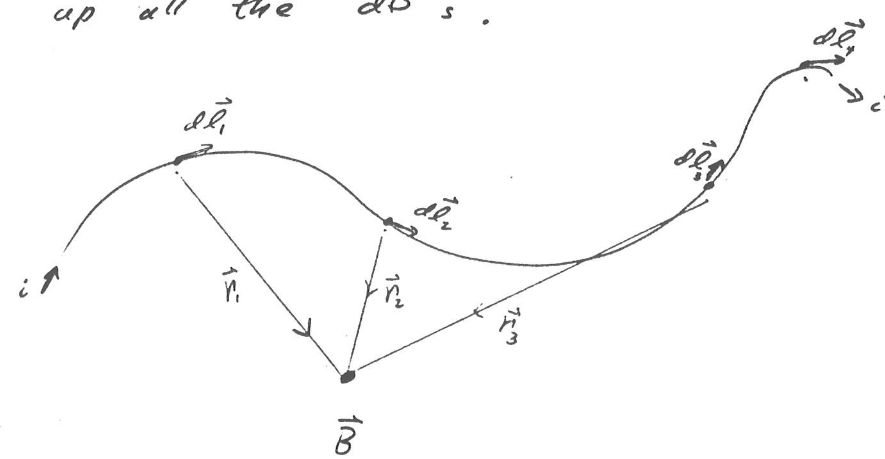
Then the piece of the magnetic field due to an infinitesimal chunk of wire (of length $d\ell$) is

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{i \, d\vec{\ell} \times \vec{r}}{r^3}$$

$\vec{r}$ points from the current element to the field point.

(Biot-Savart Law)

To get the total $\vec{B}$ field, simply integrate along the wire and add up all the $d\vec{B}$'s.



What if the wire is <u>not</u> straight?

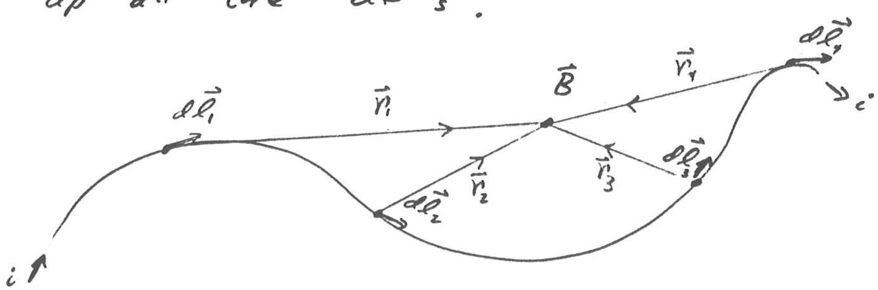
Then the piece of the magnetic field due to an infinitesimal chunk of wire (of length $d\ell$) is

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{i\, d\vec{\ell} \times \vec{r}}{r^3}$$

$\vec{r}$ points <u>from</u> the current element <u>to</u> the field point.

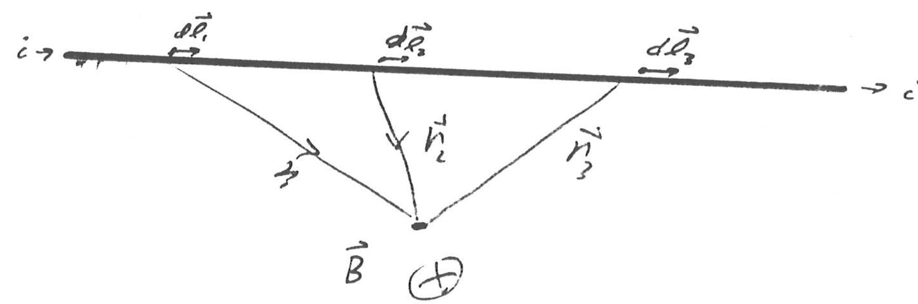
(Biot-Savart Law)

To get the total $\vec{B}$ field, simply integrate along the wire and add up all the $d\vec{B}$'s.



Does this work for a straight wire?

$\vec{B}$ ⊙

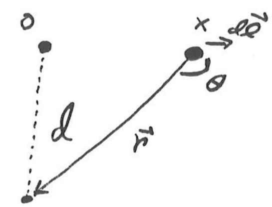


$$\vec{B} = \int_{x=-\infty}^{+\infty} d\vec{B} = \int_{-\infty}^{+\infty} \frac{\mu_0}{4\pi} \frac{i\, d\vec{\ell} \times \vec{r}}{r^3} = \sum_{i=1}^{\infty} \frac{\mu_0}{4\pi} \frac{i\, d\vec{\ell}_i \times \vec{r}_i}{(r_i)^3}$$

$$B = \frac{\mu_0}{4\pi} \frac{2i}{r}$$

See HRW pages 850-1 for a proof.

$$d\vec{\ell} \times \vec{r} = |d\vec{\ell}|\,|\vec{r}| \sin\theta$$
$$= dx\, r \sin\theta$$



$$|d\vec{\ell}| = dx$$
$$r = \sqrt{d^2 + x^2}$$

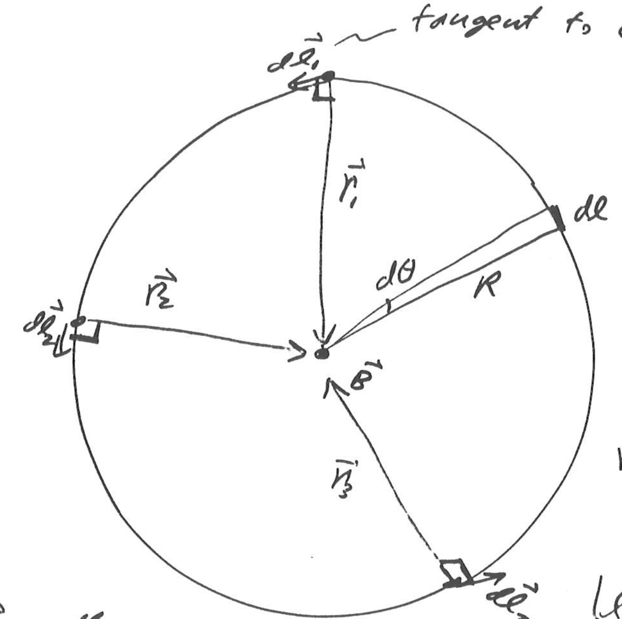
$$\vec{B} = \int d\vec{B} = \int \frac{\mu_0}{4\pi} \frac{i\, d\vec{\ell} \times \vec{r}}{r^3}$$

$$|\vec{B}| = \frac{\mu_0 i}{4\pi} \int \frac{dx\, \cancel{r} \sin\theta}{r^{3/2}} = \frac{\mu_0 i}{4\pi} \int_{-\infty}^{+\infty} \frac{dx}{d^2 + x^2} \frac{d}{r}$$

$$= \frac{d\mu_0 i}{4\pi} \int_{-\infty}^{\infty} \frac{dx}{(d^2+x^2)^{3/2}} = \frac{\mu_0}{4\pi} \frac{2i}{d}$$ check this

direction $\boxed{\text{into}}$

<u>Ex</u> $\vec{B}$ at the center at a circle of current



$$r_1 = r_2 = r_3 \cdots r_i$$
$$= R$$

$$d\vec{B}_1 = \frac{\mu_0}{4\pi}\, i\, \frac{d\vec{\ell}_1 \times \vec{r}_1}{r_1^3}$$

$$|d\vec{\ell} \times \vec{r}| =$$
$$d\ell\, r\, \cancel{\sin\theta}^{\,1}$$
$$= d\ell\, R$$

$$|\vec{B}| = \int |d\vec{B}| = \int \frac{\mu_0 i}{4\pi} \frac{d\ell\, R}{R^3} = \frac{\mu_0 i}{4\pi R^2} \int d\ell$$

$$= \text{circumference} = 2\pi R$$

$$\int d\ell = \int_0^{2\pi} R\, d\theta = R \int_0^{2\pi} d\theta = 2\pi R$$

$$|\vec{B}| = \left(\frac{\mu_0}{4\pi}\right) \frac{i\, 2\pi}{R} = K_m \frac{2\pi i}{R}$$

direction $\boxed{\text{out}}$

Gauss' Law

infinitesimal vector that points out

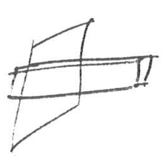
$$\oiint_S \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

closed surface

evaluated on the closed surface S

spherical Gaussian surface

cylinder



pillbox

Ampere's Law

infinitesimal line element points along the curve C

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 i_{enc}$$

evaluated on closed curve C

current within the closed curve C

closed curve

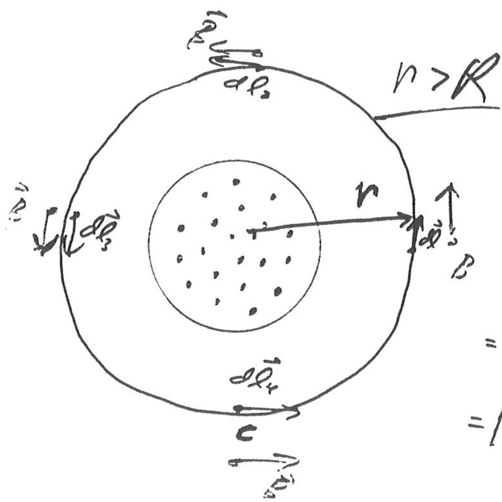
"Amperean Loops"

Useful in cases of high symmetry

Ex.

A straight wire of radius R carries a current I distributed uniformly over its cross-sectional area. Find the magnetic field everywhere.

$r > R$



$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 i_{enc} = \mu_0 I$$

$$= \oint_C |B||d\ell|$$

$$= |B| \oint_C d\ell$$

$$= |B|\, 2\pi r = \mu_0 I$$

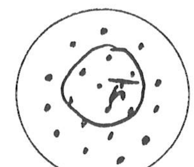
$$\therefore\ |B| = \frac{\mu_0 I}{2\pi r} = \left(\frac{\mu_0}{4\pi}\right)\frac{2I}{r}$$

Ex.

A straight wire of radius R carries a current I distributed uniformly over its cross-sectional area. Find the magnetic field everywhere.

$r < R$

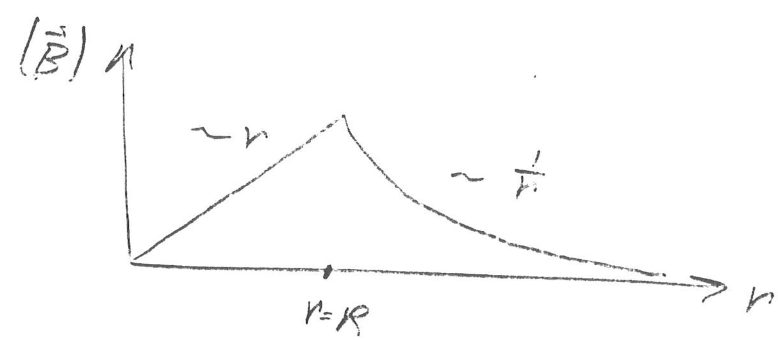
$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 i_{enc}$$



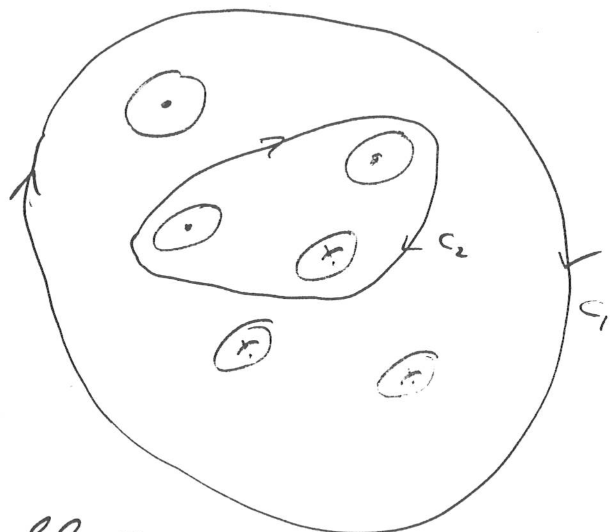
$$|B| \oint_C d\ell = \mu_0\left(\frac{I}{R^2} r^2\right)$$

$$|B|\, 2\pi r = \frac{\mu_0 I r^2}{R^2}$$

$$\therefore\ |B| = \left(\frac{\mu_0}{4\pi}\right)\frac{2Ir}{R^2}$$



Ex



$$\oint_{C_1} \vec{B}\cdot d\ell = 0$$

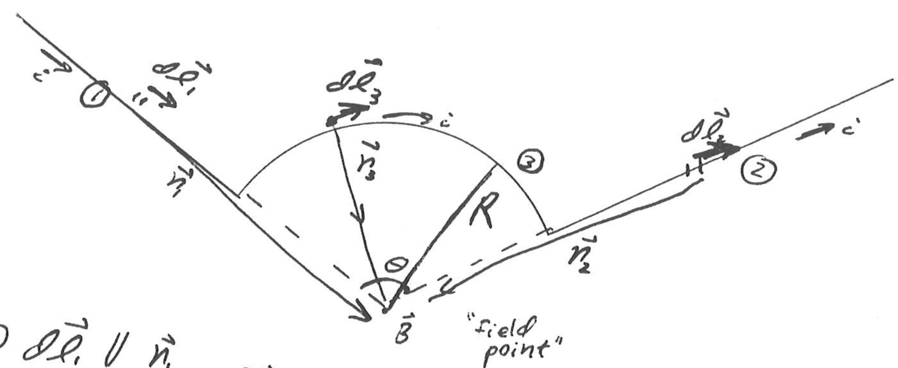
$$\oint_{C_2} \vec{B}\cdot d\vec{\ell} = \mu_0 I = \mu_0 (I+I-I)$$

If the direction of C_2 is reversed

then $i_{enc} = -I = (-I-I+I)$

Biot - Savart Examples

Ex Circular Arc



① $d\vec{l}_1 \parallel \vec{r}_1 \Rightarrow d\vec{l}_1 \times \vec{r}_1 = 0$

② $d\vec{l}_2 \parallel \vec{r}_2 \Rightarrow d\vec{l}_2 \times \vec{r}_2 = 0$

Biot - Savart Law: $d\vec{B} = \frac{\mu_0}{4\pi} \frac{i\, d\vec{l} \times \vec{r}}{r^3}$

③ $d\vec{l}_3 \perp \vec{r}_3 \Rightarrow |d\vec{l}_3 \times \vec{r}_3| = |dl|\,|r|\, \cancel{\sin\theta}\,1$

Direction of $\vec{B}$ is ⊗ into page

Magnitude

$dB = \frac{\mu_0}{4\pi} \frac{i\, dl\, r}{r^3}$ $\qquad r = R$

$\qquad dl = R\, d\theta$

$dB = \frac{\mu_0}{4\pi} \frac{i\, d\theta}{R}$

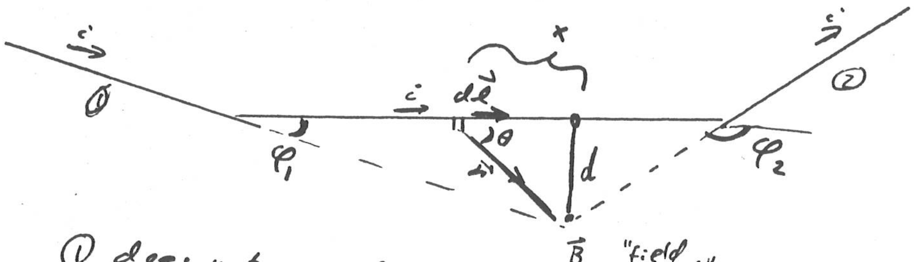
$$B = \int_0^\theta dB = \int_0^\theta \frac{\mu_0}{4\pi} \frac{i\, d\theta}{R}$$

$$\frac{\mu_0 i}{4\pi R} \int_0^\theta d\theta = \boxed{\frac{\mu_0}{4\pi} \frac{i\theta}{R}}$$

$\theta \to 2\pi$ full circle

$$B = \frac{\mu_0}{4\pi} \frac{i\, 2\pi}{R}$$

Ex: Magnetic field due to a finite length of current-carrying wire.



1, 2 does not contribute to $\vec{B}$

3 directions $\otimes$ into page

magnitude

Biot-Savart $\quad d\vec{B} = \frac{\mu_0}{4\pi} \frac{i\, d\vec{\ell} \times \vec{r}}{r^3}$

$|d\vec{\ell}| = dx$

$|d\vec{\ell} \times \vec{r}| = dx\, r \sin\theta$

$$dB = \frac{\mu_0}{4\pi} \frac{i\, dx \sin\theta}{r^2}$$

$$d = r \sin\theta \Rightarrow r = \frac{d}{\sin\theta}$$

$$dB = \frac{\mu_0}{4\pi} \frac{i\, dx \sin^3\theta}{d^2}$$

$$x = \frac{-d}{\tan\theta} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{d}{x}$$

$$dx = \frac{+d}{\sin^2\theta}\, d\theta \qquad \frac{d}{d\theta} \cot\theta = -\csc^2\theta$$

$$\frac{d}{d\theta}\left(\frac{1}{\tan\theta}\right) = \frac{-1}{\sin^2\theta}$$

$$dB = \frac{\mu_0}{4\pi}\, i \left(\frac{d}{\sin^2\theta}\right) \frac{\sin^3\theta}{d^2}\, d\theta = \frac{\mu_0}{4\pi} \frac{i \sin\theta}{d}\, d\theta$$

$$B = \int dB = \int_{\theta=\varphi_1}^{\varphi_2} \frac{\mu_0}{4\pi} \frac{i \sin\theta\, d\theta}{d}$$

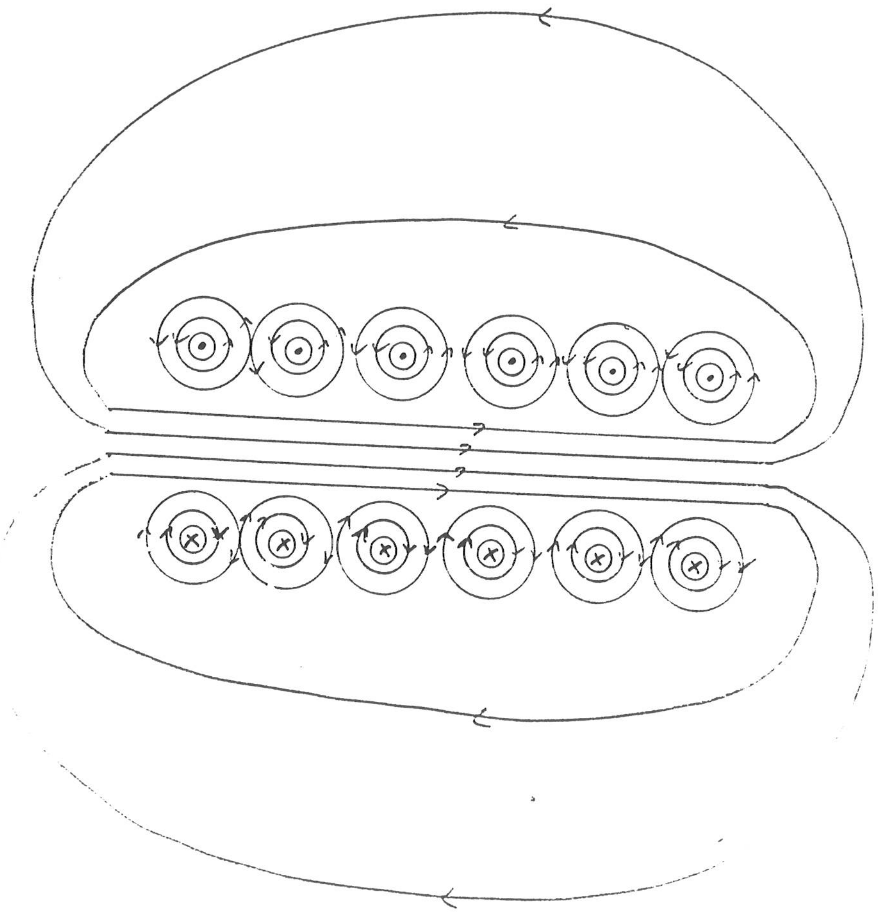
$$B = \frac{\mu_0}{4\pi} \frac{i}{d} \int_{\varphi_1}^{\varphi_2} \sin\theta\, d\theta$$

$$B = \frac{\mu_0}{4\pi} \frac{i}{d} \left[\cos\varphi_1 - \cos\varphi_2\right]$$

Check: Infinite wire $\varphi_1 = 0 \quad \varphi = 180$

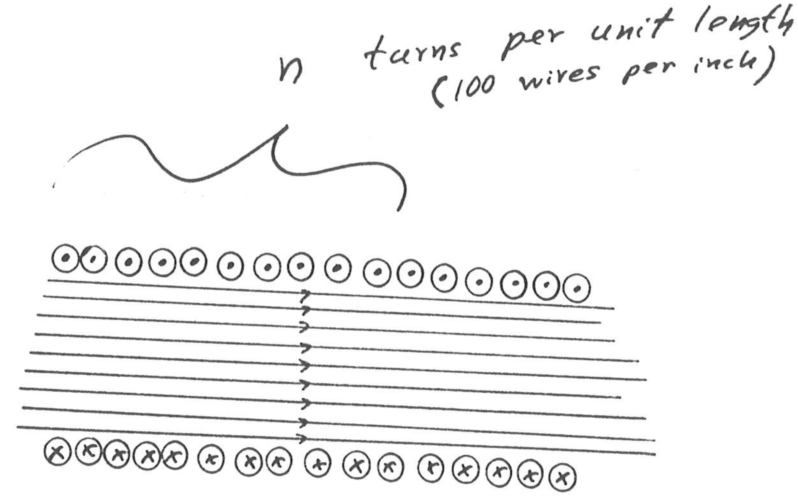
$$B = \frac{\mu_0}{4\pi} \frac{i}{d} [2] \checkmark$$

The Solenoid



$\vec{B}$ field lines are <u>closed</u> loops.

The Solenoid



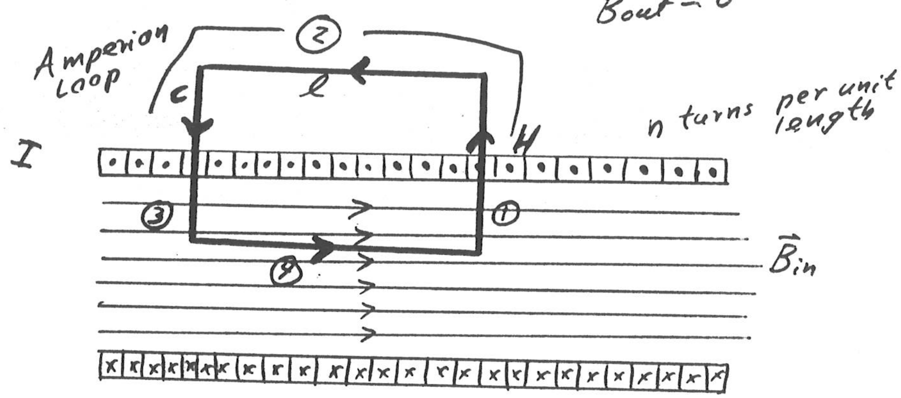
If the solenoid is very long compared to its radius and if the coils are closely spaced then:

$\vec{B}_{inside} \approx$ constant

$\vec{B}_{outside} \approx 0$

Well, not really, but the $\vec{B}$ field is <u>much</u> less dense outside.

Magnetic field inside a solenoid by Ampere's Law:



Ampere's Law: $\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 \, i_{enc}$

$$= \underbrace{\int_{①} \cancel{\vec{B}\cdot d\vec{\ell}}^{\,0}}_{\vec{B}\perp d\vec{\ell}} + \underbrace{\int_{②} \cancel{\vec{B}\cdot d\vec{\ell}}^{\,0}}_{B=0} + \underbrace{\int_{③} \cancel{\vec{B}\cdot d\vec{\ell}}^{\,0}}_{\vec{B}\perp d\vec{\ell}} + \int_{④} \vec{B}\cdot d\vec{\ell}$$

$$= \int_{④} \vec{B}\cdot d\vec{\ell} = |B| \int d\ell = B\ell$$

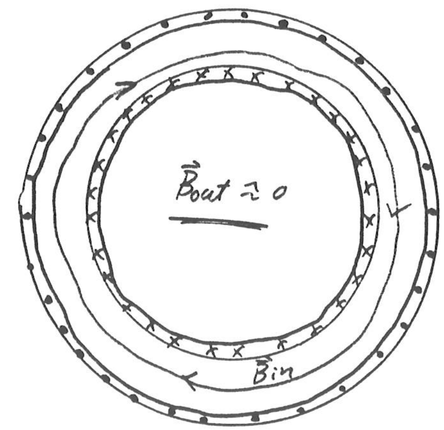
$$= \mu_0 \, i_{enc} = I \mu_0 \boxed{n\ell} \leftarrow \text{\# wires carrying } I$$

$$B\cancel{\ell} = \mu_0 I n \cancel{\ell} \qquad \boxed{B_{in} = \mu_0 I n}$$

The Toroid

Instead of making the solenoid infinitely long to get a very small $\vec{B}$ field outside, one can attach the open ends to each other to make a doughnut shape.

Total of N turns



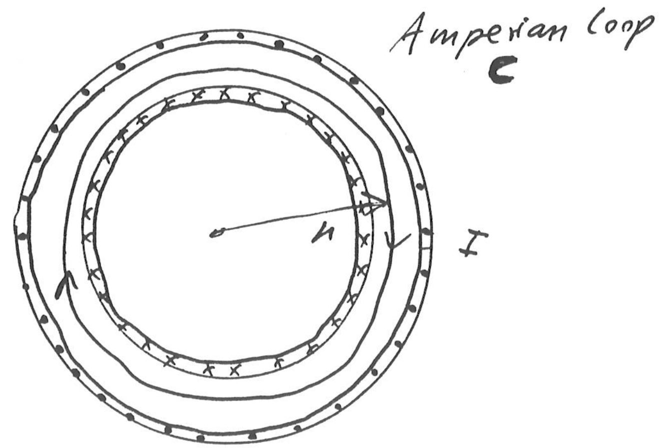
$\vec{B}_{out} \simeq 0$

This time, $\vec{B}_{in} \neq$ constant.

The Toroid

Instead of making the solenoid infinitely long to get a very small $\vec{B}$ field outside, one can attach the open ends to each other to make a doughnut shape.

Total of N turns



Ampere's law: $\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 i_{enc}$

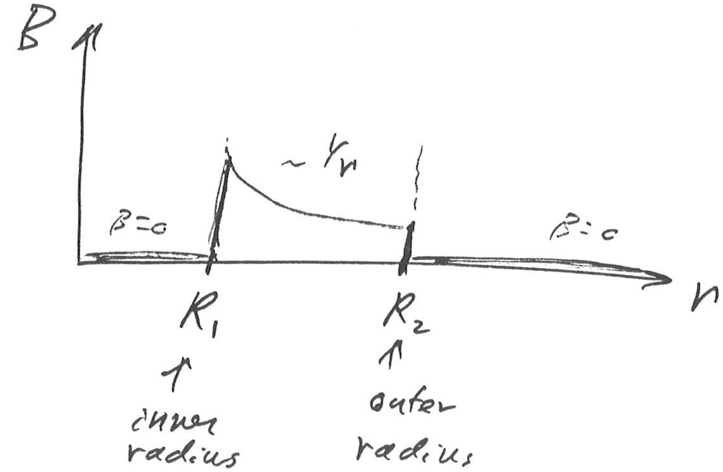
$\vec{B} \parallel d\vec{\ell}$ on Amperian Loop C

$\oint_C |B|\,|d\ell| = B \oint_C d\ell = B(2\pi r)$

$= \mu_0 i_{enc} = \mu_0 (N I)$

$$B_{inside} = \frac{\mu_0 N I}{2\pi r} = \boxed{\frac{\mu_0}{4\pi}\,\frac{2NI}{r}}$$

$B_{outside} = 0$



n = wires per unit lengt

$= \dfrac{N}{2\pi r}$

$$B_{toroid} = \frac{\mu_0}{4\pi}\,\frac{2I}{r}(2\pi r n) = \mu_0 I n$$

$= B_{solenoid}$